UNIVERSITY OF CALIFORNIA PUBLICATIONS

PUBLICATIONS

OF THE

LICK OBSERVATORY

PRINTED BY AUTHORITY OF THE REGENTS OF THE UNIVERSITY

VOLUME VII

SACRAMENTO
FRIEND WM. RICHARDSON . . . . SUPERINTENDENT OF STATE PRINTING
1913

ORGANIZATION OF THE BERKELEY ASTRONOMICAL DEPARTMENT.

(STUDENTS' OBSERVATORY.)

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BERKELEY ASTRONOMICAL DEPARTMENT,

(STUDENTS' OBSERVATORY.)

UNIVERSITY OF CALIFORNIA,

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PREFACE.

Parts 1, 2 and 3 of this volume were completed and printed in the fall of 1902. A number of separates were distributed by the Students' Observatory at that time. Parts 4, 5 and 6 were completed in May of 1905, spring of 1906, and March of 1907, respectively, but on account of pressure of work in the State Printing Office at Sacramento the last of these parts did not come off the press until the spring of 1909. Unfortunately, since the material for the completion of the volume was then ready and the prospects for having the remainder printed without delay seemed bright, no separates of Parts 4 to 6 were distributed at the proper time. The manuscript of Part 7 and practically also of Part 9, with which as Part 8 it was originally intended to conclude the volume was ready to be printed in April, 1909, but owing to the lack of University funds at the State Printing Office the printing of these papers was unavoidably delayed. It, therefore, seemed expedient to extend the scope of the volume so as to include the material which in the mean time had become available, as Parts 8 and 10, and to add the tables at the end of Part 7.

It may be of interest to point out that the opportunities for research are greatly limited in this department by our extensive duties of instruction, but this disadvantage is offset to some extent by the voluntary assistance given by graduate students in the numerical work incident to the investigations contained in these pages, particularly in the construction of tables and determination of orbits. The orbits included in Parts 8 and 10 as typical examples of the Short Methods form only a fraction of those derived with the assistance of students, mostly in connection with the regular courses of instruction.

The *Short Method*, originally proposed in Part 1, appears in a revised and greatly extended form in Parts 7 and 9.

I desire to take this opportunity to express to Director W. W. CAMPBELL of the Lick Observatory my appreciation of his coöperation in providing for the publication of papers by the staff of the Berkeley Astronomical Department by giving them space in the established publications of the Lick Observatory. By this arrangement it has become unnecessary to maintain a separate series of astronomical publications for the Berkeley Astronomical Department.

JUNE, 1911.

A. O. LEUSCHNER.

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By A. O. LEUSCHNER,

Director of the Students' Observatory,

Berkeley Astronomical Department.

UNIVERSITY OF CALIFORNIA PUBLICATIONS.

PUBLICATIONS OF THE LICK OBSERVATORY, VOL. VII, PART 1.*

A SHORT METHOD OF DETERMINING ORBITS FROM THREE OBSERVATIONS.

BY A. O. LEUSCHNER.

A few years ago Professor HARZER developed a very elegant method of determining orbits, without previous hypothesis as to the eccentricity, on the basis of a limited number of observations; five, in general, representing the greatest advantages (A. N. 3371). At least two essential points are gained by HARZER over LAPLACE'S method contained in Méc. cél., T. I., première partie, livre II, Chap. IV: the first is a practical gain, and involves an adjustment of the effects on the residuals, due to the coefficients of the higher powers, when the geocentric coördinates are developed as a function of the time by TAYLOR'S theorem; the second is a theoretical advantage, and confines the effects of perturbations to the accelerations and higher derivatives by the use of heliocentric coördinates, by means of which the higher derivatives may be expressed in terms of the coördinates, velocities, and accelerations at the normal date more readily than by means of geocentric coördinates.

From a theoretical point of view the method gives perfect satisfaction in the case to which it has been applied by HARZER. There are, however, practical difficulties involved which make it doubtful whether astronomers will adopt this method in preference to the short methods of determining circular elements for planets, to OLBERS'S and v. OPPOLZER'S methods of deriving parabolic orbits for comets, or even to the more general methods of GAUSS and OPPOLZER for the determination of orbits without previous hypothesis as to the eccentricity. Some of the difficulties, all of them of the utmost importance to computers, are as follows:

(A.) It is necessary to solve a system of five simultaneous equations for each of the two observed coördinates, before and after corrections for parallax and

* It has been thought best by all concerned that the results of astronomical researches made under the auspices of the University of California should be published in as few mediums as possible. I shall therefore be glad to give appropriate places in the established Lick Observatory publications to standard papers written by members of the Berkeley Astronomical Department.—W. W. CAMPBELL, Director.

aberration have been applied. In all, therefore, at least four such systems will have to be solved, and more if the first attempt does not give the geocentric distances with sufficient accuracy for the final determination of the parallax and aberration. This is all the more serious, as the greatest accuracy must be observed in determining the unknowns. In HARZER'S example the first attempt at the geocentric distances gave the final corrections for parallax and aberration. Further, not four, but only two systems of simultaneous equations needed to be solved, as it was possible to avoid the solution of the third and fourth systems by a somewhat less laborious process. This process is applicable whenever the parallax corrections in α and $\tan\delta$ are expressible in the linear forms

$$\delta\alpha_a = \eta_0 + \theta_a\,\eta_1\,; \quad \delta\tan\delta_a = \zeta_0 + \theta_a\,\zeta_1$$

where θ_a is the interval between an observed date t_a and the normal date. If the observations are made at observatories widely separated in latitude, or if the hour-angles differ greatly, the linear relations will not be sufficiently accurate, particularly not when, as in HARZER'S example, the calculation is conducted with seven-place logarithms. In such cases—which are as likely to occur as not—recourse must be had to the solution of a third and fourth system of five simultaneous equations.

(B.) No hints are given for the solution of the equation of the seventh degree which gives the geocentric distance at the normal date. This equation will have to be solved at least twice, before and after parallax and aberration are applied, and the necessary approximations will, in general, involve a large amount of numerical work.

(C.) After the residuals of the five observations have been determined on the basis of the last approximation for the geocentric distances, it becomes necessary to determine the coefficients of the six differential corrections to the rectangular heliocentric coördinates and their velocities at the normal date for ten observation equations. The total number of these coefficients is, therefore, sixty. If it is intended that the results shall satisfy only three of the available observations, thirty-six coefficients are still necessary. Although the solution for the six unknown quantities from the observation equations has been rendered quite simple, the amount of labor involved in determining the coefficients themselves does not seem warranted when the results are not based on normal places.

Freed from these disadvantages the methods of LAPLACE and HARZER will permit of the determination of an orbit, without previous hypothesis as to the eccentricity, more readily than OLBERS'S method for the determination of parabolic elements in the case of comets.

The essential improvements to be sought for are:

I. The restriction of the number of observations to three, the minimum number necessary for the solution of the problem. The slight gain in accuracy

resulting from the introduction of two additional observations is more than offset by the shortening of the numerical work.

II. A reduction of the number of fundamental data to be approximated. These data consist in (*a*) the two observed geocentric coördinates, their velocities and accelerations; (*b*) the rectangular heliocentric coördinates and their velocities—all for the normal date. This reduction involves no loss in accuracy.

III. A short method of obtaining preliminary values of the quantities to be approximated, which will do away with the solution of simultaneous equations.

IV. A short method of solving the equation of the seventh degree.

V. A short method of determining the final corrections to the data from which the elements are computed.

HARZER has based the determination of the fundamental data (*a*) on five observations to secure a greater accuracy in them by taking into account the third and fourth derivatives of the right ascension and declination with regard to the time. This procedure assumes that the errors of observation are considerably less appreciable than the effect of these higher derivatives. For, in cases where these effects are comparable with the errors of observation, less accurate values of the required fundamental data (*b*) will result from the accurate representation of all five observations than are attained by neglecting these derivatives and by deriving the most probable values of the required normal coördinates, velocities, and accelerations from the five observation equations available for each coördinate.

It is of course impossible to decide beforehand when the effect of the higher derivatives and the errors of observation are comparable. But it seems that for short intervals three positions can be made to give approximations for the fundamental data (*a*) sufficiently accurate to permit of a rough approximation of the fundamental data (*b*), so that it is advisable to restrict the number of observations to the minimum number necessary for a solution of the problem. This restriction (I) may be made to include a reduction in the number of fundamental data to be approximated (II) and a short method of obtaining first approximations for the same (III). The first three improvements are thus secured at one and the same time. For, let us assume for a moment that we have at our disposal an ephemeris giving α and δ referred to the mean equinox at the beginning of the year for every w mean solar days. Then the velocity α' in α at an instant midway between two ephemeris epochs is given in terms of successive differences by the following formula of numerical differentiation, the unit of the time being chosen as $1/k$ mean solar days:

$$\alpha' = \frac{d\alpha}{d\theta} = \frac{1}{wk} \left\{ f^{\text{i}}\left(a + [i+\tfrac{1}{2}]\,w\right) - \tfrac{1}{24} f^{\text{iii}}\left(a + [i+\tfrac{1}{2}]\,w\right) + \tfrac{3}{640} f^{\text{v}}\left(a + [i+\tfrac{1}{2}]\,w\right) - \ldots \right\} \quad (1)$$

and δ' is given by a similar formula. Let us further assume that the successive differences diminish numerically, which is equivalent to the assumption that the

curves representing the right ascension and declination as a function of the time are free from maxima, minima, points of inflection, etc., within the range of the ephemeris.

As a first approximation to (1) we may then write:

$$\alpha' = \frac{1}{wk} f^1\,(a + [i + \tfrac{1}{2}]\,w).$$

We now proceed to apply this expression to the observations. Let $\theta_{\prime\prime\prime}$ and $\theta_{\prime}$ denote the time intervals between the first and second, and second and third observations, respectively, expressed in $1/k$ mean solar days, so that

$$\theta_{\prime\prime\prime} = k\,(t_{\prime\prime} - t_{\prime}); \qquad \theta_{\prime} = k\,(t_{\prime\prime\prime} - t_{\prime\prime}),$$

where $t_{\prime}$, $t_{\prime\prime}$, $t_{\prime\prime\prime}$, are the dates of the three given observations. Assume $w = t_{\prime\prime} - t_{\prime}$, $t_{\prime\prime\prime} - t_{\prime\prime}$ successively, and let $\alpha_{\prime}$, $\alpha_{\prime\prime}$, $\alpha_{\prime\prime\prime}$, $\delta_{\prime}$, $\delta_{\prime\prime}$, $\delta_{\prime\prime\prime}$ be the observed right ascensions and declinations referred to the mean equinox at the beginning of the year. Then we shall have

$$f^1_{\prime\prime\prime}\,(a + [i + \tfrac{1}{2}]\,w) = \alpha_{\prime\prime} - \alpha_{\prime} = (\varDelta\alpha)_{\prime\prime\prime} \quad ; \qquad f^1_{\prime}\,(a + [i + \tfrac{1}{2}]\,w) = \alpha_{\prime\prime\prime} - \alpha_{\prime\prime} = (\varDelta\alpha)_{\prime} \quad ,$$

and we may write as a first approximation

$$\alpha_{\prime\prime\prime}' = \frac{(\varDelta\alpha)_{\prime\prime\prime}}{\theta_{\prime\prime\prime}} \; ; \quad \alpha_{\prime}' = \frac{(\varDelta\alpha)_{\prime}}{\theta_{\prime}}. \tag{2a}$$

Similarly

$$\delta_{\prime\prime\prime}' = \frac{\delta_{\prime\prime} - \delta_{\prime}}{\theta_{\prime\prime\prime}} = \frac{(\varDelta\delta)_{\prime\prime\prime}}{\theta_{\prime\prime\prime}}; \qquad \delta_{\prime}' = \frac{\delta_{\prime\prime\prime} - \delta_{\prime\prime}}{\theta_{\prime}} = \frac{(\varDelta\delta)_{\prime}}{\theta_{\prime}}. \tag{2b}$$

$\alpha_{\prime\prime\prime}'$, $\delta_{\prime\prime\prime}'$; $\alpha_{\prime}'$, $\delta_{\prime}'$ are approximations to the velocities in α and δ at the instants

$$T_{\prime\prime\prime} = \frac{t_{\prime} + t_{\prime\prime}}{2}\; ; \qquad T_{\prime} = \frac{t_{\prime\prime} + t_{\prime\prime\prime}}{2}.$$

In general, these velocities differ from their true values, first, on account of the errors of observation; secondly, because they are not corrected for parallax and aberration; thirdly, because the third and higher differences have been neglected. No allowance can be made for the errors of observation, the problem before us being the exact representation of three observations without previous hypothesis regarding the eccentricity. Parallax, aberration, third and higher differences, if appreciable, may be allowed for subsequently.

Letting, further,

$$T = \tfrac{1}{2}(T_{\prime\prime\prime} + T_{\prime}) = \tfrac{1}{4}(t_{\prime} + 2t_{\prime\prime} + t_{\prime\prime\prime}),$$

we have for this instant from (2*a*) and (2*b*)

$$\alpha' = \tfrac{1}{2}(\alpha_{\prime\prime\prime}' + \alpha_{\prime}'); \qquad \delta' = \tfrac{1}{2}(\delta_{\prime\prime\prime}' + \delta_{\prime}'), \tag{3}$$

with the same degree of accuracy as (2*a*) and (2*b*).

Let us next assume that we have at our disposal a table giving α' and δ' for every w mean solar days. The accelerations α'', δ'' at the instant T are then obtained from (2*a*) and (2*b*) by replacing the coördinates by the velocities. Accordingly we write

$$wk = \theta = k\,(T_{,} - T_{,,,}) = k\left(\frac{t_{,,} - t_{,}}{2} + \frac{t_{,,,} - t_{,,}}{2}\right) = \tfrac{1}{2}(\theta_{,,,} + \theta_{,}),$$

$$\varDelta\alpha' = \alpha_{,}' - \alpha_{,,,}' \; ; \qquad \varDelta\delta' = \delta_{,}' - \delta_{,,,}'$$

and obtain

$$\alpha'' = \frac{\varDelta\alpha'}{\theta}\, ; \qquad \delta'' = \frac{\varDelta\delta'}{\theta}\, . \tag{4}$$

α'', δ'' differ from their true values for the same three reasons that were given in the case of α' and δ'. It should be observed that whenever the effect of the third and higher differences is inappreciable within the range of the observations, the values of α', δ', α'', δ'' computed by (3) and (4) require correction only on account of parallax and aberration.

By adopting the second observation as the normal place and correcting α' and δ' for the acceleration during the interval from T to $t_0 = t_{,,}$ by the formulæ

$$\alpha_0' = \alpha' + k\,(t_{,,} - T)\,\alpha''\, ; \qquad \delta_0' = \delta' + k\,(t_{,,} - T)\,\delta'' \tag{5}$$

we obtain the following system of fundamental data (*a*), with which we may proceed to solve the equation of the seventh degree for ρ or z:

$$\alpha_0 = \alpha_{,,}\, ,\ \delta_0 = \delta_{,,}\, ; \qquad \alpha_0'\, ,\ \delta_0'\, ; \qquad \alpha_0'' = \alpha''\, ,\ \delta_0'' = \delta'' \tag{6}$$

The choice of the second observation for the normal, or better zero position, has the effect of reducing the number of quantities to be approximated from six to four, for both groups (*a*) and (*b*), and the method here given for obtaining preliminary values of the velocities and accelerations does away with the necessity of solving any system of simultaneous equations.

The nearer the actual velocities and accelerations determined by (4) and (5) are to their true values, the smaller will be the corrections to the fundamental data (*b*) which will have to be derived later. By plotting the observations we may determine readily whether the assumptions on which formulæ (4) and (5) are based hold approximately in a given case. If they do not, the velocities and accelerations may be estimated from the plot with sufficient accuracy to furnish the basis for the determination of the final values. Formulæ (4) and (5) can not, therefore, be used mechanically.

The first three of the improvements for which we were seeking having been determined, it now becomes necessary to facilitate the solution of the equation of the seventh degree (see A. N. 3371, pages 181–182).

Mr. ROGER SPRAGUE has already shown in A. N. 3669, at my suggestion, that a first approximation to the roots of the equation of the seventh degree may

be obtained from OPPOLZER'S table XIIIa, *Bahnbestimmung*, Vol. I. In its present form the table gives $z=\frac{\rho}{R}$ at most to three decimal places. I hope, however, soon to be able to extend the table so that z may be interpolated more accurately. When this shall have been accomplished, the geocentric distance at the normal date may be taken directly from the table on the basis of two arguments depending on the fundamental data (a). The so-called trials or successive approximations to the geocentric or heliocentric distances will then, in general, no longer be necessary.

In the meantime differential relations may be used for the correction of the approximate value of z taken from table XIIIa.

The equation of the seventh degree is given in the following form in A. N. 3669, page 387, equation (12):

$$-\frac{m'}{R^4}-z=\frac{-\frac{m'}{R^4}}{(1-2z\cos\psi+z^2)^{\frac{3}{2}}}$$

in which

$$m'=\frac{\varkappa}{\cos\delta}\,; \qquad \varkappa=-R\cos D\,\frac{(\tan\delta\cos(A-\alpha)-\tan D)\,\alpha'+\sin(A-\alpha)\,(\tan\delta)'}{\alpha'^3\tan\delta-\alpha''(\tan\delta)'+\alpha'(\tan\delta)''}$$

R, A, and D being the sun's geocentric polar coördinates at the normal date, referred to the equator. It is more convenient, however, to use the form

$$f(z)-m^2=(z^2-2z\cos\psi+1)^3(z-m)^2-m^2=0 \tag{7}$$

or

$$f(z)-m^2=\mu^3\nu^2-m^2=0 \tag{8}$$

where

$$m=-\frac{m'}{R^4}=-\frac{\varkappa}{\cos\delta\, R^4}\,. \tag{9}$$

The arguments with which z is taken from table XIIIa are $\frac{1}{m}$ and ψ, the geocentric angle between the planet or comet and the sun. Let the approximate value of z taken from the table be denoted by z_1 and let

$$(z_1^2-2z_1\cos\psi+1)^3(z_1-m)^2-m^2=\varDelta f(z_1)$$

or

$$\mu_1^3\nu_1^2-m^2=M_1 \tag{10}$$

It will then be necessary to determine a correction $\varDelta z_1$ to z_1 in such a manner that $z_2=z_1+\varDelta z_1$ will satisfy (7). By differentiating $f(z)$ with respect to z and by identifying $df(z)$ with M_1, we obtain for the required correction

$$\varDelta z_1=\frac{-M_1}{2\mu_1^2\nu_1\,[\mu_1+3\nu_1(z_1-\cos\psi)]} \tag{11}$$

In many cases $z_2=z_1+\varDelta z_1$ will be found to satisfy equation (7), so that $M_2=0$. If M_2 is not equal to zero, then the process must be repeated. It is easily seen that the numerical operations involved in solving for the geocentric distance according

to the foregoing method are almost insignificant in comparison with the corresponding operations even in OLBERS'S method.

From the final value of $z = z_{,,} = z_0$ we find:

$$\rho_0 = \rho_{,,} = R_{,,} z_0 \tag{12}$$

The next step will consist in the determination of the geocentric distances $\rho_{,}$ and $\rho_{,,,}$ or $\sigma_{,} = \rho_{,} \cos \delta_{,}$ and $\sigma_{,,,} = \rho_{,,,} \cos \delta_{,,,}$. We shall need the velocity σ_0' and the acceleration σ_0''. These may be computed by HARZER'S formulæ. After the approximate values of the three geocentric distances have been computed, the observations are to be corrected for parallax and aberration.

As it is intended to make use of the heliocentric rectangular coördinates and their velocities at the normal date (fundamental data (*b*)) for the purpose of deriving the elements and an ephemeris, we stop to consider whether we shall at once proceed to the determination of these quantities, or whether the whole process of determining ρ_0, ρ_0' or σ_0, σ_0' shall be repeated on the basis of the corrected observations. The latter course has been adopted by HARZER, and also in the two examples which follow. Some labor, however, could have been saved by omitting the second approximation, particularly in HARZER'S example and in the comet orbit, part 3, as in both these cases the first approximation of the geocentric distances furnished sufficiently accurate corrections for parallax and aberration. The small changes in the fundamental data (*b*) resulting from these corrections could have been determined just as well from the observation equations as part of the final corrections of the heliocentric coördinates and their velocities.

The determination of these final corrections can rarely be avoided. Even in HARZER'S example, in which the fourth derivatives of the right ascension and declination with respect to the time were taken into account, it was possible to improve the heliocentric coördinates and their velocities. There is every reason to believe that this was due to the small errors of observation rather than to the neglect of the fifth and higher derivatives. In the process outlined in these pages even the third derivatives are neglected, and, although this method is intended only for short intervals, a second approximation of the geocentric distances would rarely lead to heliocentric coördinates and velocities of such accuracy that the first and third places leave no residuals. This need not trouble us so long as we are able to set up simple formulæ for the determination of such corrections to the fundamental data (*b*) as will reduce the residuals to zero. These formulæ constitute the last of the improvements for which we are seeking.

I have so far assumed that it is intended to satisfy the given observations within the limits of accuracy of a six- or seven-place computation, but this may not always be desirable. In cases of newly-discovered comets, for example, the computer generally will aim at results only sufficiently accurate to furnish a fairly close finding ephemeris. In such cases a direct approximation of the geocentric

distances may frequently take the place of the determination of corrections to the fundamental data (*b*) by the differential formulæ derived below. We find this to be true in the case of our second example, in which, with two-day intervals, the residuals of the first and third places, resulting from the second approximation for the geocentric distances, are only

$$(O-C)\left\{\begin{array}{llll}\cos\delta_, \,\Delta\alpha_, &= -0^s.30\,, & \Delta\delta_, &= +0''.1\\ \cos\delta_{,,,}\,\Delta\alpha_{,,,} &= -0^s.28\,, & \Delta\delta_{,,,} &= +0''.8\end{array}\right.$$

When two approximations have been made it is sometimes possible to improve the corrections for parallax and aberration. This should not be neglected, if it is intended to accurately represent the observations.

From the value of $\rho_{,,}=\rho_0$ resulting from the first or second approximation of the geocentric distances, as the case may be, and from the corrected values of α_0, δ_0, the rectangular geocentric coördinates ξ_0, η_0, ζ_0 are now to be derived. To compute the velocities ξ_0', η_0', ζ_0' we avail ourselves, in addition, of the last values of ρ_0' or δ_0', α_0', δ_0' or $(\tan\delta)_0'$.

ξ_0, η_0, ζ_0 give the heliocentric rectangular coördinates x_0, y_0, z_0 with the aid of the sun's rectangular coördinates X_0, Y_0, Z_0 which have already been used in determining R_0, A_0, D_0. To obtain the velocities x_0', y_0', z_0' we determine X_0', Y_0', Z_0' for the zero (second) date, uncorrected for aberration, by numerical differentiation on the basis of the X, Y, Z, tabulated in the *American Ephemeris and Nautical Almanac*, or the *Berlin Jahrbuch*.

$X_,$, $Y_,$, $Z_,$, $X_{,,,}$, $Y_{,,,}$, $Z_{,,,}$ may be interpolated at the same time, care being taken to use the times $t_,$ and $t_{,,,}$, uncorrected for aberration, as it is supposed that in reducing the three given right ascensions and declinations to the beginning of the year the aberration terms have been taken into account. From the heliocentric coördinates and velocities we easily derive r_0 and r_0'.

We now have all the necessary data to compute $\alpha_,$, $\delta_,$, $\alpha_{,,,}$, $\delta_{,,,}$ by HARZER'S formulæ (A. N. 3371, pages 183–184) and to compare them with the observed values. Let

$$\left.\begin{array}{ll} f_, = 1 - \theta_{,,,}^2\frac{1}{2r_0^3} - \theta_{,,,}^3\frac{r_0'}{2r_0^4} + \dots\,; & g_, = -\theta_{,,,} + \theta_{,,,}^3\frac{1}{6r_0^3} - \dots\dots \\ f_{,,,} = 1 - \theta_,^2\frac{1}{2r_0^3} + \theta_,^3\frac{r_0'}{2r_0^4} + \dots\,; & g_{,,,} = \theta_, - \theta_,^3\frac{1}{6r_0^3} - \dots\dots\dots \end{array}\right\}\qquad(13)$$

then

$$x_, = f_, x_0 + g_, x_0'\,; \qquad x_{,,,} = f_{,,,}x_0 + g_{,,,}x_0'$$

and

$$\left.\begin{array}{lllll} \rho_,\cos\delta_,\cos\alpha_, = \xi_, &= X_, + x_, &= X_, &+ f_, x_0 &+ g_, x_0' \\ \rho_,\cos\delta_,\sin\alpha_, = \eta_, &= Y_, + y_, &= Y_, &+ f_, y_0 &+ g_, y_0' \\ \rho_,\sin\delta_, = \zeta_, &= Z_, + z_, &= Z_, &+ f_, z_0 &+ g_, z_0' \\ \rho_{,,,}\cos\delta_{,,,}\cos\alpha_{,,,} = \xi_{,,,} &= X_{,,,} + x_{,,,} &= X_{,,,} &+ f_{,,,}x_0 &+ g_{,,,}x_0' \\ \rho_{,,,}\cos\delta_{,,,}\sin\alpha_{,,,} = \eta_{,,,} &= Y_{,,,} + y_{,,,} &= Y_{,,,} &+ f_{,,,}y_0 &+ g_{,,,}y_0' \\ \rho_{,,,}\sin\delta_{,,,} = \zeta_{,,,} &= Z_{,,,} + z_{,,,} &= Z_{,,,} &+ f_{,,,}z_0 &+ g_{,,,}z_0' \end{array}\right\}\qquad(14)$$

Let the residuals in α and δ, $(O - C)$, be denoted by $\partial_{,}\alpha_{,} = \cos \delta_{,}\partial\alpha_{,}$, $\partial\delta_{,}$, $\partial_{,}\alpha_{,,,} = \cos \delta_{,,,}\partial\alpha_{,,,}$, $\partial\delta_{,,,}$. We proceed to derive linear differential relations which shall enable us to determine such corrections to our fundamental data (b) as will reduce these residuals to zero. Through the adoption of the second as the zero observation the number of these fundamental data has been reduced to four. The quantities requiring correction in HARZER'S method are x_0, y_0, z_0, x_0', y_0', z_0', while here they are ρ_0, x_0', y_0', z_0'.

For the sake of convenience in deriving our formulæ we shall omit for the present the subscripts referring to the first and third places. We commence with the well-known relations:

$$\left.\begin{aligned} \rho\partial_{,}\alpha &= -\sin\alpha\,\partial x + \cos\alpha\,\partial y \\ \rho\partial\delta &= -\sin\delta\,(\cos\alpha\,\partial x + \sin\alpha\,\partial y) + \cos\delta\,\partial z \end{aligned}\right\} \quad (15)$$

By differentiating (13), we have $(x, y, z = \omega)$,

$$\left.\begin{aligned} \partial\omega &= f\,\partial\omega_0 + \omega_0\partial f + g\,\partial\,\omega_0' + \omega_0'\partial g \\ \partial f &= \tfrac{3}{2}\,\theta^2\,\frac{\partial r_0}{r_0^4} + \theta^2\,\frac{\partial r_0'}{2r_0^4} - \frac{2\theta^2 r_0'}{r_0^5}\,\partial r_0 - \ldots\ldots \\ \partial g &= \tfrac{1}{2}\,\frac{\theta^2\partial r_0}{r_0^4} + \ldots\ldots \end{aligned}\right\} \quad (16)$$

In general, it will be sufficient to neglect all but the first term of ∂f and ∂g. With ten-day intervals and a correction $\partial r_0' = .01$, which is larger than we must expect, the greatest of the neglected terms will average less than .00005, a quantity which ordinarily may be neglected in determining differential corrections. Should our starting values of ρ_0, x_0', y_0', z_0' be so far from the truth as to make appreciable higher terms than those considered, it will be more convenient to make a second approximation to the differential corrections. Since α_0, δ_0 are constants, we also have

$$\partial x_0 = \cos\alpha_0\cos\delta_0\,\partial\rho_0 = \frac{\xi_0}{\rho_0}\,\partial\rho_0\,, \qquad \partial y_0 = \sin\alpha_0\cos\delta_0\,\partial\rho_0 = \frac{\eta_0}{\rho_0}\,\partial\rho_0 \quad (17)$$

$$\partial z_0 = \sin\delta_0\,\partial\rho_0 = \frac{\zeta_0}{\rho_0}\,\partial\rho_0\,,$$

and

$$\partial r_0 = \frac{x_0\partial x_0 + y_0\partial y_0 + z_0\partial z_0}{r_0} = \left\{\frac{x_0}{r_0}\frac{\xi_0}{\rho_0} + \frac{y_0}{r_0}\frac{\eta_0}{\rho_0} + \frac{z_0}{r_0}\frac{\zeta_0}{\rho_0}\right\}\partial\rho_0 \quad . \quad (18)$$

It is not advisable to introduce $\frac{\partial \log \rho_0}{\text{Mod.}}$ in place of $\frac{\partial\rho_0}{\rho_0}$ into these formulæ, as the neglected terms of the development of $\partial \log \rho_0$ in terms of $\partial\rho_0$ are apt to be appreciable. The coefficient of $\partial\rho_0$ is the cosine of the angle at the comet or planet included between ρ_0 and r_0. If we denote this angle by β, we have

$$\cos\beta = \frac{\rho_0 - R_0\cos\psi}{r_0}\,; \quad \partial r_0 = \cos\beta\,\partial\rho_0\,. \quad (19)$$

Introducing (17) and (19) into (16), we obtain ($\xi_0, \eta_0, \zeta_0, = \omega_0$)

$$\partial\omega = f_\omega\,\partial\rho_0 + g\,\partial\omega_0' \tag{20}$$

in which

$$f_\omega = f\,\frac{\omega_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}\,(\,3\theta^2\omega_0 + \theta^3\omega_0'). \tag{21}$$

or separately for the three coördinates

$$\left.\begin{aligned}
f_x &= \frac{f}{\rho_0}\,\xi_0 + \frac{\cos\beta}{2r_0^4}\,(3\theta^2x_0 + \theta^3x_0'), & \partial x &= f_x\partial\rho_0 + g\partial x_0'\,. \\
f_y &= \frac{f}{\rho_0}\,\eta_0 + \frac{\cos\beta}{2r_0^4}\,(3\theta^2y_0 + \theta^3y_0'), & \partial y &= f_y\partial\rho_0 + g\partial y_0'\,. \\
f_z &= \frac{f}{\rho_0}\,\zeta_0 + \frac{\cos\beta}{2r_0^4}\,(3\theta^2z_0 + \theta^3z_0'), & \partial z &= f_z\partial\rho_0 + g\partial z_0'\,.
\end{aligned}\right\} \tag{22}$$

Generally, the last of the three terms in the f expressions may be neglected.

From (15) and (22) we readily deduce for the first and third places:

$$\left.\begin{aligned}
A_{,} &= \frac{1}{\rho_{,}}\,[\cos\alpha_{,}f_{y_{,}} - \sin\alpha_{,}f_{x_{,}}], \\
&\qquad B_{,} = -\frac{1}{\rho_{,}}\,[\sin\delta_{,}(\sin\alpha_{,}f_{y_{,}} + \cos\alpha_{,}f_{x_{,}}) - \cos\delta_{,}f_{z_{,}}], \qquad C_{,} = \frac{g_{,}}{\rho_{,}}, \\
A_{,,,} &= \frac{1}{\rho_{,,,}}\,[\cos\alpha_{,,,}f_{y_{,,,}} - \sin\alpha_{,,,}f_{x_{,,,}}], \\
&\qquad B_{,,,} = -\frac{1}{\rho_{,,,}}\,[\sin\delta_{,,,}(\sin\alpha_{,,,}f_{y_{,,,}} + \cos\alpha_{,,,}f_{x_{,,,}}) - \cos\delta_{,,,}f_{z_{,,,}}], \qquad C_{,,,} = \frac{g_{,,,}}{\rho_{,,,}}\,;
\end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned}
\partial_{,}\alpha_{,} &= A_{,}\,\partial\rho_0 - C_{,}\,\sin\alpha_{,}\,\partial x_0' + C_{,}\,\cos\alpha_{,}\,\partial y_0'\,, \\
\partial_{,}\alpha_{,,,} &= A_{,,,}\,\partial\rho_0 - C_{,,,}\sin\alpha_{,,,}\,\partial x_0' + C_{,,,}\cos\alpha_{,,,}\partial y_0'\,;
\end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned}
\partial\delta_{,} &= B_{,}\,\partial\rho_0 - C_{,}\,\sin\delta_{,}\,\cos\alpha_{,}\,\partial x_0' - C_{,}\,\sin\delta_{,}\,\sin\alpha_{,}\,\partial y_0' + C_{,}\,\cos\delta_{,}\,\partial z_0' \\
\partial\delta_{,,,} &= B_{,,,}\,\partial\rho_0 - C_{,,,}\sin\delta_{,,,}\cos\alpha_{,,,}\,\partial x_0' - C_{,,,}\sin\delta_{,,,}\sin\alpha_{,,,}\,\partial y_0' + C_{,,,}\cos\delta_{,,,}\,\partial z_0'
\end{aligned}\right\} \tag{25}$$

Eliminating $\partial z_0'$ from equations (25), $\partial x_0'$, $\partial y_0'$ from equations (24) we have:

$$\begin{aligned}
C_{,,,}\cos\delta_{,,,}\,\partial\delta_{,} - C_{,}\cos\delta_{,}\,\partial\delta_{,,,} = (B_{,}C_{,,,}\cos\delta_{,,,} &- B_{,,,}C_{,}\cos\delta_{,})\,\partial\rho_0 \\
&- C_{,}C_{,,,}(\sin\delta_{,}\cos\delta_{,,,}\cos\alpha_{,} - \sin\delta_{,,,}\cos\delta_{,}\cos\alpha_{,,,})\,\partial x_0' \\
&- C_{,}C_{,,,}(\sin\delta_{,}\cos\delta_{,,,}\sin\alpha_{,} - \sin\delta_{,,,}\cos\delta_{,}\sin\alpha_{,,,})\,\partial y_0'.
\end{aligned} \tag{26}$$

$$\left.\begin{aligned}
C_{,,,}\sin\alpha_{,,,}\,\partial_{,}\alpha_{,} - C_{,}\sin\alpha_{,}\,\partial_{,}\alpha_{,,,} &= (A_{,}C_{,,,}\sin\alpha_{,,,} - A_{,,,}C_{,}\sin\alpha_{,})\,\partial\rho_0 + C_{,}C_{,,,}\sin(\alpha_{,,,} - \alpha_{,})\,\partial y_0', \\
C_{,,,}\cos\alpha_{,,,}\,\partial_{,}\alpha_{,} - C_{,}\cos\alpha_{,}\,\partial_{,}\alpha_{,,,} &= (A_{,}C_{,,,}\cos\alpha_{,,,} - A_{,,,}C_{,}\cos\alpha_{,})\,\partial\rho_0 + C_{,}C_{,,,}\sin(\alpha_{,,,} - \alpha_{,})\,\partial x_0'.
\end{aligned}\right\} \tag{27}$$

From (26) $C_{,}C_{,,,}\partial x_0'$, and $C_{,}C_{,,,}\partial y_0'$ may be eliminated by means of (27). In the eliminant the coefficients of $-\frac{C_{,,,}\partial_{,}\alpha_{,}}{\sin(\alpha_{,,,}-\alpha_{,})}$ and $\frac{A_{,}C_{,,,}}{\sin(\alpha_{,,,}-\alpha_{,})}$ on the one, and of $\frac{C_{,}\partial_{,}\alpha_{,,,}}{\sin(\alpha_{,,,}-\alpha_{,})}$ and $-\frac{A_{,,,}C_{,}}{\sin(\alpha_{,,,}-\alpha_{,})}$ on the other hand, are

$$\sin\delta_{,}\cos\delta_{,,,}\cos(\alpha_{,,,} - \alpha_{,}) - \sin\delta_{,,,}\cos\delta_{,}$$

and

$$\sin\delta_{,}\cos\delta_{,,,} - \sin\delta_{,,,}\cos\delta_{,}\cos(\alpha_{,,,} - \alpha_{,}),$$

respectively. As $(\alpha_{\prime\prime\prime} - \alpha_{\prime})$ is known, we can, of course, decide in a given case whether or not the accuracy which is being observed in determining the differential corrections warrants the assumption $\cos(\alpha_{\prime\prime\prime} - \alpha_{\prime}) = 1$. For the present we shall introduce in all four coefficients

$$\cos(\alpha_{\prime\prime\prime} - \alpha_{\prime}) = 1 - 2\sin^2\tfrac{1}{2}(\alpha_{\prime\prime\prime} - \alpha_{\prime}).$$

Solving the eliminant for $\partial\rho_0$ we finally obtain

$$\left.\begin{aligned}
&\partial\rho_0 = \frac{\sin(\alpha_{\prime\prime\prime} - \alpha_{\prime})[C_{\prime\prime\prime}\cos\delta_{\prime\prime\prime}\partial\delta_{\prime} - C_{\prime}\cos\delta_{\prime}\partial\delta_{\prime\prime\prime}] - \sin(\delta_{\prime\prime\prime} - \delta_{\prime})[C_{\prime\prime\prime}\partial_{\prime}\alpha_{\prime} - C_{\prime}\partial_{\prime}\alpha_{\prime\prime\prime}] - \varDelta N}{\sin(\alpha_{\prime\prime\prime} - \alpha_{\prime})[C_{\prime\prime\prime}\cos\delta_{\prime\prime\prime}B_{\prime} - C_{\prime}\cos\delta_{\prime}B_{\prime\prime\prime}] - \sin(\delta_{\prime\prime\prime} - \delta_{\prime})[C_{\prime\prime\prime}A_{\prime} - C_{\prime}A_{\prime\prime\prime}] - \varDelta D}\\
&\text{in which}\\
&\qquad \varDelta N = 2\sin^2\tfrac{1}{2}(\alpha_{\prime\prime\prime} - \alpha_{\prime})\,[C_{\prime\prime\prime}\cos\delta_{\prime\prime\prime}\sin\delta_{\prime}\,\partial_{\prime}\alpha_{\prime} + C_{\prime}\cos\delta_{\prime}\sin\delta_{\prime\prime\prime}\,\partial_{\prime}\alpha_{\prime\prime\prime}]\\
&\text{and}\\
&\qquad \varDelta D = 2\sin^2\tfrac{1}{2}(\alpha_{\prime\prime\prime} - \alpha_{\prime})\,[C_{\prime\prime\prime}\cos\delta_{\prime\prime\prime}\sin\delta_{\prime}A_{\prime} + C_{\prime}\cos\delta_{\prime}\sin\delta_{\prime\prime\prime}A_{\prime\prime\prime}]
\end{aligned}\right\} \quad (28)$$

in which the last term in the numerator and in the denominator may be omitted whenever $2\sin^2\frac{1}{2}(\alpha_{\prime\prime\prime} - \alpha_{\prime})$ is less than one unit of the last place carried in computing $\partial\rho_0$ from this formula.

With the value of $\partial\rho_0$ resulting from (28), $\partial x_0'$ and $\partial y_0'$ may be computed by means of (27). As a check on the computation of the differential corrections, $\partial z_0'$ should be obtained from each of the formulæ (25).

From $\partial\rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$, we next compute

$$\partial\xi_{\prime} = \partial x_{\prime}, \quad \partial\eta_{\prime} = \partial y_{\prime}, \quad \partial\zeta_{\prime} = \partial z_{\prime} \qquad \partial\xi_{\prime\prime\prime} = \partial x_{\prime\prime\prime}, \quad \partial\eta_{\prime\prime\prime} = \partial y_{\prime\prime\prime}, \quad \partial\zeta_{\prime\prime\prime} = \partial z_{\prime\prime\prime},$$

in which the corrections to the heliocentric coördinates are given by (22). The corrected geocentric coördinates give the values of $\rho_{\prime}$, $\alpha_{\prime}$, $\delta_{\prime}$, $\rho_{\prime\prime\prime}$, $\alpha_{\prime\prime\prime}$, $\delta_{\prime\prime\prime}$, which correspond to the corrected values of ρ_0, x_0', y_0', z_0'.

If the computed right ascensions and declinations agree with their observed values, we may proceed to the computation of the heliocentric coördinates at the zero date. From the corrected heliocentric coördinates and velocities we then derive the elements by ENCKE'S formulæ in the form in which OPPOLZER gives them, *Bahnbestimmung*, Vol. II, pp. 93 and 99.

If the agreement of the observed and computed right ascensions and declinations is not satisfactory, the corrections to the fundamental data (b) should be redetermined by substituting the new residuals in place of the old ones in (28), (27), (25); but, as a rule, it will not be necessary to recompute any of the auxiliary quantities (23).

It will rarely happen that the value of $\partial\rho_0$, determined from (28), is so large as to make necessary a recomputation of the parallax and aberration. Should such a case arise, however, it will usually be sufficient to apply the new parallax to α_0 and δ_0 and to recompute x_0, y_0, z_0. The epoch $t_{\prime\prime} = t_0$ should be recorrected for aberration.

In the majority of cases the first approximation of the geocentric distances

will be a little nearer to the truth, if, at the beginning of the calculation, the second observation be corrected for the parallax at distance unity.

As the resulting elements are referred to the equator, they must be transformed to the ecliptic, in order that they may be compared readily with those of previously known planets or comets.

The whole computation has been referred to the equator so as to avoid transformations. In some rare cases, however, it may become advisable to refer the observations to some arbitrary fundamental plane.

An ephemeris may be computed in the usual way or by (14), which formulæ sometimes will be more convenient.

Chief among the advantages of the method here outlined are the ease with which such corrections to ρ_0, x_0', y_0', z_0', may be determined as will cause the residuals due to the original values of these quantities to disappear, and the possibility of determining these corrections directly from the residuals. On that account, it is of no great consequence if the originally adopted velocities and accelerations in α and δ are only approximate. By means of the differential formulæ here introduced, it is generally possible to determine the final values of the fundamental data with greater numerical accuracy than would be possible by applying the original integral formulæ.

For the sake of convenience in following the foregoing directions for the computation of an orbit from three observations, $t_{,}$, $\alpha_{,}$, $\delta_{,}$; $t_{,,}$, $\alpha_{,,}$, $\delta_{,,}$; $t_{,,,}$, $\alpha_{,,,}$, $\delta_{,,,}$, the necessary formulæ are collected below. It is assumed that the observations are referred to the beginning of the year.

I.

From one of the Astronomical Ephemerides interpolate the solar coördinates for the dates $t_{,}$, $t_{,,}=t_0$, $t_{,,,}$.

$$X_{,}\,,\ Y_{,}\,,\ Z_{,}\,; \quad X_{,,}\,,\ Y_{,,}\,,\ Z_{,,}\,; \quad X_{,,,}\,,\ Y_{,,,}\,,\ Z_{,,,}\,.$$

At the same time obtain $X_0'=X_{,,}'$, $Y_0'=Y_{,,}'$, $Z_0'=Z_{,,}'$, by either of the following formulæ of numerical differentiation:

$$kw\frac{df(l)}{dl}=f^{\text{I}}\,(a+iw)+nf^{\text{II}}\,(a+iw)+N_1^{3}\,(n)\,f^{\text{III}}\,(a+iw)+\ldots\ldots;$$

$$l=a+[i+n]\;w=t_0\,.$$

$$kw\frac{df(l)}{dl}=f^{\text{I}}\,(a+[i+\tfrac{1}{2}]\;w)+mf^{\text{II}}\,(a+[i+\tfrac{1}{2}]\;w)+M_1^{3}\,(m)\,f^{\text{III}}\,(a+[i+\tfrac{1}{2}]\;w)+\ldots\ldots;$$

$$l=a+[i+\tfrac{1}{2}+m]\;w=t_0\,.$$

according as to whether t_0 lies nearer to a tabulated argument or to the mean of two such arguments. m and n are $<\pm 0.25$. $N_1^3\,(n)$ or $M_1^3\,(m)$ may be taken from table I or II, OPPOLZER, *Bahnbestimmung*, etc., Vol. II, page 515 or 523, but,

in general, the terms in which they occur may be neglected. When the solar coördinates are tabulated for every 12 hours, $w = \frac{1}{2}$; $\log 1/kw = 2.0654486$.

$$R\cos D\cos A = X_{\prime\prime}\,;\quad R\cos D\sin A = Y_{\prime\prime}\,;\quad R\sin D = Z_{\prime\prime}\,.$$

It will do no harm to correct the second observation for the parallax corresponding to distance unity.

II.*

$$\alpha_{\prime\prime\prime}' = \frac{15\sin 1''}{k}\,\frac{\alpha_{\prime\prime}-\alpha_{\prime}}{t_{\prime\prime}-t_{\prime}}\,;\quad \alpha_{\prime}' = \frac{15\sin 1''}{k}\,\frac{\alpha_{\prime\prime\prime}-\alpha_{\prime\prime}}{t_{\prime\prime\prime}-t_{\prime\prime}}\,;\quad \delta_{\prime\prime\prime}' = \frac{\sin 1''}{k}\,\frac{\delta_{\prime\prime}-\delta_{\prime}}{t_{\prime\prime}-t_{\prime}}\,;\quad \delta_{\prime}' = \frac{\sin 1''}{k}\,\frac{\delta_{\prime\prime\prime}-\delta_{\prime\prime}}{t_{\prime\prime\prime}-t_{\prime\prime}}$$

$$\alpha_0'' = \frac{30\sin 1''}{k^2}\,\frac{\dfrac{\alpha_{\prime\prime\prime}-\alpha_{\prime\prime}}{t_{\prime\prime\prime}-t_{\prime\prime}} - \dfrac{\alpha_{\prime\prime}-\alpha_{\prime}}{t_{\prime\prime}-t_{\prime}}}{t_{\prime\prime\prime}-t_{\prime}}\,;\qquad \delta_0'' = \frac{2\sin 1''}{k^2}\,\frac{\dfrac{\delta_{\prime\prime\prime}-\delta_{\prime\prime}}{t_{\prime\prime\prime}-t_{\prime\prime}} - \dfrac{\delta_{\prime\prime}-\delta_{\prime}}{t_{\prime\prime}-t_{\prime}}}{t_{\prime\prime\prime}-t_{\prime}}$$

$$\alpha_0' = \tfrac{1}{2}(\alpha_{\prime\prime\prime}' + \alpha_{\prime}') + \tfrac{k}{4}\left[(t_{\prime\prime}-t_{\prime}) - (t_{\prime\prime\prime}-t_{\prime\prime})\right]\alpha_0''\,;\quad \delta_0' = \tfrac{1}{2}(\delta_{\prime\prime\prime}' + \delta_{\prime}') + \tfrac{k}{4}\left[(t_{\prime\prime}-t_{\prime}) - (t_{\prime\prime\prime}-t_{\prime\prime})\right]\delta_0''$$

$$(\tan\delta)_0' = \sec^2\delta_{\prime\prime}\,\delta_0'\,;\qquad (\tan\delta)_0'' = \sec^2\delta_{\prime\prime}\,[2\tan\delta_{\prime\prime}\,(\delta_0')^2 + \delta_0''].$$

$$\log\frac{15\sin 1''}{k} = 7.626\,084\,7 - 10\,;\quad \log\frac{\sin 1''}{k} = 6.449\,993\,4 - 10\,;\quad \log\frac{k}{4} = 7.633\,521\,4 - 10\,;$$

$$\log\frac{30\sin 1''}{k^2} = 9.691\,533\,3 - 10\,;\qquad \log\frac{2\sin 1''}{k^2} = 8.515\,442\,0 - 10.$$

III.

$$N = (\alpha_0')^3\tan\delta_{\prime\prime} - \alpha_0''\,(\tan\delta)_0' + \alpha_0'\,(\tan\delta)_0''\,;$$

$$\varkappa = -\frac{R\cos D}{N}\left\{[\tan\delta_{\prime\prime}\cos(A-\alpha_{\prime\prime}) - \tan D]\,\alpha_0' + \sin(A-\alpha_{\prime\prime})\,(\tan\delta)_0'\right\}$$

$$\cos\psi = \sin\delta_{\prime\prime}\sin D + \cos\delta_{\prime\prime}\cos D\cos(A-\alpha_{\prime\prime})\,;\qquad \frac{1}{m} = -\frac{R^4\cos\delta}{\varkappa}\,.$$

IV.

With $\frac{1}{m}$ and ψ take $z = z_1$ from table XIIIa, OPPOLZER, *Bahnbestimmung*, Vol. I.

$$\mu_1 = z_1^2 - 2z_1\cos\psi + 1\,;\qquad \nu_1 = z_1 - m\,;\qquad \mu_1^3\nu_1^2 - m^2 = M_1\,;$$

$$z_2 = z_1 + \frac{-M_1}{2\,\mu_1^2\nu_1\,[\mu_1 + 3\,\nu_1\,(z_1 - \cos\psi)\,]}$$

Continue these approximations until $M = 0$; z, without subscript, being the final value.

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* When the geocentric motion is such that the higher differences in α and δ are very appreciable, these formulæ must receive proper modification, depending upon the special conditions prevailing in individual cases. See page 7.

V.

$$\rho_0 = Rz\ ; \qquad \sigma_0 = \rho_0 \cos \delta_{,,}\ ;$$

$$\lambda = \frac{R \cos D}{2N} \left\{ [\tan \delta_{,,} \cos (A - \alpha_{,,}) - \tan D]\, \alpha_0'' + \sin (A - \alpha_{,,})\, [(\alpha_0')^2 \tan \delta_{,,} + (\tan \delta)_0''] \right\}$$

$$\sigma_0' = \frac{\lambda}{\varkappa}\, \sigma_0\ ; \quad \sigma_0'' = \frac{\sigma_0}{\varkappa} \left[R \cos D \cos (A - \alpha_{,,}) - \sigma_0 \right] - \sigma_0 \left[\frac{1}{R^3} - (\alpha_0')^2 \right] ;$$

$$\theta_{,} = k(t_{,,,} - t_{,,})\ ; \qquad \theta_{,,,} = k(t_{,,} - t_{,})\ ;$$

$$\rho_{,} = \frac{1}{\cos \delta_{,}} \left[\sigma_0 - \theta_{,,,}\, \sigma_0' + \theta_{,,,}^2 \frac{\sigma_0''}{2} \right] ; \quad \rho_{,,,} = \frac{1}{\cos \delta_{,,,}} \left[\sigma_0 + \theta_{,}\, \sigma_0' + \theta_{,}^2 \frac{\sigma_0''}{2} \right] .$$

With ρ_0 correct the second observation for parallax and with $\rho_{,}$, ρ_0, $\rho_{,,,}$ correct the three dates of observation for aberration, also $\theta_{,}$ and $\theta_{,,,}$.

VI.

$$x_0 = \sigma_0 \cos \alpha_{,,} - X_{,,}\ ; \quad y_0 = \sigma_0 \sin \alpha_{,,} - Y_{,,}\ ; \quad z_0 = \sigma_0 \tan \delta_{,,} - Z_{,,}\ ; \quad r_0^2 = x_0^2 + y_0^2 + z_0^2$$

$$= \xi_0 - X_{,,}\ ; \qquad = \eta_0 - Y_{,,}\ ; \qquad = \zeta_0 - Z_{,,}\ ;$$

$$x_0' = \cos \alpha_{,,} \sigma_0' - \sin \alpha_{,,} \sigma_0 \alpha_0' - X_0'; \qquad y_0' = \sin \alpha_{,,} \sigma_0' + \cos \alpha_{,,} \sigma_0 \alpha_0' - Y_0';$$

$$z_0' = \tan \delta_{,,} \sigma_0' + \sigma_0 (\tan \delta)_0'; \qquad r_0 r_0' = x_0 x_0' + y_0 y_0' + z_0 z_0';$$

$$f_{,} = 1 - \theta_{,,,}^2 \frac{1}{2r_0^3} - \theta_{,,,}^3 \frac{r_0'}{2r_0^4} + \ldots\ldots\ ; \qquad g_{,} = - \theta_{,,,} + \theta_{,,,}^3 \frac{1}{6r_0^3} - \ldots\ldots$$

$$f_{,,,} = 1 - \theta_{,}^2 \frac{1}{2r_0^3} + \theta_{,}^3 \frac{r_0'}{2r_0^4} + \ldots\ldots\ ; \qquad g_{,,,} = \theta_{,} - \theta_{,}^3 \frac{1}{6r_0^3} + \ldots\ldots$$

$$\rho_{,} \cos \delta_{,} \cos \alpha_{,} = X_{,} + f_{,} x_0 + g_{,} x_0' = \xi_{,}\ ; \qquad \rho_{,,,} \cos \delta_{,,,} \cos \alpha_{,,,} = X_{,,,} + f_{,,,} x_0 + g_{,,,} x_0' = \xi_{,,,}$$

$$\rho_{,} \cos \delta_{,} \sin \alpha_{,} = Y_{,} + f_{,} y_0 + g_{,} y_0' = \eta_{,}\ ; \qquad \rho_{,,,} \cos \delta_{,,,} \sin \alpha_{,,,} = Y_{,,,} + f_{,,,} y_0 + g_{,,,} y_0' = \eta_{,,,}$$

$$\rho_{,} \sin \delta_{,} = Z_{,} + f_{,} z_0 + g_{,} z_0' = \zeta_{,}\ ; \qquad \rho_{,,,} \sin \delta_{,,,} = Z_{,,,} + f_{,,,} z_0 + g_{,,,} z_0' = \zeta_{,,,}$$

With these last values of $\rho_{,}$, $\rho_{,,,}$ correct the second and third observations for parallax. Form the residuals, $(O-C)$, $\partial_{,}\alpha_{,} = \partial\alpha_{,} \cos \delta_{,}$, $\partial\delta_{,}$, $\partial_{,}\alpha_{,,,} = \partial\alpha_{,,,} \cos \delta_{,,,}$, $\partial\delta_{,,,}$. If these residuals are sufficiently small for the purpose in hand, the elements and an ephemeris may be computed at once by VIII. If it is intended, however, to represent the observations more accurately within the limits of the tables, the necessary corrections to ρ_0, x_0', y_0', z_0' may be determined by means of [VII].

[VII.]

$$\cos\beta = \frac{\rho_0 - R_0\cos\psi}{r_0};$$

$$f_{x_{,}} = f_{,}\frac{\xi_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}(3\theta_{,,,}^2 x_0 - \theta_{,,,}^3 x_0'); \qquad f_{x_{,,,}} = f_{,,,}\frac{\xi_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}(3\theta_{,}^2 x_0 + \theta_{,}^3 x_0')$$

$$f_{y_{,}} = f_{,}\frac{\eta_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}(3\theta_{,,,}^2 y_0 - \theta_{,,,}^3 y_0'); \qquad f_{y_{,,,}} = f_{,,,}\frac{\eta_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}(3\theta_{,}^2 y_0 + \theta_{,}^3 y_0')$$

$$f_{z_{,}} = f_{,}\frac{\zeta_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}(3\theta_{,,,}^2 z_0 - \theta_{,,,}^3 z_0'); \qquad f_{z_{,,,}} = f_{,,,}\frac{\zeta_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}(3\theta_{,}^2 z_0 + \theta_{,}^3 z_0').$$

$$A_{,} = \frac{1}{\rho_{,}}\left[\cos\alpha_{,} f_{y_{,}} - \sin\alpha_{,} f_{x_{,}}\right],$$

$$B_{,} = -\frac{1}{\rho_{,}}\left[\sin\delta_{,}\left(\sin\alpha_{,} f_{y_{,}} + \cos\alpha_{,} f_{x_{,}}\right) - \cos\delta_{,} f_{z_{,}}\right], \qquad C_{,} = \frac{g_{,}}{\rho_{,}},$$

$$A_{,,,} = \frac{1}{\rho_{,,,}}\left[\cos\alpha_{,,,} f_{y_{,,,}} - \sin\alpha_{,,,} f_{x_{,,,}}\right],$$

$$B_{,,,} = -\frac{1}{\rho_{,,,}}\left[\sin\delta_{,,,}\left(\sin\alpha_{,,,} f_{y_{,,,}} + \cos\alpha_{,,,} f_{x_{,,,}}\right) - \cos\delta_{,,,} f_{z_{,,,}}\right], \qquad C_{,,,} = \frac{g_{,,,}}{\rho_{,,,}}.$$

$$\partial\rho_0 = \frac{N - \Delta N}{D - \Delta D},$$

in which

$$\begin{aligned}
N &= \sin(\alpha_{,,,} - \alpha_{,})[C_{,,,}\cos\delta_{,,,}\partial\delta_{,} - C_{,}\cos\delta_{,}\partial\delta_{,,,}] - \sin(\delta_{,,,} - \delta_{,})[C_{,,,}\partial\alpha_{,} - C_{,}\partial\alpha_{,,,}]\\
D &= \sin(\alpha_{,,,} - \alpha_{,})[C_{,,,}\cos\delta_{,,,} B_{,} - C_{,}\cos\delta_{,} B_{,,,}] - \sin(\delta_{,,,} - \delta_{,})[C_{,,,} A_{,} - C_{,} A_{,,,}]\\
\Delta N &= 2\sin^2\tfrac{1}{2}(\alpha_{,,,} - \alpha_{,})[C_{,,,}\cos\delta_{,,,}\sin\delta_{,}\partial\alpha_{,} + C_{,}\cos\delta_{,}\sin\delta_{,,,}\partial\alpha_{,,,}]\\
\Delta D &= 2\sin^2\tfrac{1}{2}(\alpha_{,,,} - \alpha_{,})[C_{,,,}\cos\delta_{,,,}\sin\delta_{,} A_{,} + C_{,}\cos\delta_{,}\sin\delta_{,,,} A_{,,,}]
\end{aligned}$$

or also

$$\partial\rho_0 = \frac{\sin(\alpha_{,,,} - \alpha_{,})[N_{,}^1 - N_{,,,}^1] - \sin(\delta_{,,,} - \delta_{,})[N_{,}^2 - N_{,,,}^2] - 2\sin^2\frac{1}{2}(\alpha_{,,,} - \alpha_{,})[N_{,}^3 + N_{,,,}^3]}{\sin(\alpha_{,,,} - \alpha_{,})[D_{,}^1 - D_{,,,}^1] - \sin(\delta_{,,,} - \delta_{,})[D_{,}^2 - D_{,,,}^2] - 2\sin^2\frac{1}{2}(\alpha_{,,,} - \alpha_{,})[D_{,}^3 + D_{,,,}^3]}.$$

$$\begin{aligned}
C_{,}C_{,,,}\sin(\alpha_{,,,} - \alpha_{,})\,\partial x_0' &= C_{,,,}\cos\alpha_{,,,}\partial\alpha_{,} - C_{,}\cos\alpha_{,}\partial\alpha_{,,,} -\\
&\quad (C_{,,,}\cos\alpha_{,,,} A_{,} - C_{,}\cos\alpha_{,} A_{,,,})\,\partial\rho_0 = (1) - (2) - [(3) - (4)]\,\partial\rho_0\\
C_{,}C_{,,,}\sin(\alpha_{,,,} - \alpha_{,})\,\partial y_0' &= C_{,,,}\sin\alpha_{,,,}\partial\alpha_{,} - C_{,}\sin\alpha_{,}\partial\alpha_{,,,} -\\
&\quad (C_{,,,}\sin\alpha_{,,,} A_{,} - C_{,}\sin\alpha_{,} A_{,,,})\,\partial\rho_0 = (1) - (2) - [(3) - (4)]\,\partial\rho_0
\end{aligned}$$

$$\left.\begin{aligned}
C_{,}\cos\delta_{,}\,\partial z_0' &= \partial\delta_{,} - B_{,}\,\partial\rho_0 + C_{,}\sin\delta_{,}\cos\alpha_{,}\,\partial x_0' - C_{,}\sin\delta_{,}\sin\alpha_{,}\,\partial y_0'\\
C_{,,,}\cos\delta_{,,,}\,\partial z_0' &= \partial\delta_{,,,} - B_{,,,}\,\partial\rho_0 + C_{,,,}\sin\delta_{,,,}\cos\alpha_{,,,}\,\partial x_0' + C_{,,,}\sin\delta_{,,,}\sin\alpha_{,,,}\,\partial y_0'
\end{aligned}\right\}\ \textit{check}$$

Multiply $\partial\rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$ by $\sin 1''$ to reduce them to circular measure. As a test on the accuracy of the corrections derived from these linear differential relations, compute

$$\begin{aligned}
\rho_{,}\cos\delta_{,}\cos\alpha_{,} &= \xi_{,} + f_{x_{,}}\partial\rho_0 + g_{,}\,\partial x_0'; &\quad \rho_{,,,}\cos\delta_{,,,}\cos\alpha_{,,,} &= \xi_{,,,} + f_{x_{,,,}}\partial\rho_0 + g_{,,,}\,\partial x_0'\\
\rho_{,}\cos\delta_{,}\sin\alpha_{,} &= \eta_{,} + f_{y_{,}}\partial\rho_0 + g_{,}\,\partial y_0'; &\quad \rho_{,,,}\cos\delta_{,,,}\sin\alpha_{,,,} &= \eta_{,,,} + f_{y_{,,,}}\partial\rho_0 + g_{,,,}\,\partial y_0'\\
\rho_{,}\sin\delta_{,} &= \zeta_{,} + f_{z_{,}}\partial\rho_0 + g_{,,,}\,\partial z_0'; &\quad \rho_{,,,}\sin\delta_{,,,} &= \zeta_{,,,} + f_{z_{,,,}}\partial\rho_0 + g_{,,,}\,\partial z_0'
\end{aligned}$$

If the computed values of $\alpha_{\prime}$, $\delta_{\prime}$, $\alpha_{\prime\prime\prime}$, $\delta_{\prime\prime\prime}$ represent the observed values within the desired limits of accuracy, the elements and an ephemeris may be computed by VIII. Otherwise, the differential corrections must be recomputed by [VII]. It will not, however, be necessary to change any of the auxiliary quantities.

Whenever new values of $\rho_{\prime}$, $\rho_{\prime\prime}=\rho_0$, $\rho_{\prime\prime\prime}$ have been derived, the corrections for parallax and aberration should be recomputed, if necessary. If the observations as previously corrected are properly represented, and if the final values of $\rho_{\prime}$, ρ_0, $\rho_{\prime\prime\prime}$ indicate the possibility of further improving the corrections for parallax and aberration, the second observation alone needs to be thus improved. In this case compute the final values:

$$x_0=\rho_0\cos\delta_{\prime\prime}\cos\alpha_{\prime\prime}-X_{\prime\prime};\qquad y_0=\rho_0\cos\delta_{\prime\prime}\sin\alpha_{\prime\prime}-Y_{\prime\prime};\qquad z_0=\rho_0\sin\delta_{\prime\prime}-Z_{\prime\prime},$$

using the final values for ρ_0, $\alpha_{\prime\prime}$, $\delta_{\prime\prime}$.
Otherwise correct the original values of x_0, y_0, z_0 by

$$\partial x_0=\xi_0\frac{\partial\rho_0}{\rho_0};\qquad \partial y_0=\eta_0\frac{\partial\rho_0}{\rho_0};\qquad \partial z_0=\zeta_0\frac{\partial\rho_0}{\rho_0},$$

respectively, and in either case correct the original values of x_0', y_0', z_0' by $\partial x_0'$, $\partial y_0'$, $\partial z_0'$, respectively.

VIII.

In the following formulæ all coördinates refer to the zero (second) date. The subscripts are omitted throughout:

$$\begin{aligned}
&\sqrt{p}\cos i = xy'-yx'; && r\sin u=\frac{z}{\sin i}\\
&\sqrt{p}\sin i\sin\Omega = yz'-zy'; && r\cos u = x\cos\Omega+y\sin\Omega\\
&\sqrt{p}\sin i\cos\Omega = xz'-zx'; && r^2=x^2+y^2+z^2,\ \text{check}\\
&r'=\frac{1}{r}(xx'+yy'+zz'); && \omega=u-v\\
&e\sin v=r'\sqrt{p}; && \pi=\omega+\Omega\\
&e\cos v=\frac{p}{r}-1; && a=\frac{p}{1-e^2}
\end{aligned}$$

Ellipse.

$$\tan\tfrac{1}{2}E=\sqrt{\frac{1-e}{1+e}}\tan\tfrac{1}{2}v$$

$$M=E-\frac{e}{\sin 1''}\sin E$$

$$\mu=k''a^{-\frac{3}{2}}$$

$\log k''=3.550\ 006\ 6$

Epoch $=t_{\prime\prime}$

Parabola.

$$q=\frac{p}{2}$$

with v as argument, take $M_{\prime\prime}$ from OPPOLZER, vol. I, table IV.

$$T=t_{\prime\prime}-M_{\prime\prime}q^{\frac{3}{2}}$$

Hyperbola.

$$\tan\tfrac{1}{2}F=\sqrt{\frac{e-1}{e+1}}\tan\tfrac{1}{2}v$$

$$T=t_{\prime\prime}+\frac{a^{\frac{3}{2}}}{k}\left[e\tan F-\frac{\log\tan(45^\circ+\frac{1}{2}F)}{\text{Mod.}}\right]$$

An ephemeris may be computed in the usual way, or, when the intervals are not too long, by means of the proper formulæ of VI. The number of terms to be taken into account in f and g depends, of course, on the intervals between ephemeris date and epoch. Additional terms may be written out easily on the basis of HARZER'S formulæ, A. N. 3371, pages 183–184.

When the orbit computed by the foregoing formulæ does not represent later observations with sufficient accuracy, the elements may be corrected easily so as to satisfy a single additional observation (or normal place) falling within the limits within which an ephemeris may be computed from formulæ VI. Since $\theta = k\ (t_\alpha - t_0)$ and $k = 0.017\ldots$, θ will be very nearly equal to unity for an interval of a little over 50 mean solar days. In general, for this interval, the f and g formulæ will no longer be applicable for determining an ephemeris. Just what the limiting value is in a given case depends also on r_0 and r_0', but as these quantities are already known, the applicability of the f and g expressions, and the number of terms to be considered in them, is easily decided in each individual case.

To correct the elements so as to represent a later observation, we compute for this observation the residuals due to the final elements, as in VI. If, then, these residuals be introduced in [VII] in place of $\partial_{,}\alpha_{,,,}$ and $\partial\delta_{,,,}$ we shall obtain such corrections $\partial\rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$ to the final values of ρ_0, x_0', y_0', z_0', of the original calculation, as will cause the residuals of the new observation to vanish. The auxiliary quantities contained in VI and [VII] should be computed with those values of x_0, y_0, z_0, r_0, x_0', y_0', z_0', r_0', etc., which correspond to the elements to which corrections are sought. If these elements exactly satisfied the original observations, then we put $\partial_{,}\alpha_{,} = \partial\delta_{,} = 0$ in the formulæ which give the differential corrections $\partial\rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$. If the first observation was not accurately represented by the finally adopted orbit, the proper residuals $\partial'\alpha'$ and $\partial\delta'$ must be introduced in [VII]. With the newly corrected heliocentric coördinates and velocities the elements are computed as in the ordinary case.

The method here outlined for computing an orbit is principally intended to aid in the rapid determination of the best orbit that can be passed through three observations made at short intervals. Such observations are now almost always available within a few days after the discovery of a new planet or comet. In the case of an object, however, remaining above the horizon the greater part of the

night, it is quite practicable to apply our formulæ to observations made in a single night, provided that the object be not too difficult to measure and that the geocentric motion be sufficiently rapid to enable the observer to determine observationally the velocity and acceleration in α and δ by making settings three times during the night as many hours apart as may be possible. In cases where the necessary observations can not be obtained in a single night, two nights may be sufficient, etc. The point images of asteroids secured by photography seem to be particularly promising in this direction.

ELEMENTS OF ASTEROID 1900 GA.

BY A. O. LEUSCHNER AND ADELAIDE M. HOBE,
BERKELEY ASTRONOMICAL DEPARTMENT.

UNIVERSITY OF CALIFORNIA PUBLICATIONS.

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By A. O. LEUSCHNER and ADELAIDE M. HOBE.

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About a month after discovery, Director KEELER sent us the positions of the middle of the trail of June 28th, and of the point images, and expressed the opinion that an indeterminate case seemed to be involved in the solution for the orbit from these observations. Unfortunately, no more observations could then be obtained, as the asteroid had moved out of the range of the reflector, and as it was too faint for visual observations, besides being in the thick of the Milky Way.

It should be stated here that, although the asteroid was photographed on four nights, the available data for a determination of the orbit consist only of three positions, which lie very nearly in the same great circle. Very little, if any, weight can be attached to the position derived from the trail of June 29th, as this trail fell near the edge of the plate and was exceedingly difficult to measure. On June 28th, the time which corresponds to the middle of the trail is in doubt by over a minute. Owing to the slow motion of the planet, this uncertainty of the time is of no great consequence.

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It should be stated here that, although the asteroid was photographed on four [illegible] nights, the available data for a determination of the orbit consist [illegible] positions, which lie very nearly in the same great circle. Very [illegible] weight can be attached to the position derived from the trail of June [illegible] trail fell near the edge of the plate and was exceedingly [illegible]. On June 28th, the time which corresponds to the middle of the [illegible] by over a minute. Owing to the slow motion of the planet [illegible] the time is of no great consequence.

The available data are as follows:

Mt. Hamilton M. T.		Exposure Time.	Asteroid's App. α.	Asteroid's App. δ.	No. of Comp. Stars.
1900, June 28^d	9^h 54^m 25^s	2 hours	18^h 3^m $19^s.80$	$-22°$ $28'$ $59''.4$	2
	11 54 25		18 3 15 .60	-22 29 50 .4	
June 29	10 13	1 hour	18 2 36 .66	-22 40 49 .5	2
	11 13		18 2 35 .43	-22 41 11 .3	
June 30	10 30.0	1 hour	18 1 54 .48	-22 52 42 .6	5
	11 30.0		18 1 53 .12	-22 53 7 .6	
June 30	11 52 4	5 minutes	18 1 52 .16	-22 53 22 .1	5
July 2	10 15 30	1 hour	18 0 34 .46	-23 15 58 .9	7
	11 15 30		18 0 33 .31	-23 16 32 .8	
July 2	11 22 50	10 minutes	18 0 32 .28	-23 16 37 .1	7

The problem of determining the orbit of the asteroid presents many points of interest, and has led to the derivation of "A Short Method of Determining Orbits from Three Observations," published in Part I of this volume.

A few preliminary tests, in which the calculations were made by Miss A. M. Hobe and Mr. S. C. Phipps, made it evident that the existing methods for determining orbits from three observations, without hypothesis as to the eccentricity, could not be used to advantage in this case.

In Gauss's and Oppolzer's methods, for instance, we find log K' (the logarithm of the coefficient of the geocentric distances) $= 4.62_n$, so that with six-place logarithms this coefficient is comparable with the uncertainty of the calculation. The geocentric distance is, therefore, practically indeterminate by these methods. In a seven-place calculation it is nearly so, and the results of the first hypothesis do not furnish data sufficiently accurate for a second hypothesis, etc.

In problems of this sort the greatest disadvantage arises from the fact that the numerical operations of the successive hypotheses are so complicated that it is difficult to form an estimate of the degree of their convergency. This is true also of Gibbs's Vector Method. In none of the existing methods are the quantities upon which a new hypothesis is to be based given directly as functions of the residuals of the preceding hypothesis.

In the present problem we encounter the additional difficulties that the observations extend over only four days, that the total motion in right ascension and declination is only 2^m 45^s and $47'$, respectively, and that owing to the faintness of the object the available positions are less reliable than is ordinarily the case.

The uncertainty of the orbit, due to the insufficiency of the available material, can not be overcome by any method of solution, no matter on what theoretical considerations it may be based. The best that can be done is to accurately represent the given material. The theoretical difficulties involved in this task can be overcome by a proper choice of method. The consideration that an indeterminate function may be evaluated by the use of derivatives, suggests the introduction of the geocentric velocities and accelerations. The methods of LAPLACE and HARZER furnish convenient expressions which contain the geocentric distance, its velocity and acceleration as a function of the corresponding right ascension and declination, their velocities and accelerations; but for reasons which are apparent from the preceding article, these methods are otherwise not suited for the solution of the present problem. The method outlined in the preceding pages has proved very efficient in this case, as well as in the cases of comet 1900 III and comet 1901 I, to which it has been applied.

It was at first hoped that it might be possible to derive the necessary velocities and accelerations in the two observed geocentric coördinates directly from the trails, and for this purpose all the necessary data were kindly furnished by Director CAMPBELL, but it was found to be impracticable.

The numerical operations involved in determining the orbit are given here in full, as they are intended to serve as an illustration of the method derived in Part 1. The arguments are in strict accordance with the notation adopted in the formulæ collected under the headings I–VIII.

Ordinarily the calculation will be much briefer than the present case would seem to indicate. Thus, systems II–V have been computed three times in order to demonstrate that nothing is gained by making three direct approximations for the geocentric distance. One approximation would have been entirely sufficient. It should be noted, also, that almost the entire calculation [VII] is based on logarithms already known. The only new logarithms which need to be taken from the tables are the addition and subtraction logarithms involved.

Of the three places on which the calculation is based, the first represents the position of the middle of the trail of June 28th. It was derived by taking the mean of the coördinates of the termini and of the corresponding times. The places for June 30th and July 2d were derived by combining the coördinates of the middle of the trail with the coördinates of the point-image observed on the same date, the latter coördinates being given twice the weight of the former. Whether this manner of combining the observations possesses the greatest advantages can only be decided by future observations, if the asteroid should be observed again. The adopted positions are:

Date, Gr. M.T.	α app.	δ app.
1900, June 28.792359	$18^h\ 3^m\ 17^s.700$	$-22^\circ\ 29'\ 24''.90$
June 30.820341	1 52 .707	53 13 .10
July 2.803450	0 32 .815	— 23 16 30 .02

For the position of the middle of the trail of June 29th we have

June 29.784431 $18^h\ 2^m\ 36^s.045$ $-22^\circ\ 41'\ 0''.40$

This observation was given zero weight. The other three places were reduced to the beginning of the year by the ordinary formulæ on the basis of the data given in the *American Ephemeris*, f' being neglected. The resulting mean places, which are not corrected for the aberration term, depending on the product of the aberration into the eccentricity of the earth's orbit, are given in II.

I.

From the *American Ephemeris and Nautical Almanac* for 1900 we obtain by interpolation the solar coördinates referred to the beginning of the year, as follows:

Date, Gr. M. T., 1900, June	28.792359_5	30.820341_5	32.803450_5
X	-0.1246766	-0.1586609_5	-0.1917110
Y	$+0.9257300$	$+0.9213667_5$	$+0.9160654_5$
Z	$+0.4015994$	$+0.3997058_5$	$+0.3974057$

For the determination of the velocities X_0', Y_0', Z_0' we have

$m = 0.1406830$ $\log m = 9.148242$

	X	Y	Z
$\log\ f^{\text{II}}(a+[i+\frac{1}{2}]w)$	5.0540	5.813414_n	5.450634_n
$\log m f^{\text{II}}(a+[i+\frac{1}{2}]w)$	4.2022	4.961656_n	4.598876_n
$\log\ f^{\text{I}}(a[i+\frac{1}{2}]w)$	7.9221465_n	7.0787286_n	6.7160869_n
B	3.7199	A 2.11707	2.11721
A	0.0000828	B 0.0033041_5	0.0033031
$\log (X_0', Y_0', Z_0')$	9.9875127_n	9.1474817_{5n}	8.7848390_n

$\log \cos A$	9.229692_n	$\log \cos D$	9.963549
$\log X_{\bullet}$	9.200470_n	$\log R \cos D$	9.970778
$\log Y_{\bullet}$	9.964432	$\log Z_{\bullet}$	9.601740
$\log \sin A$	9.993654	$\log \sin D$	9.594511
$\log \tan A$	0.763962_n	$\log \tan D$	9.630962
A	$99^\circ\ 46'\ 14''.21$	D	$23^\circ\ 8'\ 52''.59$
α	270 27 3 .44	$\log R$	0.007229
$A-\alpha$	−170 40 49 .23		

The second observation was not corrected for the parallax corresponding to the unit of distance.

II.

Date, Gr. M. T.			α (1900.0).		δ (1900.0).	
June 28.792359_5			$18^h\ 3^m\ 13^s.254$		$-22°\ 29'\ 28''.95_5$	
	2.027982			$-85^s.024_5$		$-1428''.04_5$
June 30.820341_5		0.044873	$1\ \ 48.229_5$		$53\ \ 17.00$	
	1.983109			-79.921_5		-1396.76_5
July 2.803450_5			$0\ \ 28.308$		$-23\ \ 16\ \ 33.76_5$	
	4.011091 (sum)					

$\log(\alpha_{\prime\prime}-\alpha_{\prime})$	1.929544_n	$\delta_{\prime}'-\delta_{\prime\prime\prime}'$	-0.0000446
$\log(t_{\prime\prime}-t_{\prime})$	0.307064	$\log(\alpha_{\prime}'-\alpha_{\prime\prime\prime}')$	7.836830
$\log(\delta_{\prime\prime}-\delta_{\prime})$	3.154742_n	$\log(t_{\prime\prime\prime}-t_{\prime})$	0.603262
$\log(\alpha_{\prime\prime\prime}-\alpha_{\prime\prime})$	1.902663_{5n}	$\log(\delta_{\prime}'-\delta_{\prime\prime\prime}')$	5.649335_n
$\log(t_{\prime\prime\prime}-t_{\prime\prime})$	0.297347	$\log \alpha_0''$	9.299016_5
$\log(\delta_{\prime\prime\prime}-\delta_{\prime\prime})$	3.145123_n	$\log[(t_{\prime\prime}-t_{\prime})-(t_{\prime\prime\prime}-t_{\prime\prime})]$	8.652
$\log \alpha_{\prime\prime\prime}'$	9.248565_n	$\log \delta_0''$	7.111521_{5n}
$\log \delta_{\prime\prime\prime}'$	9.297671_{5n}	$\log \frac{k}{4}\left[(t_{\prime\prime}-t_{\prime})-(t_{\prime\prime\prime}-t_{\prime\prime})\right]\alpha_0''$	5.585
$\log \alpha_{\prime}'$	9.231401_{5n}	$\log \frac{1}{2}(\alpha_{\prime\prime\prime}'+\alpha_{\prime}')$	9.240067_{5n}
$\log \delta_{\prime}'$	9.297769_{5n}	B	3.655
$\alpha_{\prime\prime\prime}'$	-0.1772412	C	9.999904
$\alpha_{\prime}'$	-0.1703732	$\log \alpha_0'$	9.239971_{5n}
$\delta_{\prime\prime\prime}'$	-0.1984593	$\log \frac{k}{4}\left[(t_{\prime\prime}-t_{\prime})-(t_{\prime\prime\prime}-t_{\prime\prime})\right]\delta_0''$	3.40_n
$\delta_{\prime}'$	-0.1985039	$\log \frac{1}{2}(\delta_{\prime\prime\prime}'+\delta_{\prime}')$	9.297721_n
$\alpha_{\prime\prime\prime}'+\alpha_{\prime}'$	-0.3476144	A	4.10
$\delta_{\prime\prime\prime}'+\delta_{\prime}'$	-0.3969632	B	0.000001
$\alpha_{\prime}'-\alpha_{\prime\prime\prime}'$	$+0.0068680$	$\log \delta_0'$	9.297722_n

δ_0'', being less than a tenth of a second of arc per day, might have been put equal to zero without diminishing the accuracy of the first approximation of σ_0, σ_0', σ_0'', but it was retained in the calculation in order to render the example more complete. It should be stated, however, that the values of α_0', α_0'', and δ_0'', used below, differ slightly from those derived above. The differences are due to the fact that in the original calculation right ascensions and declinations, differing from those printed above by a few one thousands of a second of arc, were used. These differences serve to show that the accelerations are subject, generally, to more or less uncertainty, depending on their magnitude as compared with the errors of observation. This uncertainty is of no consequence in our method, as later we may readily correct the fundamental data (*b*) so as to accurately represent the foregoing observations.

$\log \delta_0'$	9.297722_n	$\log 2 \tan \delta_{\prime\prime}(\delta_0')^2$	8.521962_n
$\log \sec^2 \delta_{\prime\prime}$	0.071228	$\log \delta_0''$	7.112040_n
$\log(\tan\delta)_0'$	9.368950_n	A	8.590078
$\log \tan \delta_{\prime\prime}$	9.625488_n	B	0.016579
$\log(\delta_0')^2$	8.595444	sum	8.538541_n
		$(\tan\delta)_0''$	8.609769_n

III.

log $(\alpha_0')^3$	7.719907_n
log tan δ_0	9.625488_n
log α_0''	9.298855
log $(\tan\delta)_0'$	9.368950_n
log α_0'	9.239969_n
log $(\tan\delta)_0''$	8.609769_n
log $\alpha_0'\,(\tan\delta)_0''$	7.849738
log $[-\alpha_0''(\tan\delta)_0']$	8.667805
A	9.181933
B	0.061464
sum	8.729269
log $(\alpha_0')^3 \tan\delta_0$	7.345395
A	8.616126
B	0.017583
N	8.746852
log cos $(A-\alpha_0)$	9.994230_n
log tan δ_0	9.625488_n
log tan $\delta_0 \cos(A-\alpha_0)$	9.619718
log tan D	9.630962
B	0.011244
A	8.418769
log$[\tan\delta_0\cos(A-\alpha_0)-\tan D]$	8.038487_n
log $\alpha_0'[\quad]$	7.278456

log sin $(A-\alpha_0)$	9.209361_n
log sin $(A-\alpha_0)\,(\tan\delta)_0'$	8.578311
A	8.700145
B	0.021245
log $R\cos D$	9.970778
colog N	1.253148
log κ	9.823482_n
log cos δ_0	9.964386
log cos D	9.963549
log sin δ_0	9.589874_n
log sin D	9.594511
log sin δ_0 sin D	9.184385_n
log cos $\delta_0 \cos\delta\cos(A-\alpha_0)$	9.922165_n
A	9.262220
B	0.072949
log cos ψ	9.995114_n
ψ	171° 25'.3
log R^4	0.028916
colog κ	0.176518_n
log $\frac{1}{m}$	0.169820
$\frac{1}{m}$	1.4785
z_1 (OPPOLZER, table XIIIa)	0.4551

IV.

log cos ψ	9.995114_n	
log z_1	9.658107	9.658345
log $2z_1\cos\psi$	9.954251_n	9.954489_n
log z_1^2	9.316214	9.316690
A	9.361963	9.362201
B	0.089949	0.089993
log (μ_1-1)	0.044200	0.044482
B	0.279492	0.279358_5
log μ_1	0.323692	0.323840_5
log m	9.830180	9.830180
B	0.172073	0.171835
A	9.686803	9.686075
log ν_1	9.344910_n	9.344420_n
log ν_1^2	8.689820	8.688840
log μ_1^3	0.971076	0.971521_5

log $\nu_1^2\mu_1^3$	9.660896	9.660361_5
log m^2	9.660360	9.660360
B	0.000536	0.000001_5
A	7.091667	4.55
log M_1	6.7520	4.21
A	9.6630	
B	0.1644	
log $(z_1-\cos\psi)$	0.1595	
log $3\frac{\nu_1}{\mu_1}$	9.4983	
log $3\frac{\nu_1}{\mu_1}(z_1-\cos\psi)$	9.6578_n	
log $\left[1+\frac{3\nu_1}{\mu_1}(z_1-\cos\psi)\right]$	0.7366	
log $2\,\mu_1^3\nu_1$	0.6170_n	
log Δz_1	6.3984	3.86
Δz_1	0.000250	0.0000007
z_2	0.455350	0.455351

V.

$\log R$	0.007229	$\log \frac{R \cos D}{2N}$	0.922896
$\log Z$	9.658346	$\log \lambda$	8.734717
$\log \rho_0$	9.665575	$\operatorname{colog} \varkappa$	0.176518_n
$\log \cos \delta_{\prime\prime}$	9.964386	$\log \sigma_0'$	8.541196_n
$\log \sigma_0$	9.629961	$\operatorname{colog} R^3$	9.9783
$\log (\alpha_0')^2$	8.479938	$\log (\alpha_0')^2$	8.4799
$\log \tan \delta_{\prime\prime}$	9.625488_n	B	1.4984
$\log (\alpha_0')^2 \tan \delta_{\prime\prime}$	8.105426_n	C	9.9860
$(\tan \delta)_0''$	8.609769_n	$\log \sigma_0 [R^{-3} - (\alpha_0')^2]$	9.5943
A	9.495657	$\log R \cos D \cos (A - \alpha_{\prime\prime})$	9.9650_n
B	0.118292	A	9.6650
$\log \sin (A - \alpha_{\prime\prime})$	9.209361_n	B	0.1650
$\log \sin (A - \alpha_{\prime\prime}) [(\alpha_0')^2 \tan \delta_{\prime\prime} + (\tan \delta)_0'']$	7.937422	$\log \frac{\sigma_0}{\varkappa} [R \cos D \cos (A - \alpha_{\prime\prime}) - \sigma_0]$	9.9365
$\log \alpha_0'' [\tan \delta_{\prime\prime} \cos (A - \alpha_{\prime\prime}) - \tan D]$	7.337342_n	B	0.3422
B	0.600080	A	0.0788
C	9.874399	$\log \sigma_0''$	9.6731

	I.	III.		I.	III.
$\log \theta$	8.5426_5	8.5329	$\log\left(\pm \theta\sigma_0' + \theta^2 \frac{\sigma_0''}{2}\right)$	7.1759	6.9600_n
$\log \theta^2$	7.0853	7.0658	A	7.5459	B 2.6700
$\log \pm \theta\sigma_0'$	7.0838	7.0741_n	B	0.00152	C 9.99907
$\log \theta^2 \frac{\sigma_0''}{2}$	6.4574	6.4379	$\log \sigma$	9.63148	9.62903
A	9.3736	B 0.6362	$\log \cos \delta$	9.96564	9.96313
B	0.0921	C 9.8859	$\log \rho$	9.66584	9.66590

Owing to the uncertainty of some of the fundamental data (a) in this example, particularly of the accelerations, we are justified in assuming that the first approximation does not result in sufficiently accurate geocentric distances to enable us to determine the final corrections for parallax and aberration. It would, therefore, be best to proceed at once to the computation of the heliocentric rectangular coördinates and their velocities, after applying a preliminary parallax correction to the second observation. Nevertheless, we shall correct all three places for parallax and aberration, and shall redetermine the geocentric velocities and accelerations, and, from these, the values of σ_0, σ_0', etc., in order that we may be able to show that little or nothing can be gained by a second approximation of this kind. With the values of ρ, derived above, the original data (coördinates of termini of trails and point-images) were separately corrected for parallax and aberration, and then recombined into three places as before, with the following result:

Date, Gr. M. T.	α (1900.0).	δ (1900.0).
June 28.789686	$18^h\ 3^m\ 13^s.065_5$	$-22°\ 29'\ 12''.87$
June 30.817670	$1\ \ 48.269_5$	$53\ \ 0.61$
July 2.800777	$0\ \ 28.277$	$-23\ \ 16\ \ 17.30$

From these we derive

log α_0'	9.239575_n	log $(\tan\delta)_0'$	9.368864_n	log $\varkappa$	9.856427_n
log α_0''	9 257362	log $(\tan\delta)_0''$	8.621354_n	log λ	8.790774

and further

log ρ_0	9.723653	log σ_0'	8.622400_n	log $\rho_{\prime}$	9.72396
log σ_0	9.688053	log σ_0''	9.7057	log $\rho_{\prime\prime\prime}$	9.72389

These values of ρ differ from the former to such an extent that it is necessary to recompute the corrections for parallax and aberration. It might, therefore, seem that the approximations should be continued until the parallax and aberration cease to vary. But, in the present case, $\varkappa$ and ρ_0 must remain uncertain, even if the observations were fully corrected. This may easily be seen from the computation of the first approximation of $\varkappa$. It is, then, evident that, after the parallax and aberration cease to vary, everything depends on whether the accelerations are constant or not within the interval between our first and last dates of observation. If they fail to be constant, even by a small amount, the value of $\partial\rho_0$, to be computed later from the residuals of the first and third places, will be considerable, and may affect the parallax and aberration. The approximations for the geocentric velocities and accelerations and for the corresponding value of ρ_0 should, therefore, now be discontinued; but in order that the reader may not be led to conclude that the large numerical value of $\partial\rho_0$, which will be found later, is due to the fact that the approximations have not been carried sufficiently far, we shall continue the same until the corrections practically cease to vary. Thus, from the last values of $\rho_{\prime}$, ρ_0, and $\rho_{\prime\prime\prime}$, printed above, we first obtain the following corrected positions:

Date, Gr. M. T.		α (1900.0).			δ (1900.0).		
June	28.789303	18^h	3^m	$13^s.089$	$-22°$	29′	14″.89
June	30.817287		1	48 .265		53	2 .66
July	2.800394_5		0	28.280_5	-23	16	19 .36

These positions give

log α_0'	9.239625_n	log $\varkappa$	9.852281_n	log σ_0'	8.612542_n
log α_0''	9.262634	log λ	8.783846	log σ_0''	9.7019
log $(\tan\delta)_0'$	9.368874_n	log ρ_0	9.716579	log $\rho_{\prime}$	9.71689
log $(\tan\delta)_0''$	8.619945_n	log σ_0	9.680977	log $\rho_{\prime\prime\prime}$	9.71683

With the foregoing values of ρ we obtain the following corrected positions:

Date, Gr. M. T.		α (1900.0).			δ (1900.0).		
June	28.789352	18^h	3^m	$13^s.086_5$	$-22°$	29′	$14''.65_5$
June	30.817336		1	48.265_5		53	2.42_5
July	2.800443		0	28 .280	-23	16	19.12_5

Further approximations would not produce any sensible change in the parallax or aberration. We, therefore, proceed to the computation of the heliocentric coördinates and velocities, and of the residuals of the first and third places, etc.

VI.

$\log \cos \alpha_{\prime\prime}$	7.8961509	$\log \cos \alpha_{\prime\prime}\, \sigma_0'$	6.5086929_n	B	1.7957751
$\log \sin \alpha_{\prime\prime}$	9.9999865_n	$\log \eta_0\, \alpha_0'$	8.9205890	A	0.0070066
$\log \tan \delta_{\prime\prime}$	9.6254026_5	A	2.4118961	$\log \tan \delta_{\prime\prime}\, \sigma_0'$	8.2379446
$\log \xi_0$	7.5771279	B	0.0016790	$\log \sigma_0 (\tan \delta)_0'$	9.0498515_n
$\log \eta_0$	9.6809635_n	$\log \sin \alpha_{\prime\prime}\, \sigma_0'$	8.6125285	B	0.8119068
$\log \zeta_0$	9.3063796_{5n}	$\log \xi_0\, \alpha_0'$	6.8167534_n	A	0.0727339

ξ_0, η_0, ζ_0		+0.0037768		−0.4796931	−0.2024788
$X_{\prime\prime\prime}, Y_{\prime\prime\prime}, Z_{\prime\prime}$		-0.1586609_5		$+0.9213667_5$	$+0.3997058_5$
x_0, y_0, z_0		$+0.1624377_5$		-1.4010598_5	-0.6021846_5
$\log (x_0, y_0, z_0)$		9.2106869_5		0.1464566_{5n}	9.7797296_{5n}
$\log (\xi_0', \eta_0', \zeta_0')$		8.9222680_n		8.6055219	8.9771176_n
$\log (X_0', Y_0', Z_0')$		9.9875127_n		9.1474817_{5n}	8.7848390_n
	B	1.0652447	A	0.5419598_5	0.1922786
	A	0.0390780	B	0.1096138_5	0.4464514
$\log (x_0', y_0', z_0')$		9.9484347		9.2570956	8.5306662_n

$\log x_0^2$	8.4213739	$\log x_0 x_0'$	9.1591216_5	$\log r_0 r_0'$	8.9472556_{5n}
$\log y_0^2$	0.2929133	$\log y_0 y_0'$	9.4035522_5	$\log r_0$	0.1857169
A	1.8715394	A	0.2444306	$\log r_0'$	8.7615387_{5n}
B	0.0057989	B	0.3661270	$\log r_0^3$	0.55715
$\log (x_0^2+y_0^2)$	0.2987122	$\log (x_0 x_0'+y_0 y_0')$	9.0374252_n	$\log r_0^4$	0.74
$\log z_0^2$	9.5594593	$\log z_0 z_0'$	8.3103958_5	$\log \frac{1}{2r_0^3}$	9.14182
A	0.7392529	B	0.7270294	$\log \frac{r_0'}{2r_0^4}$	7.72_n
B	0.0727215_5	A	0.0901696	$\log \frac{1}{6r_0^3}$	8.66470
$\log r_0^2$	0.3714337_5				

	I.	III.		I.	III.
$(t_{\prime\prime}-t_{\prime}), (t_{\prime\prime\prime}-t_{\prime\prime})$	2.0279840	1.9831070	$g x_0'$	−0.0309781	+0.0302928
$\log \Delta t$	0.3070645	0.2973461	Y	+0.9257300	$+0.9160654_5$
$\log \theta$	8.5426455	8.5329271	$f y_0$	−1.4008239	-1.4008335_5
$\log \theta^2$	7.08529	7.06585	$g y_0'$	-0.0063054_5	$+0.0061659_5$
$\log \theta^2 \frac{1}{2r_0^3}$	6.22711	6.20767	Z	+0.4015994	+0.3974057
$\log \theta^3 \frac{r_0'}{2r_0^4}$	3.35_n	3.32_n	$f z_0$	−0.6020832	−0.6020874
$\theta^2 \frac{1}{2r_0^3}$	+0.0001687	+0.0001613	$g z_0'$	+0.0011838	−0.0011576
$\theta^3 \frac{r_0'}{2r_0^4}$	2	2	ξ	+0.0067557	+0.0009933
f	0.9998315	0.9998385	η	-0.4813993_5	-0.4786021_5
$\log \theta^3 \frac{1}{6r_0^3}$	4.293	4.263	ζ	−0.1993000	−0.2058393
$\theta^3 \frac{1}{6r_0^3}$	0.0000020	0.0000018	$\log \cos \alpha$	8.1471221_5	7.3171049
θ	0.0348855_5	0.0341136	$\log \xi$	7.8296704	6.9970804
g	-0.0348835_5	+0.0341118	$\log \eta$	9.6825055_n	9.6799746_{5n}
$\log f$	9.9999268_5	9.9999298_5	$\log \sin \alpha$	9.9999572_{5n}	9.9999991_n
$\log g$	8.5426206_{5n}	8.5329047	$\log \cot \alpha$	8.1471649_n	7.3171057_{5n}
$\log f x_0$	9.2106138	9.2106168	α	270° 48′ 14″.420	270° 7′ 8″.085
$\log f y_0$	0.1463835_n	0.1463865_n	$\log \cos \delta$	9.9656541_5	9.9631460
$\log f z_0$	9.7796565_n	9.7796595_n	$\log \rho \cos \delta$	9.6825482_5	9.6799755
$\log g x_0'$	8.4910553_n	8.4813394	$\log \zeta$	9.2995073_n	9.3135283_n
$\log g y_0'$	7.799716_n	7.790000	$\log \sin \delta$	9.5826132_n	9.5966988_n
$\log g z_0'$	7.073287	7.063571_n	$\log \tan \delta$	9.6169590_{5n}	9.6335528_n
X	−0.1246766	−0.1917110	δ	−22° 29′ 15″.467	$-23° 16′ 18″.276_5$
$f x_0$	+0.1624104	+0.1624115	$\log \rho$	9.7168941	9.7168296

	I.	III.
$\log f_x\, \partial\rho_0$	6.33533_5	6.33558_5
$\log f_y\, \partial\rho_0$	8.43640_{5n}	8.43639_{5n}
$\log f_z\, \partial\rho_0$	8.06182_{5n}	8.06181_{5n}
$\log g\, \partial x_0'$	6.23794	6.22822_{5n}
$\log g\, \partial y_0'$	6.08470_{5n}	6.07499
$\log g\, \partial z_0'$	6.23874	6.22902_{5n}
$f_x\, \partial\rho_0$	$+0.0002164$	$+0.0002166$
$g\, \partial x_0'$	$+0.0001729_5$	-0.0001691
ξ	$+0.0067557$	$+0.0009933$
corrected ξ	$+0.0071450_5$	$+0.0010408$
$f_y\, \partial\rho_0$	-0.0273152	-0.0273146
$g\, \partial y_0'$	-0.0001215	$+0.0001188_5$
η	-0.4813993_5	-0.4786021_5
corrected η	-0.5088360_5	-0.5057979
$f_z\, \partial\rho_0$	-0.0115298_5	-0.0115296
$g\, \partial z_0'$	$+0.0001733$	-0.0001694_5
ζ	-0.1993000	-0.2058393

	I.	III.
corrected ζ	-0.2106565_5	-0.2175383_5
$\log \cos \alpha$	8.1473846	7.3133893_5
$\log \xi$	7.8540052_5	7.0173673
$\log \eta$	9.7065778_{5n}	9.7039770_n
$\log \sin \alpha$	9.9999572_n	9.9999991_n
$\log \cot \alpha$	8.1474274_n	7.3133903_n
α	$270^\circ\ 48'\ 16''.169$	$270^\circ\ 7'\ 4''.438_5$
$\log \cos \delta$	9.9656548_5	9.9631452
$\log \rho \cos \delta$	9.7066206_5	9.7039779
$\log \zeta$	9.3235750_n	9.3375358_n
$\log \sin \delta$	9.5826092_n	9.5967031_n
$\log \tan \delta$	9.6169543_{5n}	9.6335579_n
δ	$-22^\circ\ 29'\ 14''.673$	$-23^\circ\ 16'\ 19''.155$
$\log \rho$	9.7409658	9.7408327
$O-C$ $\Delta\alpha$	$+0''.13$	$-0''.24$
$O-C$ $\Delta\delta$	$+0\ .02$	$+0\ .03$

In view of the comparative uncertainty of the observations, these residuals are more than satisfactory. It should be noted, however, that the residuals in declination are actually larger than those derived above by about 0″.8, on account of change in the parallax, due to the considerable change which has taken place in the geocentric distances. We shall allow for the changes in parallax in computing the final heliocentric rectangular coördinates. With the values of $\log \rho_{,}$ and $\log \rho_{,,,}$, found above, and with $\log \rho_{,,} = \log(\rho_0 + \partial\rho_0) = 9.74062$, we first obtain the following newly-corrected positions:

Date, Gr. M. T.	α (1900.0).	δ (1900.0).
June 28.789181	$18^h\ 3^m\ 13^s.096$	$-22^\circ\ 29'\ 15''.43$
June 30.817165	$1\ \ 48\ .264$	$53\ \ 3\ .21$
July 2.800272	$0\ \ 28\ .281$	$-23\ \ 16\ \ 19\ .91$

With the second of these positions and with $\log \rho_{,,} = \log(\rho_0 + \partial\rho_0) = 9.7406228$ the heliocentric rectangular coördinates are now re-computed as in VI; $X_{,,}$, $Y_{,,}$, and $Z_{,,}$ remaining the same as in the original computation. The velocities x', y', z' are obtained by applying to x_0', y_0', z_0' the corrections $\partial x_0'$, $\partial y_0'$, $\partial z_0'$, determined above. We then complete the recomputation of VI, and thus obtain the new residuals. It will be noted that by following the process just outlined, the change in parallax is fully taken into account in the rectangular coördinates and in the residuals. The velocities, however, are not changed, this being considered unnecessary, as all three declinations were changed by the new parallax by about the same amount, and as the change in the right ascensions was very small. The various corrected quantities follow below:

$\log x$	9.2112614	$\log x'$	9.9460031
$\log y$	0.1548393_n	$\log y'$	9.2653868
$\log z$	9.7879651_n	$\log z'$	8.5899909_n
$\log r$	0.1939908	$\log r'$	8.7867019_n

	I.	III.
α	$270^\circ\ 48'\ 16''.13$	$270^\circ\ \ 7'\ \ 4''.40$
δ	$-\ 22\ \ 29\ \ 15\ .49$	$-\ 23\ \ 16\ \ 19\ .98$
$O-C$ $\cos\delta\ \Delta\alpha$	$+\ 0''.28$	$-\ 0''.16$
$O-C$ $\Delta\delta$	$+\ 0\ .06$	$+\ 0\ .07$

A further reduction of the residuals is unwarranted, and would not lead to more accurate elements.

VIII.

A	1.6241942	$\log \sqrt{p}$	0.1486498	φ	16° 22′ 55″.02
$\log xy'$	8.4766482	$\log p$	0.2972996	$\frac{1}{2}\varphi$	8 11 27 .51
$\log yx'$	0.1008424_n	B	0.4416705	$\log \cot (45° + \frac{1}{2}\varphi)$	9.8740997
B	0.0101972	$\log x \cos \Omega$	9.1907378	$\log \tan \frac{1}{2} v$	9.1945212_n
A	0.3085217	$\log y \sin \Omega$	9.6324083_n	$\log \tan \frac{1}{2} E$	9.0686210_n
$\log yz'$	8.7448302	A	0.1949644	$\frac{1}{2} E$	− 6° 40′ 47″.62₅
$\log zy'$	9.0533519_n	$\log \cos u$	9.2434531_n	E	− 13 21 35 .25
B	0.1736083	$\log r \cos u$	9.4374439_n	$\log \cos \varphi$	9.9820010
B	1.9327159	$\log r \sin u$	0.1872236_n	$\log \sin E$	9.3637345_n
$\log xz'$	7.8012523_n	$\log \sin u$	9.9932328_n	$\log e''$	4.7647346
$\log zx'$	9.7339682_n	$\log \tan u$	0.7497797	$\log \Delta M$	4.1284691_n
A	0.0051005	u	259° 54′ 41″.78	ΔM	− 3° 44′ 2″.16
$\log \sin \Omega$	9.4775690	$\log r$	0.1939908 check	M	− 9 37 33 .09
$\log \sqrt{p} \sin i \sin \Omega$	9.2269603	$\log \frac{p}{r}$	0.1033088	ω	277 42 4 .19
$\log \sqrt{p} \sin i \cos \Omega$	9.7288677	B	0.6742779	π	295 10 38 .48
$\log \cos \Omega$	9.9794764	$\log \sin v$	9.4850422_n	$\log \cos^2 \varphi$	9.9640020
$\log \tan \Omega$	9.4980926	$\log \sin \varphi \sin v$	8.9353517_n	$\log a$	0.3332976
Ω	17° 28′ 34″.29	$\log \sin \varphi \cos v$	9.4290308	a	2.1542574
$\log \sin i$	9.6007415	$\log \cos v$	9.9787213	$\log a^{\frac{3}{2}}$	0.4999464
$\log \sqrt{p} \sin i$	9.7493913	$\log \tan v$	9.5063208_n	$\log \mu$	3.0500602
$\log \sqrt{p} \cos i$	0.1110396	v	− 17° 47′ 22″.41₅	$\log P$	3.0625448
$\log \cos i$	9.9623898	$\frac{1}{2}v$	− 8 53 41 .21	μ	1122″.174
$\log \tan i$	9.6383516	$\log \sin \varphi$	9.4503095	P	1154ᵈ.901
i	23° 30′ 8″.64	e	0.2820392		

The elements which determine the position of the orbit in space are referred to the equator. After transforming them to the ecliptic in the usual way and determining the reductions to the equinox of 1902.0, we obtain the following collected results, where the reductions of the constants to the equator for 1901.0 are one half of those for 1902.0. The quantities referring to 1900.0 are given more accurately than the others, as it is intended to use them in an accurate redetermination, by means of an ephemeris, of the residuals of the places on which the calculation was based.

Epoch 1900, June 30.817165, Gr. M. T.

M	350° 22′ 16″.91
φ	16 22 55 .01
a	2.154257
μ	1122″.174
P	1154ᵈ.901

	1900.0.	Reduction to 1902.0.
Ω	97° 36′ 55″.57	+ 1′ 32″.8
i	6 56 23 .06	− 0 .2
ω	196 8 5 .52	+ 7 .7

	1900.0.	Reduction to 1901.0.	Reduction to 1902.0.
A'	23° 48′ 21″.04	+ 50″.6	+ 1′ 41″.2
B'	296 38 58 .86	+ 49 .4	+ 1 38 .8
C'	277 42 4 .19	+ 48 .4	+ 1 36 .9
$\log \sin a$	9.9968631	+ 0.000000	+ 0.000000
$\log \sin b$	9.9660620	+ 5	+ 10
$\log \sin c$	9.6007415	− 29	− 58

According to these elements, opposition and perihelion occurred in 1900 on June 23.09 and July 31.70, Gr. M. T., respectively.

As a check on the calculation, the residuals of all three places were computed directly from the constants to the equator by the usual ephemeris method. The residuals of the middle of the trail, photographed on June 29th, are also here given. For the sake of comparison, the final residuals derived by VI are also printed below. To make the latter strictly comparable with the former, the aberration terms depending on the product of the aberration into the eccentricity of the earth's orbit were computed according to OPPOLZER, *Bahnbestimmung*, Vol. I, page 115. In this final comparison of computed and observed apparent geocentric places f' was also taken account of.

	$\cos\delta\,\Delta\alpha\,(O-C)$.		$\Delta\delta\,(O-C)$.	
	From Ephemeris.	From VI.	From Ephemeris.	From VI.
June 28	+0".42	+0".40	+0".06	+0".06
[June 29	+1 .74	——	+2 .78	——]
June 30	+0 .11	+0 .16	−0 .02	±0 .00
July 2	+0 .19	+0 .15	+0 .06	+0 .07

The agreement of the two sets of residuals is all that can be desired. As the residuals of the adopted positions are far smaller than the unavoidable errors of observation, a further correction of the elements would have no practical value. In this connection it may be stated that Mr. PALMER writes in a report, kindly furnished by Director CAMPBELL to aid me in determining the orbit, that "the point-image of June 30th is so faint that the measures are less accurate than on many very faint nebulæ with no nucleus." Under these circumstances, no practical gain will result from a further reduction of the residuals.

With reference to the manner in which the trails were measured, Mr. PALMER states that "the intersection of the cross-wires was set on what seemed to be the points which would be occupied by the centers of two short exposures taken such that the first should begin at the same time as the long (trail) exposure and the last end at the same time as the long exposure." "The trail of July 2d had two good termini, but the others each had one very poor terminus." Let us, then, assume for a moment that the instants of the beginning and end of the trail-exposures were accurately known, that the trails were so distinct as to enable the observer to estimate accurately the positions on the trails of the centers of two short exposures as defined above, and that, in addition, all other conditions which might affect the accuracy of the measured coördinates were the same for the four trails. Then, by forming the mean of the beginning and end of the exposure time and of the coördinates of the termini for each trail, we should obtain positions of the middle of the trail which are free from error, provided the motion of the asteroid during the time of exposure is proportional to the time. An ephemeris which has been computed for the dates under consideration, shows this proportionality to be true within 0".01. Under these ideal conditions the residuals relative to the middle of the trail of the coördinates of the short exposure point

estimated at the beginning of the trail should be the same for the four trails, and, furthermore, the corresponding relative residuals for the ends should be numerically the same as the former, but of opposite sign. The algebraic sign of these relative residuals should be negative for the beginning of the trails, for both right ascension and declination.

In order to determine how nearly these ideal conditions are fulfilled in the present case, the positions of the asteroid were computed by VI for June 28.75, June 29.75, June 30.75, and July 2.75, Gr. M. T., the necessary solar coördinates for the same dates being interpolated from the *American Ephemeris*. From these positions and from the positions already computed by means of the constants to the equator for the four adopted observation-times, the coördinates of the asteroid were determined by proportional parts for the epochs of the beginning, middle, and end of the trails, and of the point-images, the results being as follows:

Date.	Point Measured.	Epoch.	app. α_o	app. α_c	$\cos\delta$ $(\alpha_o - \alpha_c)$	app. δ_o	app. δ_c	$\delta_o - \delta_c$
1900		Gr. M. T.	h m s	s	"	° ′ "	"	"
June 28	(1) Beginning of trail.	June 28.747514	18 3 19.408	19.282	+1.75	−22 28 46.10	42.22	−3.88
	(2) Middle of trail.... = adopted position.	28.789181	17.542	17.512	+0.42	29 11.37	11.43	+0.06
	(3) End of trail	28.830848	15.676	15.742	−0.91	36.64	40.64	+4.00
June 29	(4) Beginning of trail.	June 29.760421	2 36.356	36.604	−3.43	40 35.97	34.86	−1.11
	(5) Middle of trail....	29.781255	35.857	35.731	+1.74	46.74	49.52	+2.78
	(6) End of trail	29.802088	35.358	34.858	+6.92	57.51	4.18	+6.67
June 30	(7) Beginning of trail.	June 30.772228	1 54.255	54.583	−4.53	52 28.91	27.66	−1.25
	(8) Middle of trail....	30.793061	53.695	53.725	−0.41	41.34	42.33	+0.99
	(9) End of trail	30.813895	53.135	52.867	+3.70	53.76	57.00	+3.24
	(10) Point image.....	30.829219	52.264	52.237	+0.37	53 8.29	7.77	−0.52
	(11) Adopted position.	30.817165	52.741	52.733	+0.11	52 59.31	59.29	−0.02
July 2	(12) Beginning of trail.	July 2.762156	0 34.214	34.280	−0.91	−23 15 45.20	49.38	+4.18
	(13) Middle of trail....	2.782990	33.759	33.457	+4.16	16 2.06	4.05	+1.99
	(14) End of trail	2.803823	33.304	32.634	+9.23	18.92	18.72	−0.20
	(15) Point image......	2.808916	32.303	32.432	−1.76	23.22	22.31	−0.91
	(16) Adopted position.	2.800272	32.788	32.774	+0.19	16.16	16.22	+0.06

By subtracting the residuals for the middle of each trail from those for the termini, we obtain the following residuals of the termini relatively to the middle of the trails:

Trail.	$\cos\delta\,\Delta\alpha$		$\Delta\delta$	
	Beginning.	End.	Beginning.	End.
June 28	+1".3	−1".3	−3".9	+3".9
June 29	−5 .2	+5 .2	−3 .9	+3 .9
June 30	−4 .1	+4 .1	−2 .2	+2 .2
July 2	−5 .1	+5 .1	+2 .2	−2 .2

Three of the four residuals in each coördinate correspond to the ideal conditions remarkably well, but one in each coördinate has the wrong sign. The inconsistency, in this respect, of the declinations of the termini of the trail on July 2d can not affect the resulting orbit to any great extent, as a point-image is available for the same date, to which twice the weight of the middle of the trail has been assigned. But the inconsistency of the right ascensions of June 28th, for which date we have no point-image at our disposal, will affect the accuracy of the orbit which has been derived, unless, accidentally, both termini are given too long in right ascension by approximately the same amount.

The magnitude of the asteroid was estimated by Mr. PALMER to be between 15 and 16½. Adopting 15.75, we find $g = 16.1$ and $m_0 = 18.1$. The asteroid is, therefore, the faintest so far observed.

According to our elements, the next opposition will take place 1902, January $4^d.3$, Gr. M. T. The magnitude of the asteroid at opposition will be 19.5 ± 0.75. It is, therefore, extremely unlikely that the asteroid will be observed at this opposition. An ephemeris computed by Miss HOBE, to aid astronomers who have the necessary equipment in searching for this interesting object, has been published in *Astronomische Nachrichten*, No. 3752.

PRELIMINARY ELEMENTS OF COMET 1900 III.

By R. H. CURTISS and C. G. DALL,
Berkeley Astronomical Department.

UNIVERSITY OF CALIFORNIA PUBLICATIONS.

PUBLICATIONS OF THE LICK OBSERVATORY, VOL. VII, PART 3.

PRELIMINARY ELEMENTS OF COMET 1900 III.

BY R. H. CURTISS AND C. G. DALL.

In order to furnish an example for the application to comets of the short method of determining orbits from three observations, contained in Part 1, we have, at the suggestion and under the direction of Professor LEUSCHNER, derived preliminary elements of comet 1900 III from the first three observations secured by Assistant Astronomer AITKEN at the Lick Observatory.

The observations are published in the *Astronomical Journal* No. 489, and were reduced to the beginning of the year in the usual manner.

As the numerical operations in the case of comet and asteroid orbits are practically identical, and as the details of the computation are given in full in connection with the asteroid example in Part 2, it has been deemed sufficient, in the present case, to reproduce only the principal features of the numerical work.

The arrangement of the calculations is the same as in Part 2.

I.

From the *American Ephemeris and Nautical Almanac* for 1900:

Date, Gr. M. T., 1901, Dec.	24.602199	26.627974	28.619826
X	+ 0.049167	+ 0.084519	+ 0.119168
Y	— 0.901044	— 0.898757	— 0.895386
Z	— 0.390889	— 0.389896	— 0.388435

$$m = -0.244052$$

X_0'	Y_0'	Z_0'
+ 1.013059	+ 0.082182	+ 0.035647

A	275° 22′ 20″.20	D	— 23° 21′ 36″.32
α	347 50 2 .40	log R	9.992697
$(A - \alpha)$	— 72 27 42 .20		

The second observation was not corrected for the parallax corresponding to the unit distance.

II.

Date, Gr. M. T.	α (1900.0).	δ (1900.0).
1900 Dec. 24.602199	$22^h\ 59^m\ 6^s.63$	$-22^\circ\ 44'\ 59''.4$
26.627974	23 11 20.16	-22 58 20.2
28.619826	23 23 14.97	-23 7 44.9

$\log \alpha_0'$	0.182958	$\log (\tan \delta)_0'$	9.051878_n
$\log \alpha_0''$	9.595824_n	$\log (\tan \delta)_0''$	0.027986

III.

$\log \varkappa$ 0.805116_n $\log \lambda$ 0.044684_n

ψ $65^\circ.83$ $\log \frac{1}{m}$ 9.129785 z_1 0.916 (Oppolzer, Table XIIIa)

IV.

$z_0 = 0.919233$

V.

$\log \rho_0$	9.956122	$\log \sigma_0'$	9.1958	$\log \rho_{,}$	9.9531
$\log \sigma_0$	9.9202	$\log \sigma_0''$	0.0532	$\log \rho_{,,,}$	9.9595

With the foregoing values of ρ we obtain the following positions, corrected for parallax and aberration:

Date, Gr. M. T.	α (1900.0).	δ (1900.0).
1900 Dec. 24.597018	$22^h\ 59^m\ 6^s.853$	$-22^\circ\ 44'\ 51''.14$
26.622757	23 11 20.456	-22 58 12.16
28.614568	23 23 15.231	-23 7 36.82

It will be seen later that another correction of the observations will not be necessary, the values of ρ, derived above, being sufficiently accurate for the determination of the final corrections for parallax and aberration.

$\log \alpha_0'$	0.182972	$\log (\tan \delta)_0'$	9.051928_n	$\log \varkappa$	0.798948_n
$\log \alpha_0''$	9.604040_n	$\log (\tan \delta)_0''$	0.028506	$\log \lambda$	0.046199_n
z_0	0.920807	$\log \sigma_0$	9.920986	$\log \sigma_0'$	9.168237

As indicated in Part 1, page 9, the second approximation of z_0, on the basis of these correct positions, might have been avoided by applying the parallax correction to the middle place only and then proceeding at once, by VI, to the computa-

tion of the heliocentric coördinates and velocities of the middle place, and to the determination of the residuals of the first and third places. In determining these residuals the observations should be corrected by means of the values of $\rho_{,}$ and $\rho_{,,,}$ obtained by VI.

VI.

ξ_0	η_0	ζ_0
+ 0.814936	— 0.175671	— 0.353352
x_0	y_0	z_0
+ 0.730414	+ 0.723086	+ 0.036544
ξ_0'	η_0'	ζ_0'
+ 0.411712	+ 1.210885	— 0.156393
x_0'	y_0'	z_0'
— 0.601340	+ 1.128703	— 0.192040

log r_0 0.012181 log r_0' 9.555908

log $\theta_{,,,}$	8.542164_n	log $f_{,}$	9.999755	log $g_{,}$	8.54208_n
log $\theta_{,}$	8.534829	log $f_{,,,}$	9.999768	log $g_{,,,}$	8.53475
$\xi_{,}$	+ 0.800124	$\eta_{,}$	— 0.217690	$\zeta_{,}$	— 0.347675
$\xi_{,,,}$	— 0.828596	$\eta_{,,,}$	— 0.134022	$\zeta_{,,,}$	— 0.358489
$(\alpha_{,})_c$	344° 46′ 47″.77	$(\delta_{,})_c$	— 22° 44′ 51″.27	$(\log \rho_{,})_c$	9.9538
$(\alpha_{,,,})_c$	350 48 43 .97	$(\delta_{,,,})_c$	— 23 7 37 .59	$(\log \rho_{,,,})_c$	9.9603

A comparison of these values with those derived in V shows that in the present case further correction of the observations for parallax and aberration is unnecessary.

O — C	I.	III.
$\Delta\alpha$	— 4″.97	+ 4″.50
$\Delta\delta$	+ 0 .13	+ 0 .77

VII.

log cos β = 9.6893

	I.	III.		I.	III.
log f_x	9.9542	9.9543	log A	8.7428	8.7133_n
log f_y	9.2862_n	9.2863_n	log B	8.5516	8.5667
log f_z	9.5910_n	9.5911_n	log C	8.5883_n	8.5744
$\alpha_{,,,} - \alpha_{,}$	6° 1′ 56″.20		log $\partial_{,}\alpha$	0.6616_n	0.6168
$\delta_{,,,} - \delta_{,}$	— 0 22 46 .32		log $\partial\delta$	9.1139	9.8865

$\log \partial\rho_0 \sin 1'' = 7.9198$

$\log \partial x_0' \sin 1''$	$\log \partial y_0' \sin 1''$	$\log \partial z_0' \sin 1''$
7.8388	8.0421	9.4286_n

Corrected Geocentric Co-ordinates.

$\log \xi_{,}$	9.907071	$\log \eta_{,}$	9.341794_n	$\log \zeta_{,}$	9.545090_n
$\log \xi_{,,,}$	9.922371	$\log \eta_{,,,}$	9.131145_n	$\log \zeta_{,,,}$	9.558498_n

$(\alpha_{,})_c$	344° 46′ 42″.65	$(\delta_{,})_c$	−22° 44′ 51″.36	$(\log \rho_{,})_c$	9.958
$(\alpha_{,,,})_c$	350 48 48 .42	$(\delta_{,,,})_c$	−23 7 36 .86	$(\log \rho_{,,,})_c$	9.964

O — C	I.	III.
$\cos\delta\,\Delta\alpha$	+ 0″.1	0″.0
$\Delta\delta$	+ 0 .2	0 .0

VIII.

$\log x$	$\log y$	$\log z$
9.867999	9.858220	8.522433
$\log x'$	$\log y'$	$\log z'$
9.774107_n	0.056801	9.289419_n

Elements.

M	3° 22′ 34″
$\log a$	0.60780
φ	50° 12′ 51″
μ	434″.81

	Equator 1900.0.	Equator 1901.0.	Ecliptic 1901.0.
i	9° 42′ 27″	9° 42′ 43″	30° 0′ 11″
Ω	235 13 38	235 15 31	196 5 39
ω	131 12 37	131 11 29	172 1 58

Constants to the Equator.

1900.0.	1901.0.
$x = r\,[9.99579] \sin(96°\ 2'\ 58'' + v)$	$x = r\,[9.99579] \sin(96°\ 3'\ 42'' + v)$
$y = r\,[9.99798] \sin(6\ 49\ 25 + v)$	$y = r\,[9.99798] \sin(66\ 50\ 11 + v)$
$z = r\,[9.22690] \sin(131\ 12\ 37 + v)$	$z = r\,[9.22710] \sin(131\ 11\ 29 + v)$

The positions computed for the three dates of observation by means of the constants to the equator agree with those computed in VII, so that the calculation of the elements is fully checked.

The method as applied to this case seems to have accomplished all that can be desired—an exact representation of the given observations with less expendi-

ture of time than would have been required by using the ordinary methods of computing preliminary parabolic elements. As judged from published results based on longer arcs, the foregoing elements bring out the characteristic features of the orbit better than would a parabolic orbit computed from the same observations.

By comparison with an observation taken at the Lick Observatory, 1901, February 15th, about seven weeks after the mean epoch of the given observations, we find the following corrections to the ephemeris:

$$O - C \qquad \cos\delta\,\Delta\alpha = +\,7^{s}.3 \qquad \Delta\delta = -\,55''$$

As the motion in the interval was about four hours in right ascension and eight degrees in declination, the method appears to be very satisfactory for ephemeris purposes.

TABLES FOR THE REDUCTION OF PHOTOGRAPHIC MEASURES

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UNIVERSITY OF CALIFORNIA PUBLICATIONS.

PUBLICATIONS OF THE LICK OBSERVATORY, VOL. VII, PART 4.

TABLES FOR THE REDUCTION OF PHOTOGRAPHIC MEASURES.*

BY BURT L. NEWKIRK.

The main obstacle to a wider application of the photographic method in the determination of celestial positions is the labor involved in the measurement and reduction of the star plates. Especially is this true in the case of plates made with the modern photographic telescopes which cover a field five degrees or more in radius. Special tables† have been prepared to facilitate the reduction of the measures made for the astrographic catalogue, and the Paris Observatory has recently issued a set of general tables‡ for use in the reduction of the *Eros* plates. These are applicable in the reduction of plates covering a field of two degrees square within 55° 20′ of the equator. Professor HAROLD JACOBY§ has just published tables for the reduction of plates covering the same area of the sky up to 75° of declination.

Before the appearance of Professor JACOBY'S tables, work had been begun at this Observatory upon a set of tables designed for the same purpose, but of wider application and based on different formulas. They apply to the reduction of measures of stars whose right ascensions differ from that of the center of the plate by ten degrees or less, and whose declinations differ from that of the center of the plate by five degrees or less, and hold good for any declination. These tables are intended primarily for use in the reduction of measures of plates made with the two portrait lenses which form part of the equipment of the Students' Observatory, but astrophotographic lenses covering a field from five to ten degrees in

*These tables, with text, were completed and prepared for publication in May, 1905. The section on refraction was rewritten in June, 1906.—BURT L. NEWKIRK, Lick Observatory, March 23, 1907.

†*Bull du Com. Int. Perm. pour l'exécution photographique de la Carte du Ciel.* Tome II, p. 303, Gauthier-Villars.

‡*Conférence Astrographique International de Juillet*, 1900. Circ. No. 10.

§*Contributions from the Observatory of Columbia College*, No. 23.

diameter are rapidly becoming popular and it is hoped that the tables will find a wider sphere of usefulness than is afforded by the needs of this Observatory. (See general notes, p. 63.)

The process of reducing measured rectangular coördinates to intervals of right ascension and declination involves a preliminary determination of the constants of the plate by a comparison of the measured coördinates of catalogue stars found on the plate, with computed coördinates found with the help of the catalogue positions. The plate constants are used to correct the measured coördinates of stars whose positions are unknown, giving the so-called "standard coördinates"* which are to be reduced to intervals of right ascension and declination. Corrections for refraction must be applied either to the rectangular coördinates or to the intervals of right ascension and declination. Aside from the solution of equations of condition for the determination of plate constants and the correction of measured coördinates, the whole work of derivation of celestial positions from measured rectangular coördinates and known star places consists in the transformation of intervals of right ascension and declination into rectangular coördinates, and its converse, and the introduction of corrections for refraction. It is these transformations and the introduction of the correction for refraction that the present tables as well as those prepared by Professor JACOBY and by the Paris Observatory are designed to facilitate.

Of the various plans for the reduction of photographic measures, three have acquired especial prominence, namely: those of Professor TURNER, the late PROSPER HENRY, and Professor JACOBY. Professor JACOBY uses expressions for the rectangular coördinates in the form of series of ascending powers of the intervals of right ascension and declination, involving coefficients which depend only upon the coördinates of the center of the plate, and are, therefore, constant for any one plate. Analogous series express the intervals of right ascension and declination in terms of the measured coördinates. It is, therefore, necessary to compute four sets of coefficients for each plate, with the help of which the transformations are to be accomplished. A good deal can be said in favor of this method in the case of a plate covering only four square degrees of the sky within seventy-five degrees of the equator, especially if many stars are to be measured on a single plate. It was developed with a view to the reduction of the measures made for the astrographic catalogue, and to these cases it is entirely applicable, provided the region photographed is not too near the pole. The method is inadequate, however, in the case of plates made with the modern wide angle photographic lenses covering one hundred or more square degrees of the sky† unless the region photographed is near the celestial equator, and even in the case of plates covering a smaller area, the method fails if the region photographed is

* *Monthly Notices*, Vol. LIV, p. 13, par. 7.

† See *Monthly Notices*, Vol. LXIV, p. 608.

near the pole. The method loses its especial advantage and is cumbersome in cases where the positions of only one or two of the objects on a plate are required. A further objection to a wider application of Professor JACOBY'S method is the lack of provision for any but the linear terms of the correction for refraction. It is necessary to take account of the terms of a higher order in the *Eros* work, even on plates covering a field of four square degrees, and they become appreciable where the standard of accuracy is less severe if the field is larger.

The formulas of M. HENRY have been thrown into a form convenient for the construction of general tables by M. LOEWY, and made the basis of his tables for the facilitation of the reduction of the *Eros* observations. Certain approximations have been introduced which would not be permissible if the tables were to be extended so as to suffice for the reduction of plates covering a field five degrees in radius.*

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*M. HENRY'S rigorous formulas are: (*Publ. Institut de France, Academie des Sciences, Bulletin du Comite international permanent pour l'exécution photographique de la carte du ciel*, Tome II, pp. 315, 318),

$$\tan x = \xi \cos y; \qquad \tan y = \eta;$$
$$\sin x \sec \delta = \sin(\alpha - \alpha_0); \qquad \sin \delta = \sin(\delta_0 + y)\cos x.$$

x and y are two auxiliary quantities defined by M. HENRY on page 306 of the publication above cited. Otherwise, the notation conforms to that adopted in this paper. (Cf. page 52.) I have left out of these equations the scale value factors, and the quantities are all expressed in units of radius.

The formulas used by M. LOEWY in the construction of his tables are (*Conférence astrographique international de Juillet*, No. 10, Gauthier-Villars):

$$\begin{aligned} x &= \xi - (\tan\xi - \xi + 2\,\xi\sin^2\tfrac{\eta}{2}) = \xi - d\xi \\ y &= \eta - (\tan\eta - \eta) = \eta - d\eta \\ \sin x &= \sin(\alpha - \alpha_0)\cos\delta \\ \sin\delta &= \sin(\delta_0 + y)\cos x \\ (\alpha - \alpha_0)\cos\delta &= x + t_a \\ \delta_0 + y &= \delta + t_d, \end{aligned}$$

where

$$t_a = (\alpha - \alpha_0 - x\sec\delta)\cos\delta$$
$$t_d = \frac{x\tan x\sin(\delta_0 + y - t_d)}{2\cos(\delta_0 + y - \frac{t_d}{2})}.$$

$y = \eta - (\tan y - y)$ being the rigorous form, the neglected terms in the second equation are immediately seen to be:

$$\Delta\, d\eta = \tan\eta - \eta - (\tan y - y) = \frac{\eta^3 - y^3}{3} + 2\frac{\eta^5 - y^5}{3\cdot 5} + \text{etc.}$$

The neglected terms of the first equation are also found by subtracting the given value of $d\xi$ from the rigorous one. The rigorous relation being

$$x = \xi - [(\tan x - x)\sec y + x(\sec y - 1)],$$

we have

$$\Delta\, d\xi = (\tan x - x)\sec y + x(\sec y - 1) - [\tan\xi - \xi + \xi(1 - \cos\eta)].$$

Substituting

$$\sec y = 1 + \frac{y^2}{2} + \frac{5\,y^4}{2^3.3} + \text{etc.},$$

$$\tan x - x = \frac{x^3}{3} + \frac{2\,x^5}{3\cdot 5} + \text{etc.},$$

and corresponding values of $\cos\eta$ and $\tan\xi - \xi$, and neglecting terms higher than the 5th order,

$$\Delta\, d\xi = \frac{x^3 - \xi^3}{3} + \frac{(x\,y^2 - \xi\,\eta^2)}{2} + \frac{x^3\,y^2}{2\cdot 3} + \frac{5\,x\,y^4 + \xi\,\eta^4}{24} + 2\frac{x^5 - \xi^5}{3\cdot 5} + \text{etc.}$$

When $\xi = \eta$ is equivalent to $+5^\circ$ the values of these terms are

$$\Delta\, d\eta = 0''.3$$
$$\Delta\, d\xi = 1''.0$$

The introduction of a sufficient number of the neglected terms to permit of the use of these equations in the construction of more extended tables would considerably increase the numerical work involved in their application. Another disadvantage of these formulas as compared with those of Professor TURNER is that one of the two auxiliary quantities, requiring the construction of a table of double entry, can be dispensed with without increasing the extent or materially augmenting the difficulty of manipulation of the other three tables.

Professor TURNER'S plan, making use of simple rigorous relations, is free from the objection to which Professor JACOBY'S are open when applied to the reduction of plates of large area near the pole (for which, I repeat, they were not intended), and possess an advantage over M. LOEWY'S adaptation of HENRY'S plan, in the simplicity of the formulas involved and in the saving of one auxiliary quantity as stated above. I have, therefore, adopted Professor TURNER'S formulas as the basis of these tables.

One of the essential points of difference between the plan for the reduction of photographic measures proposed by Professor TURNER and those proposed by HENRY, JACOBY, and others is in the number of the plate constants. TURNER recommends the use of six constants, whereas the others use only four. If only four constants are employed, it is necessary to compute a part of the linear terms of the refraction correction, which is unnecessary if the six-constant method be employed. In the tables designed to facilitate the computation of the refraction, I have included terms of higher orders and those terms of the first order which must be included if the four-constant solution is used. Any one computing the refraction corrections with the help of these tables, is, therefore, not committed to the use of the six-constant method.

Professor TURNER'S fundamental relations are*:

$$(1) \qquad \eta = \tan\,(d - \delta_0)$$
$$(2) \qquad \tan d = \tan \delta \sec\,(\alpha - \alpha_0)$$
$$(3) \qquad \xi \sec d = \tan\,(\alpha - \alpha_0) \sec\,(d - \delta_0)$$

where

α, δ = coördinates of a star,

ξ, η = rectangular coördinates of image of above, the unit of measurement being the focal length of the telescope,

α_0, δ_0 = coördinates of center of plate, *i. e.*, of the origin of rectangular coördinates,

d = the declination of the point of intersection of the hour circle passing through the center of the plate with a great circle perpendicular to it passing through the star.

The quantity $d - \delta_0$ is M. HENRY'S auxiliary quantity y. The introduction of M. HENRY'S second auxiliary quantity, x, is avoided by the choice of different trigonometric relations between the parts of the spherical triangle involved.

**Monthly Notices*, Vol. LIV, p. 13.

To facilitate the application of these formulas, I have prepared three tables. If the tabulated quantities be represented by A, B, and C, the following relations hold (α_0, δ_0 are supposed given in both cases):

Given α and δ,

$$d - \delta = B, \qquad \text{arg.} \begin{Bmatrix}(\alpha - \alpha_0)\\ \delta\end{Bmatrix}$$

$$k\,\eta = d - \delta_0 + A, \text{ arg. } (d - \delta_0)$$

$$k\,\xi \sec d = \alpha - \alpha_0 + C, \text{ arg.} \begin{Bmatrix}(\alpha - \alpha_0)\\ (d - \delta_0)\end{Bmatrix}$$

Given ξ and η,

$$d - \delta_0 = k\,\eta - A, \qquad \text{arg. } k\,\eta$$

$$\alpha - \alpha_0 = k\,\xi \sec d - C, \text{ arg.} \begin{Bmatrix}k\,\xi \sec d\\ (d - \delta_0)\end{Bmatrix}$$

$$d - \delta = B, \qquad \text{arg.} \begin{Bmatrix}(\alpha - \alpha_0)\\ d\end{Bmatrix}$$

$$\delta - \delta_0 = d - \delta_0 - B$$

in which all angles are expressed in degrees, ξ and η are expressed in units of the scale of the measuring apparatus, and k is a factor equal to the number of degrees in a radian divided by the focal length of the lens with which the plate was made, expressed in units of the scale of the measuring apparatus. $k\,\xi$ and $k\,\eta$ are, therefore, the measured coördinates, expressed in units of length equivalent to one degree of arc at the center of the plate. In the case of our 5-inch lens, the focal length is such that one unit of the scale of the measuring apparatus is equivalent to $0^\circ.1$. The factor k is, therefore, 0.1.

The three tables are based on TURNER'S fundamental formulas, numbers (1), (2), and (3), respectively. I insert here some special notes concerning each table with the modified forms of the fundamental relations which seemed best adapted to the computation of the tabulated quantities.

CONSTRUCTION OF TABLES. ARGUMENTS.

TABLE I.

If η be expressed in units of the scale of the measuring apparatus and k be taken as defined in the preceding paragraph, then:

$$k\,\eta = R \tan (d - \delta_0) = (d - \delta_0)^\circ + A,$$

where

$$R = 57^\circ.29578,$$

$$A = (d - \delta_0)^\circ \left\{ \frac{(d - \delta_0)^2}{3} + \frac{6}{5} \left[\frac{(d - \delta_0)^2}{3} \right]^2 + \text{etc.} \right\}$$

The quantity A is tabulated for each $0^\circ.1$ of the argument $d - \delta_0$ up to 3° and for every $0^\circ.01$ from 3° to 5° of the argument. The table may also be entered with the argument $k\,\eta = d - \delta_0 + A$. For values of the argument up to one degree, the quantity A is given in units of $0^\circ.000001 = 0''.0036$, but for larger values of the argument, the unit is ten times as large.

TABLE II.

Equation (2) yields the relation:

$$\tan (d - \delta) = \frac{\tan \delta}{1 + \tan^2 \delta \sec (\alpha - \alpha_0)} [\sec (\alpha - \alpha_0) - 1].$$

Table II gives $(d - \delta)^\circ = B$ with the arguments $(\alpha - \alpha_0)^\circ$ and δ, when $0 < \delta < 45^\circ$ and $0 < (\alpha - \alpha_0) < 10^\circ$. It is evidently unnecessary to tabulate for negative values of δ or $(\alpha - \alpha_0)$. B is independent of the sign of $(\alpha - \alpha_0)$ and changes sign with δ. It is convenient to remember that d is always numerically greater than δ. The table may also be entered with the arguments $(\alpha - \alpha_0)^\circ$ and d, it being remembered that the value $d_{,}$ corresponding to any tabulated value $B_{,}$ is $\delta_{,} + B_{,}$, where $\delta_{,}$ is the argument corresponding to the tabulated quantity $B_{,}$. See example* p. 67.

This table suffices also for the cases in which

$$\delta > 45^\circ$$
$$d > 45^\circ.$$

In equation (2), let

$$d = 90^\circ - q$$
$$\delta = 90^\circ - p,$$

whence

$$p - q = d - \delta.$$

Formula (2) then becomes:

$$\tan p = \tan q \sec (\alpha - \alpha_0),$$

which is of the same form as (2).

$(\alpha - \alpha_0)$ and $d > 45^\circ$ being given, enter Table II with the arguments $(\alpha - \alpha_0)$ and $q = (90^\circ - d)$ and obtain $p - q = d - \delta$. It is to be noted that q would have to be found in the column of arguments at the left of the page. The addition of

* For example the quantity $B = 0.01608$ corresponds to the arguments $\alpha - \alpha_0 = 2^\circ.8$ and $d = 14. + .01608$. See table.

a column of arguments at the right of each page permits the use of d directly as an argument.

When δ and $(\alpha - \alpha_0)$ are given, $\delta > 45°$, the arguments on the right are to be employed, it being remembered that the value $\delta_,$ corresponding to any tabulated value $B_,$ is $d_, - B_{,,}$ where $d_,$ is the argument corresponding to the tabulated quantity $B_,$. See example p. 67.

Tabulated quantities are expressed in units of $0°.000001 = 0''.0036$ for $0 < (\alpha - \alpha_0) < 2°$. For larger values of $(\alpha - \alpha_0)$, the unit is ten times as large.

The following working directions for the use of Table II summarize the above:

		GIVEN $(\alpha - \alpha_0)$, δ:
1.	$\delta < 45°$	Argument δ at left of page.
2.	$\delta > 45°$	Argument $(d - B) = \delta$, at right of page.
		GIVEN $(\alpha - \alpha_0)$, d:
3.	$d < 45°$	Argument $(\delta + B) = d$, at left of page.
4.	$d > 45°$	Argument d at right of page.

See examples pp. 67–71.

TABLE III.

If in equation (3) we develop $\tan(\alpha - \alpha_0)$ and $\sec(d - \delta_0)$ into power series in $(\alpha - \alpha_0)$ and $(d - \delta_0)$ respectively, and perform the multiplication, neglecting terms higher than the seventh order in $(\alpha - \alpha_0)$ and those higher than the fifth order in $(d - \delta_0)$ we have:

$$k\,\xi \sec d = (\alpha - \alpha_0)° + C,$$

where

$$C = (\alpha - \alpha_0)° \, [L + M + N]$$

$$L = \frac{(\alpha - \alpha_0)^2}{3} + \frac{(d - \delta_0)^2}{2}$$

$$M = \frac{5}{6}\left[\frac{(d - \delta_0)^2}{2}\right]^2 + \frac{6}{5}\left[\frac{(\alpha - \alpha_0)^2}{3}\right]^2 + \frac{(\alpha - \alpha_0)^2}{3} \cdot \frac{(d - \delta_0)^2}{2}$$

$$N = \frac{17}{14} \cdot \frac{6}{5}\left[\frac{(\alpha - \alpha_0)^2}{3}\right]^2 \frac{(\alpha - \alpha_0)^2}{3} + \frac{6}{5}\left[\frac{(\alpha - \alpha_0)^2}{3}\right]^2 \frac{(d - \delta_0)^2}{2} + \frac{(\alpha - \alpha_0)^2}{3} \cdot \frac{5}{6}\left[\frac{(d - \delta_0)^2}{2}\right]^2.$$

N amounts to less than $0''.1$ for $(\alpha - \alpha_0) = 10°$, $(d - \delta_0) = 5°$.

C is tabulated with the arguments $(\alpha - \alpha_0)$ and $(d - \delta_0)$ for

$$0 < (\alpha - \alpha_0) < 10°$$
$$0 < (d - \delta_0) < 5°$$

C changes sign with $(\alpha - \alpha_0)$ and is independent of the sign of $(d - \delta_0)$.

$k\,\xi \sec d$ is always numerically greater than $(\alpha - \alpha_0)$.

This table may be entered with the argument $k\,\xi \sec d$, it being remembered that the argument $(k\,\xi \sec d)_,$ corresponding to any tabulated value $C_,$ is $(\alpha - \alpha_0)_, + C_,$ where $(\alpha - \alpha_0)_,$ is the argument corresponding to the tabulated quantity $C_,$.* The quantity C is expressed in units of $0°.000001 = 0''.0036$ for

$$0 < (\alpha - \alpha_0) < 2°$$
$$0 < (d - \delta_0) < 1°.$$

For larger values of the arguments, the unit is ten times as large.

*The quantity $C = 0.05928$ corresponds to the arguments $k\,\xi \sec \delta = 7.9 + 0.05928$ and $d - \delta_0 = 2.7$. See example p. 68.

REFRACTION.

The correction of the measured coördinates for differential refraction has been the subject of considerable discussion. The larger part of the correction is a linear function of the measured coördinates, and is, therefore, provided for in Professor TURNER'S six-constant method of solution. Some authorities prefer the four-constant solution, with the necessity for computing some of the linear terms of the refraction correction. That part of the refraction which depends on higher powers of the measured coördinates* becomes quite appreciable in the case of plates made with the modern wide angle telescopes, where the field of good definition contains a hundred square degrees or more.

I have taken the formulas produced by Professor TURNER† as the basis of Tables IV–XI. They are:

$$\begin{aligned} \Delta x &= (X - x)\, t \\ \Delta y &= (Y - y)\, t \\ t &= \mu \frac{1 + x^2 + y^2}{1 + xX + yY}, \end{aligned} \qquad (1)$$

where

$\left.\begin{matrix} X, Y \\ x, y \end{matrix}\right\}$ = Coördinates of zenith and star projected on "standard" plate.

$\left.\begin{matrix} \Delta x \\ \Delta y \end{matrix}\right\}$ = Projection on plate of total displacement of star due to refraction.

μ = Constant of refraction.

It is necessary to insert some remarks regarding the details of the application of these relations before the construction of the tables is taken up. This is all the more desirable since certain difference of opinion§ seems to prevail regarding this matter. The situation can perhaps be most easily made clear by outlining the following rigorously correct process.

Suppose the center of the plate (that is, the point on the plate where a radius of the projected sphere meets it perpendicularly) has been ascertained, plotted on a star chart by means of the neighboring stars on the plate, and its right ascension and declination (α_0, δ_0) read off accurately from the chart. Let $\Delta\alpha_0$, $\Delta\delta_0$ represent the components of the displacement of the point α_0, δ_0 due to refraction. The spherical coördinates of the "center of the plate" are then:‡

$$\begin{aligned} &\alpha_0 + \Delta\alpha_0 \\ &\delta_0 + \Delta\delta_0 \end{aligned}$$

and these values must be used in computing X and Y according to the formulas given on p. 60.

*See WILSON. *Popular Astronomy*, Vol. XII, p. 159. † *Monthly Notices*, Vol. LVII, p. 136.

§ Dissertation of WALTER ZURHELLEN. Bonn 1904, p. 33–48.

‡ *Cf.* ÖSTEN BERGSTRAND. *Untersuchungen über das Doppelstern system 61 Cygni.* Upsala 1905.

The refraction corrections must first be computed for the catalogue stars so as to introduce the effect of refraction into the standard coördinates. For this purpose we should employ for x and y in the above formulas, the true standard coördinates on a plate tangent at the point $\alpha_0 + \varDelta\,\alpha_0$, $\delta_0 + \varDelta\,\delta_0$, and the appropriate refraction constant: $\mu = \alpha' \beta'^{A'} \gamma^{\lambda'}$. (CHAUVENET, *Sph. Astr.*, Vol II, Table II, Col. B.) The corrections are the total displacements due to refraction. Applying these corrections we should have what may be called *the apparent standard coördinates on a plate tangent to the sphere at the point* $\alpha_0 + \varDelta\,\alpha_0$, $\delta_0 + \varDelta\,\delta_0$. The constants of the plate can then be determined in the ordinary manner by comparison of these computed with the measured coördinates.

The constants of the plate having been determined in this way and the apparent standard coördinates of the unknown stars having been derived with their help, the refraction corrections would be computed for these stars. This would be accomplished by means of TURNER'S formulas given above, using the values of X and Y previously employed, the "apparent standard coördinates" x and y and the appropriate refraction constant: $\mu = \alpha \beta^{A} \gamma^{\lambda}$ (CHAUVENET, Table II, Col. A). The true standard coördinates on a plate tangent at $\alpha_0 + \varDelta\,\alpha_0$, $\delta_0 + \varDelta\,\delta_0$ would be obtained by applying these corrections to the apparent standard coördinates.

It is to be noticed that the quantities x, y, defined above as "the coördinates of the star projected on the 'standard' plate," may designate either the true or the apparent coördinates, depending on which value of the constant μ is employed.

It is to be noted further that the constant of refraction μ is in every case the one corresponding to the mean or true zenith distance of the particular star for which the correction is being computed.

This process may be shortened by certain modifications, involving the omission of some terms of the refraction correction, and the use of tables. These modifications and approximations are indicated in the following paragraphs.

Since an important simplification is obtained by omitting certain large terms of the refraction correction which may be included in the plate constants, it is necessary to consider in this connection the process of determining these constants. The tables have been so constructed as to be available for the computation of the refraction corrections when the four-constant method is employed, and consequently are equally available for the six-constant method, and for cases in which further unknowns are introduced to provide for other corrections.

Let

x, y = True rectangular (standard) coördinates of a star on a plate tangent to the celestial sphere at the point $\alpha_0 + \varDelta\,\alpha_0$, $\delta_0 + \varDelta\,\delta_0$.

x', y' = Measured coördinates.

$\mu = \mu_0 + \varDelta\mu = \alpha' \beta'^{A'} \gamma^{\lambda'}$ = The constant of refraction for the star whose measured coördinates are x', y'.

$\mu_0 = \alpha' \beta'^{A'} \gamma^{\lambda'}$ for point α_0, δ_0

$= \alpha\ \beta^{A}\ \gamma^{\lambda}$ for point $\alpha + \varDelta\alpha_0$, $\delta + \varDelta\delta_0$.

The four-constant formulas for the determination and correction of errors of orientation, scale value and assumed center are:

(2)
$$ax' + by' + c = x - x' + \Delta x$$
$$-bx' + ay' + d = y - y' + \Delta y,$$

where

(3)
$$\Delta x = \mu (X - x) \frac{1 + x^2 + y^2}{1 + Xx + Yy}$$
$$= \mu (X - x - xX^2 - yXY + \delta x),$$
$$\Delta y = \mu (Y - y) \frac{1 + x^2 + y^2}{1 + xX + yY}$$
$$= \mu (Y - y - yY^2 - xXY + \delta y),$$

the expressions in series of powers of x and y being obtained by developing the fraction $\left(\frac{1 + x^2 + y^2}{1 + Xx + Yy}\right)$ in powers of $(Xx + Yy)$ and multiplying out. δx and δy represent terms involving second and higher powers of x and y.

Adding $\mu_\circ x'$ to both sides of the first of equations (2) and $\mu_\circ y'$ to both sides of the second, substituting values of Δx and Δy from (3), putting $\mu = \mu_\circ + \Delta\mu$, and substituting for $x - x'$ and $y - y'$ in the coefficients of $\mu_\circ$ the values

(4)
$$x - x' = ax' + by' + c - \Delta x$$
$$y - y' = -bx' + ay' + d - \Delta y,$$

we have

(5)
$$a'x' + b'y' + c' = (x + r_x) - x'$$
$$-b'x' + a'y' + d' = (y + r_y) - y',$$

where the primed letters indicate new constants* and

(6)
$$r_x = \mu (-xX^2 - yXY + \delta x) + \Delta\mu (X - x) + \mu_\circ (\Delta x - \mu_\circ X)$$
$$r_y = \mu (-yY^2 - xXY + \delta y) + \Delta\mu (Y - y) + \mu_\circ (\Delta y - \mu_\circ Y).$$

The values of the expressions:

(7)
$$-xX^2 - yXY + \delta x$$
$$-yY^2 - xXY + \delta y$$

are obtained with the help of the tables. The last term of each of equations (6) is negligible, even in extreme cases.

Having computed the right-hand members of equations (5) and solved for the plate constants, we determine $x + r_x$ and $y + r_y$ for each of the unknown stars by means of the equations (5):

$$x + r_x = x' + a'x' + b'y' + c'$$
$$y + r_y = y' + b'x' + a'y' + d',$$

and it remains to determine x and y by computing and applying the corrections r_x and r_y. This can be done most simply by means of the relations (6), using the quantities $(x + r_x)$ and $(y + r_y)$ as approximate values of the quantities x and y, and making a second approximation if necessary. If r_x and r_y are

*$a' = a + \mu_\circ a$; $b' = b + \mu_\circ b$; $c' = c + \mu_\circ (X + c) - \mu_\circ^2 \Delta x$; $d' = d + \mu_\circ (Y + d) - \mu_\circ^2 \Delta y$.

computed in this way the values of μ corresponding to *true* zenith distance are to be employed.

When it becomes necessary to compute a number of values of μ for the same plate the approximate true zenith distances of points whose standard coördinates (referred to the center $\alpha_0 + \Delta\alpha_0$, $\delta_0 + \Delta\delta_0$) are x and y may be obtained with the help of the following relation:*

$$z = z_0 - \frac{Xx + Yy}{\tan z_0}$$

which gives a good approximation, provided the zenith distance is large. z_0 is the true zenith distance of the point $\alpha_0 + \Delta\alpha_0$, $\delta_0 + \Delta\delta_0$.

The process just outlined differs from the rigorous one in the transfer of the large terms $\mu_0\,(X - x)$, $\mu_0\,(Y - y)$ from the refraction correction to the plate constants, which reduces the size of the computed corrections, and renders possible the further modification involving the use of the constant μ (eqs. 6) obtained with the argument true zenith distance in computing the corrections for the unknown, as well as for the fiducial stars. This principle may be carried further if a six-constant plate solution is employed, all of the linear terms of the refraction being omitted. The non-linear terms are given by the expressions

$$r_x + \mu_0\,X\,(Xx + Yy)$$
$$r_y + \mu_0\,Y\,(Xx + Yy).$$

If the refractions are taken in this form, a second approximation will seldom if ever be necessary in computing the refractions for the unknown stars.

To put the relations (1) in a form suitable to the construction of tables I have developed the right hand members in a series of powers of the expression $(Xx + Yy)$.

$$\Delta x = \mu\left[(X-x) + (X-x)\,\{D + (x^2+y^2)\,(1+D)\}\right]$$
$$\Delta y = \mu\left[(Y-y) + (Y-y)\,\{D + (x^2+y^2)\,(1+D)\}\right],$$

where

$$D = -(Xx + Yy) + (Xx + Yy)^2 - (Xx + Yy)^3 + \text{etc.}$$
$$= \frac{1}{1 + Xx + Yy} - 1.$$

Subtracting the terms $\mu_0(X - x)$ and $\mu_0(Y - y)$, which, save for the negligible terms $\mu_0(\Delta x - \mu_0 X)$ and $\mu_0(\Delta y - \mu_0 Y)$, are included in the four plate constants, we have:

$$r_x = \mu\,(X - x)\,[D + (x^2+y^2)\,(1+D)] + \Delta\mu\,(X - x)$$
$$r_y = \mu\,(Y - y)\,[D + (x^2+y^2)\,(1+D)] + \Delta\mu\,(Y - y).$$

$$\mu = \frac{1.0154}{3600}\,\alpha'\beta^{A'}\gamma^{\lambda'} \quad \text{or}\ \mu = \frac{1.0154}{3600}\,\alpha\beta^{A}\gamma^{\lambda}. \quad \text{(Expressed in degrees)}$$

$$\log\frac{1.0154}{3600} = 6.4503.$$

1.0154 is the ratio of the photographic to the visual constant.† The focal length of

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* See note p. 61.

† *Bull. du Com. Int. Perm.* T. I., p. 465.

the telescope is the unit for X, Y, x, y. The quantities r_x and r_y as obtained from the above formulas are expressed in decimals of a degree and are to be added algebraically to the computed standard coördinates in order to *introduce* the effect of refraction. The effect of refraction is eliminated from the measured coördinates by applying the above corrections with the opposite sign.

Tables IV and V give values of X and Y, respectively, as functions of the hour angle and declination of the center of the plate for

$$-45^\circ < (s - \alpha) < +45^\circ$$
$$-40 < \delta < +90^\circ.$$

They are constructed for the latitude of Berkeley (37° 52'.4) by means of the formulas*

$$\tan \triangle = \tan \varphi \sec (s - \alpha)$$
$$Y = \tan (\triangle - \delta)$$
$$X = \frac{\tan (s - \alpha)}{\cos (\triangle - \delta)} \cos \triangle,$$

where

φ = Latitude of the Students' Observatory = 37° 52'.4,

s = Sidereal time of exposure,

α, δ = Coördinates of center of plate.

It is stated above that $\alpha_0 + \varDelta \alpha_0$ and $\delta_0 + \varDelta \delta_0$ are to be used as arguments in taking out values of X and Y. μ_0 is to be computed with the argument true zenith distance of the point α_0, δ_0. X, Y and μ_0 may be conveniently computed as follows: Enter tables for X and Y with arguments α_0, δ_0. Obtain Z from Table XI. Compute μ_0.

$$\varDelta \alpha_0 = \mu_0 X \sec \delta$$
$$\varDelta \delta_0 = \mu_0 Y$$

Enter tables again for final values of X and Y with arguments $\alpha_0 + \varDelta \alpha_0$, $\delta_0 + \varDelta \delta_0$. In very extreme cases (see example p. 72) the zenith distance of the center of the plate and the coördinates X and Y must be computed by means of the rigorous formulas.

Tables VI and VII are to facilitate the computation of X and Y when the center of the plate is more than three hours from the meridian. $\triangle$ is independent of the sign of $(s - \alpha)$, always $\geqq \varphi$, and $> 90^\circ$ when $(s - \alpha)$ is numerically greater than 90°. Y will therefore always be positive when $\delta < \varphi$ or $(s - \alpha)$ numerically $> 90^\circ$. $X \cos (\triangle - \delta)$ has the same sign as $(s - \alpha)$, and since $(\triangle - \delta)$ is always numerically less than 90° (Y would be ∞ and the center of the plate on the horizon for $\triangle - \delta = 90^\circ$) X has the same sign as the hour angle of the center of the plate, being positive when the center of the plate is west of the meridian.

* TURNER. *Monthly Notices*, Vol. LIV, p. 19.

ABERRATION.

TURNER'S formulas* for the introduction of the correction for differential aberration give only terms of the first power in the measured coördinates. The complete expressions may be obtained in a manner analogous to the one employed by him, *Monthly Notices*, Vol. LVII, p. 134, ff., in developing expressions for the refraction corrections. If W represent the apex of the Earth's way, S a star, and C the center of the sphere which is projected on the plate, the direction cosines, with respect to a rectangular coördinate system having its origin in C, of lines connecting C with S and W are proportional to $x, y, 1$, and $X, Y, 1$, respectively, x, y, X, Y being coördinates of the projections of the points S and W on the plate. We have then

$$\cos SCW = \frac{1 + xX + yY}{\sqrt{(1 + x^2 + y^2)(1 + X^2 + Y^2)}}$$

$$\sin SCW = \sqrt{\frac{(X-x)^2 + (Y-y)^2 + (xY - yX)^2}{(1 + x^2 + y^2)(1 + X^2 + Y^2)}} .$$

The coördinates of S' which represents the position of the star S as displaced by aberration are

$$x + \Delta x = x + t(X - x)$$
$$y + \Delta y = y + t(Y - y).$$

Similarly,

$$\sin SCS' = t \sqrt{\frac{(X-x)^2 + (Y-y)^2 + (xY - yX)^2}{(1 + x^2 + y^2)\,[1 + (x + \Delta x)^2 + (y + \Delta y)^2]}} .$$

But,

$$\sin SCS' = \beta \sin SCW,$$

where β is the constant of aberration.

Therefore $$t = +\beta \sqrt{\frac{1 + (x + \Delta x)^2 + (y + \Delta y)^2}{1 + X^2 + Y^2}}$$

$$= \frac{+\beta}{\sqrt{1 + X^2 + Y^2}}\,[1 + \tfrac{1}{2}(x^2 + y^2) + \ldots\ldots],$$

and $$\Delta x = \frac{+\beta}{\sqrt{1 + X^2 + Y^2}}\,[X - x + \tfrac{1}{2}(X - x)(x^2 + y^2) + \ldots\ldots]$$

$$\Delta y = \frac{+\beta}{\sqrt{1 + X^2 + Y^2}}\,[Y - y + \tfrac{1}{2}(Y - y)(x^2 + y^2) + \ldots].$$

The first term within the bracket in each case yields the displacement of the center of the plate due to aberration and should be omitted in computing differential aberration. The quantities $\frac{X}{\sqrt{1 + X^2 + Y^2}}$ and $\frac{Y}{\sqrt{1 + X^2 + Y^2}}$ can never exceed unity. The factor

$$\tfrac{1}{2}\beta(x^2 + y^2) = 20''\,(\tfrac{5}{57})^2 = 0''.15$$

when $x = y$ is equivalent to $5°$. The terms of the first degree in x and y are provided for in either method of determination of the plate constants.

* *Monthly Notices*, Vol. LIV, p. 20.

Abbreviated tables for these transformations are given here. (See last page.) The multiplication of the measured coördinates by the factor k can be accomplished with the help of a Rechentafel, three places being sufficiently accurate for the preliminary scale value. A six-place number can be accurately multiplied by a three-place factor by entering the table twice with three digits of the multiplicand each time.

The number of decimals to which the tabulated quantities are given in Tables I–III makes it possible to reduce measures of stars within one degree of the center of the plate in declination and two degrees in right ascension, with an accuracy of about 0″.01. (One unit of the last decimal place of the tabulated quantities is equivalent to 0″.0036.) Throughout the remainder of the tables, one decimal place less is given. The error of interpolation due to second differences amounts to less than two units of the last place, except in parts of Table II. In a portion of this table, horizontal second differences are appreciable, and in another portion vertical second differences should be taken into account.* In each case, the second differences are nearly constant along a horizontal line across the page so that it is possible to use one approximate value for the whole row. One half this value has been tabulated in a column at the left of the page, headed "$\frac{h}{2}$" or, "$\frac{v}{2}$". The formula to be used in the interpolation is:

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\frac{f''(a)}{2}\right]$$

Where $f'(a)$ is the average of the tabulated first differences preceding and following the number from which the interpolation is made, and $\frac{f''(a)}{2}$ is the quantity given in the column at the left.

The tables of proportional parts for the interpolations are on the last page.

These tables have been computed to one decimal place more than is given, where this was necessary to insure the accuracy of all tabulated values within one unit of the last place. The checking has been done in part by duplicate computation, but mainly by comparison of differences. The methods of computation and checking are not such as to exclude in all cases errors of one unit in the last decimal place. This is evidently immaterial since, in Tables I–III one unit of the last place is equivalent to 0″.0036, or 0″.036, in portions of the tables where the accuracy aimed at is 0″.01, and 0″.1, respectively, and in the refraction tables a sufficient number of decimal places is given so that the last place of the computed refraction correction may be dropped without committing an error greater than the uncertainties of the value due to varying states of the atmosphere.

NOTE ON THE ASSUMED CENTER OF THE PLATE. In developing formulas for the reduction of measured rectangular coördinates to standard, and their transformation into intervals of right ascension and declination, it is generally assumed that the right ascension and declination of the point in the sky corresponding to the point at which the optic axis of the lens meets the plate are

*In a part of the table on page 81 both horizontal and vertical second differences are appreciable.

known, and that the plane of the plate is perpendicular to the optic axis. It is important to know how accurately these conditions must be fulfilled in the case of plates covering fields several degrees in diameter.

It is to be noted that errors introduced by the non-fulfillment of these two conditions can not be treated separately, if the following fundamental condition is fulfilled. It is supposed, namely, that the angle between the axis of two cones of light due to two stars, upon leaving the lens, is equal to the angle between the two bundles of parallel rays coming to the lens from the two stars. If this condition is fulfilled the impression on the plate is the central projection of a spherical surface, the center of projection being the optical center of the lens, the radius of the sphere being its focal length. If this is the case the optic axis is of no more importance than any other radius of the sphere except that the field of good definition is more symmetrically situated about it. If the plate is not perpendicular to the optic axis we should choose as the "center of the plate" the point where a radius of the sphere which is perpendicular to the plate meets it. The formulas for the transformation from rectangular coördinates to intervals of right ascension and declination are applicable to coördinates measured with this point as origin. It is sufficient to know the coördinates, α, δ, corresponding to the point where a radius of the sphere perpendicular to the plate meets it, this single condition replacing the two previously mentioned.

In the reduction of a set of measures this point is known only approximately and the coördinates, α, δ, of the "center of the plate" are taken somewhat arbitrarily. Professor JACOBY has developed expressions for the corrections to the measured coördinates due to error in the assumed value of the coördinates of the center of the plate. The terms of first degree in the measured coördinates are provided for along with the other linear corrections in the six-constant method of solution. The terms of second degree are:

Corrections to the x coördinate: $(x^2 \cos \delta)\, \varDelta\, \alpha + xy\, \varDelta\, \delta$

Corrections to the y coördinate: $(xy \cos \delta)\, \varDelta\, \alpha + y^2\, \varDelta\, \delta$

where the coördinates of the assumed and true center of the plate are α, δ and $\alpha + \varDelta\, \alpha$, $\delta + \varDelta\, \delta$, respectively.

In the case of plates made with portrait lenses an error of one centimeter in locating the "center," as defined above, might easily occur unless special precautions were taken to secure a more accurate value. This would be equivalent to something like a degree of arc. To gain an idea of the error thus introduced into the computed coördinate of a catalogue star on the plate, take, as a special case, a star, declination zero, five degrees from the center of the plate along the x axis, and suppose $\cos \delta\, \varDelta\, \alpha = 1^\circ$, $\varDelta\, \delta = 0^\circ$.

Then

$$x^2 \cos \delta\, \varDelta\, \alpha + xy\, \varDelta\, \delta = 5 \cdot \frac{5}{57} \cdot \frac{1}{57} = 0^\circ.0076 = 27''.$$

For stars at a given distance from the center of the plate the effect of the second order terms is proportional to the error in the assumed values of the coördinates of the center.

It follows from the above that in reducing measured to standard coördinates over a field some degrees in diameter the right ascension and declination of the "center of the plate" (defined as the point of intersection of the plate with a perpendicular to it from the center of projection) must be known with considerable accuracy, or, failing in this, the relations between the measured and standard coördinates must be written:

$$\xi = ax + by + c + mx^2 + nxy$$
$$\eta = dx + ey + f + ny^2 + mxy,$$

where

$$m = \cos\delta\,\Delta\alpha,$$
$$n = \Delta\delta\ .\ .$$

The values of the coördinates of the true center of the plate are then:

$$\alpha + \Delta\alpha = \alpha + m\sec\delta$$
$$\delta + \Delta\delta = \delta + n.$$

A considerable proportion of the routine work involved in the preparation of these tables was done by computers at the expense of the Students' Observatory of the University of California. For this assistance, as well as for encouragement and valuable suggestions, I am indebted to the Director, Professor A. O. Leuschner.

SUMMARY OF DIRECTIONS FOR USE OF TABLES I, II, AND III.

A. Given α_0, δ_0, α, δ. Required $k\xi$, $k\eta$.

Express δ_0, δ and $(\alpha - \alpha_0)$ in decimals of a degree.

If $\delta < 45^\circ$, with arguments, $(\alpha - \alpha_0)$ and δ, take B from Table II.

If $\delta > 45^\circ$ enter Table II with argument $\delta = d - B$, d being found in the column at the right of the page. Then

$$d = \delta + B.$$

With argument $(d - \delta_0)$ take A from Table I. Then

$$k\eta = (d - \delta_0) + A.$$

With argument $(\alpha - \alpha_0)$ and $(d - \delta_0)$ take C from Table III. Then

$$k\xi \sec d = (\alpha - \alpha_0) + C.$$

B. Given α_0, δ_0, $k\xi$, $k\eta$. Required $(\alpha - \alpha_0)$, δ.

With argument $k\eta$ take A from Table I. Then

$$d - \delta_0 = k\eta - A.$$

With arguments $(d - \delta_0)$ and $k\xi \sec d$, take C from Table III. Then

$$(\alpha - \alpha_0) = k\xi \sec d - C.$$

If $d < 45^\circ$, with argument $(\alpha - \alpha_0)$ and $d = \delta + \mathrm{B}$, take B from Table II. Then

$$\delta = d - B.$$

If $d > 45^\circ$, take out B with argument $(\alpha - \alpha_0)$ and d in column at right.

I insert here the details of the interpolation in the case under A when $\delta > 45^\circ$ and in the case under B when $d < 45^\circ$. Also in the use of Table III under B.

DETAILS OF INTERPOLATION.

TABLE II.

$(\alpha - \alpha_0) + 7^\circ.91210$
$\delta + 78.88250$

Arg. $\left\{\begin{array}{l}(\alpha - \alpha_0) = 7^\circ.9 \\ \delta = 78.89721\end{array}\right\}$		0.10279
$\frac{\partial B}{\partial \delta}$	$(-.01471)$	$+13$
$\frac{\partial B}{\partial (\alpha - \alpha_0)}$	$(+.1210)$	$+32$
B		0.10324

$(\alpha - \alpha_0) + 2^\circ.81400$
$d + 14.15090$

Arg. $\left\{\begin{array}{l}(\alpha - \alpha_0) = 2^\circ.8 \\ d = 14.01608\end{array}\right\}$		0.01608
$\frac{\partial B}{\partial \delta}$	$(+.13482)$	$+14$
$\frac{\partial B}{\partial (\alpha - \alpha_0)}$	$(+.140)$	$+16$
B		$+0.01638$

TABLE III.

$k\xi$ sec d 7.97150

$(d - \delta_0)$ 2°.68570

Arg. $\begin{cases} k\xi \sec\delta = 7.95928 \\ (d - \delta_0) = 2.7 \end{cases}$		.05928
$\partial \frac{\partial C}{(k\xi \sec\delta)}$ (.1222)	. . . .	+ 25
$\partial \frac{\partial C}{(d - \delta_0)}$ (—.143)	. . . .	— 9
C	. . . .	0.05944

EXAMPLES.

The following are the data of observation of

PLATE No. 46.

Exposed 1904, Nov. 11.

	h	m	s
Sidereal time at beginning of exposure	23	21	0
Sidereal time at end of exposure - -	0	51	0

Coördinates of center of plate $\begin{cases} \alpha & 3^h\ 44^m\ 50^s. \\ \delta & +10^\circ\ 28'\ 30'' \end{cases}$

Approximate scale value 1 mm = 0°.1.

Latitude of Berkeley = 37° 52′.

Star.			M. E. 1875.					
No.	AG No.	Mag.	α			δ		
			h	m	s	°	′	″
1	1062	7.4	3	32	19.94	+13	29	7.1
2	1136	7.6	3	47	35.13	+13	24	5.8
3	1183	8.5	3	56	5.36	+14	8	4.6
4	1345	8.6	3	33	2.62	+ 6	3	25.7
5	1431	8.1	3	46	29.93	+ 7	23	57.0
6	1511	8.3	4	0	9.95	+ 7	51	9.5

COMPUTATION OF $k\xi$ AND $k\eta$ FROM CATALOGUE POSITIONS.

$k = .1$

Star.	1	2	3	4	5	6
$(\alpha - a)$	$-12^m\ 30^s.06$	$+\ 2^m\ 45^s.13$	$+11^m\ 15^s.36$	$-11^m\ 47^s.38$	$+\ 1^m\ 39^s.93$	$+15^m\ 19^s.95$
	$-\ 3^\circ.1253$	$+\ {}^\circ.6880$	$+\ 2^\circ.8140$	$-\ 2^\circ.9474$	$+\ {}^\circ.4164$	$+\ 3^\circ.8331$
δ	$+\ 13^\circ.4853$	$+\ 13^\circ.4016$	$+\ 14^\circ.1345$	$+\ 6^\circ.0571$	$+\ 7^\circ.3992$	$+\ 7^\circ.8526$
B (Table II) $= d - \delta$.	+ 193	+ 9	+ 164	+ 80	+ 2	+ 174
d	+ 13 .5046	+ 13 .4025	+ 14 .1509	+ 6 .0571	+ 7 .3994	+ 7 .8700
$d - \delta_0$ . .	+ 3 .0296	+ 2 .9275	+ 3 .6759	− 4 .4099	− 3 .0756	− 2 .6050
A (Table I) .	+ 28	+ 25	+ 50	− 85	− 30	− 18
C (Table III) .	− 74	+ 10	+ 81	− 114	+ 6	+ 97
log $k\xi$ sec d .	0.49592	9.83820	0.45057	0.47111	9.62014	0.58465
log cos d . .	9.98782	9.98801	9.98662	9.99756	9.99637	9.99589
log $k\xi$ . .	0.48374	9.82621	0.43719	0.46827	9.61651	0.58054
$k\xi$ . . .	− 3.0461	+ 0.6702	+ 2.7365	− 2.9422	+ 0.4135	+ 3.8066
$k\eta$ . . .	+ 3.0324	+ 2.9300	+ 3.6809	− 4.4184	− 3.0786	− 2.6068

I have performed this transformation also by means of Professor JACOBY'S formulas and obtain the same values to within two units of the last decimal place.

COMPUTATION OF THE CORRECTIONS FOR REFRACTION.

Table IV, $X = -\ 1.156$
Table V, $Y = +\ .943$
$\mu = 0.0164$

Star.	1	2	3	4	5	6
x Table VIII . . .	−0.0532	+0.0117	+0.0478	−0.0513	+0.0072	+0.0667
y Table VIII . . .	+ 528	+ 512	+ 642	− 771	− 537	− 455
$x^2 + y^2$ Table IX . . .	56	28	64	86	29	65
$\mu\,(X - x)$	− 180	− 192	− 197	− 180	− 190	− 200
$\mu\,(Y - y)$	+ 146	+ 146	+ 144	− 167	+ 164	+ 162
Xx	+ 615	− 135	− 553	+ 593	− 83	− 771
Yy	+ 493	+ 478	+ 599	− 720	− 501	− 425
Σ	+ .1108	+ .0343	+ .0046	− .0127	− .0584	− .1196
D	−0.0999	−0.0331	−0.0046	+0.0129	+0.0620	+0.1358
$(x^2 + y^2)\,(1 + D)$ . . .	+ 5	+ 27	+ 64	+ 84	+ 28	+ 56
r_x	+0.0017	+0.0006	−0.0000	−0.0004	−0.0012	−0.0028
r_y	− 14	− 4	0	+ 4	+ 10	+ 23

To illustrate the application of the tables in the converse transformation I have taken the values of $k\xi$ and $k\eta$ just computed and reproduced the values of $(\alpha - \alpha_0)$ and δ.

Star.	1	2	3	4	5	6
$k\xi$ . . .	$-$ 3.0461	$+$ 0.6702	$+$ 2.7365	$-$ 2.9422	$+$ 0.4135	$+$ 3.8066
$k\eta$ . . .	$+$ 3.0324	$+$ 2.9300	$+$ 3.6809	$-$ 4.4184	$-$ 3.0786	$-$ 2.6068
A (Table I)	$-$ 28	$-$ 25	$-$ 50	$+$ 85	$+$ 30	$+$ 18
$d - \delta_0$ . .	$+$ 3°.0296	$+$ 2°.9275	$+$ 3°.6759	$-$ 4°.4099	$-$ 3°.0756	$-$ 2°.6050
d	$+$ 13 .5046	$+$ 13 .4025	$+$ 14 .1509	$+$ 6 .0651	$+$ 7 .3994	$+$ 7 .8700
log sec d .	0.01218	0.01199	0.01338	0.00243	0.00363	0.00411
log $k\xi$ sec d	0.49592	9.83820	0.45057	0.47111	9.62014	0.58465
$k\xi$ sec d .	$-$ 3.1388	$+$ .6890	$+$ 2.8221	$-$ 2.9588	$+$ 4.1700	$+$ 3.8428
C (Table III)	$+$ 74	$-$ 10	$-$ 81	$+$ 114	$-$ 6	$-$ 97
$(\alpha - \alpha_0)$.	$-$ 3°.1253	$+$ 0°.6880	$+$ 2°.8140	$-$ 2°.9474	$+$ 0°.4164	$+$ 3°.8331
B (Table II)	$-$ 193	$-$ 9	$-$ 164	$-$ 80	$-$ 2	$-$ 174
δ	$+$ 13°.4853	$+$ 13°.4016	$+$ 14°.1345	$+$ 6°.0571	$+$ 7°.3992	$+$ 7°.8526

The transformation of intervals of right ascension and declination to rectangular coördinates, and its converse, for stars on another plate are inserted here to illustrate the use of the tables in the case of a plate taken near the pole.

DATA OF OBSERVATION.

Exposed 1905, April 11.

Sidereal time at beginning of exposure	10^h 30^m
Sidereal time at end of exposure . .	10 50
Coördinates of center of plate . .	α 14^h 38^m δ 76° $12'$

Approximate scale value, 1 mm = $0^\circ.1$

Latitude of Berkeley = 37° $52'$.

Star.	1	2	3	4
$(\alpha - \alpha_0)$ . .	-33^m $21^s.84$	$+31^m$ $38^s.90$	-33^m $31^s.55$	$+30^m$ $40^s.97$
	$-$ $8^\circ.3410$	$+$ $7^\circ.9121$	$-$ $8^\circ.3815$	$+$ $7^\circ.6707$
δ	$+$ $78^\circ.6831$	$+$ $78^\circ.8825$	$+$ $75^\circ.3151$	$+$ $75^\circ.0568$
B (Table II) . .	$+$ 1167	$+$ 1032	$+$ 1502	$+$ 1278
d	$+$ 78 .7998	$+$ 78 .9857	$+$ 75 .4653	$+$ 75 .1846
$d - \delta_0$	$+$ 2 .4998	$+$ 2 .6857	$-$ 0 .8347	$-$ 1 .1154
A (Table I) . .	$+$ 44	$+$ 20	$-$ 1	$-$ 1
C (Table III) . .	$-$ 673	$+$ 594	$-$ 612	$+$ 476
log $k\xi$ sec d . .	0.92471	0.90154	0.92648	0.88752
log cos d . . .	9.28834	9.28115	9.39962	9.40774
$k\xi$	$-$ 1.6332	$+$ 1.5230	$-$ 2.1188	$+$ 1.9736
$k\eta$	$+$ 2.5042	$+$ 2.6877	$-$ 0.8348	$-$ 1.1155

Star.	1	2	3	4
$k\xi$	$-$ 1.6332	$+$ 1.5230	$-$ 2.1188	$+$ 1.9736
$k\eta$	$+$ 2.5042	$+$ 2.6877	$-$ 0.8348	$-$ 1.1155
A (Table I) $d - \delta_0$	$-$ 44	$-$ 20	$+$ 1	$+$ 1
d	$+$ $78^\circ.7998$	$+$ $78^\circ.9857$	$+$ $75^\circ.4653$	$+$ $75^\circ.1846$
log sec d . . .	0.71166	0.71885	0.60038	0.59226
log $k\xi$	0.21305	0.18269	0.32610	0.29526
$k\xi$ sec d . . .	$-$ 8.4083	$+$ 7.9715	$-$ 8.4427	$+$ 7.7183
C (Table III) . .	$+$ 673	$-$ 594	$+$ 612	$-$ 476
$(\alpha - \alpha_0)$ . . .	$-$ $8^\circ.3410$	$+$ $7^\circ.9121$	$-$ $8^\circ.3815$	$+$ $7^\circ.6707$
B (Table II) . .	$-$ .1167	$-$ .1032	$-$ .1502	$-$ .1278
δ	$+$ $78^\circ.6831$	$+$ $78^\circ.8825$	$+$ $75^\circ.3151$	$+$ $75^\circ.0568$

AS A TEST OF THE APPLICATION OF THE TABLES FOR REFRACTION to a very extreme case the following computation has been made. Five points on the celestial sphere were chosen arbitrarily, having the following coördinates:

	α	δ	Zenith Distance at Sid. t. 4^h
0	$+15^\circ$ 0′ 0″.0	-30° 0′ 0″.0	79° 50′.3
1	20 0 0 .0	35 0 0 .0	81 46 .0
2	10 0 0 .0	25 0 0 .0	78 26 .4
3	11 0 0 .0	34 0 0 .0	85 3 .8
4	19 0 0 .0	-29 0 0 .0	77 5 .4

Imagining stars to be situated at these points, their apparent spherical coördinates for the latitude of Berkeley and sidereal time four hours were computed, and from these were derived rectangular coördinates as they would have been measured on a plate taken at that hour angle. The so obtained fictitious plate measures were reduced, according to the plan outlined, pages 57–61, using the zero position as the center of the plate, supposed found by plotting on a star chart (see p. 56), and stars 1 and 2 as fiducial stars. The positions 3 and 4 are accurately reproduced.

The apparent spherical coördinates with the corresponding photographic refraction constants are:

	α	δ	μ
0	15° 3′ 25″.2	-29° 55′ 41″.7	0°.015613
1	20 3 58 .1	34 54 33 .0	0 .015306
2	10 3 8 .8	24 56 21 .8	0 .015756
3	11 7 2 .2	33 52 10 .1	0 .014063
4	19 2 13 .2	-28 56 29 .0	0 .015856

From these are deduced with the help of Tables I, II, and III (see example p. 69) the following fictitious measured coördinates, the center of the plate being taken at

$$\alpha = 15^\circ\ 3'\ 25''.2$$
$$\delta = -29^\circ\ 55\ 41\ .7$$

	x	y
1	+4.12937	−5.09725
2	−4.56308	+4.91709
3	−3.28197	−4.01049
4	+3.49146	+0.92819

Taking the same center of the plate and the true coördinates of stars 1 and 2, the standard coördinates are:

	ξ	η
1	+4.07076	−5.18620
2	−4.60838	+4.85411

The corrections introducing part of the effect of refraction into these standard coördinates are computed as follows:

α	15°	$3'$
δ	-29°	$56'$
s	60°	0
$s-\alpha$	44°	$57'$
Table VI $\triangle$	47	42
$\triangle - \delta$	77	38
Table VII log X cos $(\triangle - \delta)$	9.8272	
log X cos $(\triangle - \delta)$	9.3308	
*X	$+3.136$	
Y	$+4.560$	

	1	2
X	$+3.1340$	
Y	$+4.5571$	
p. 72 μ	$+0.015306$	$+0.015756$
$\Delta\mu$	-0.000307	$+0.000143$
Table VIII x	$+0.07105$	-0.08044
y	-0.09053	$+0.08472$
Table IX, x^2+y^2	$+0.0132$	$+0.00136$
$\mu(X-x)$	$+0.04688$	$+0.05065$
$\mu(Y-y)$	$+0.07114$	$+0.07047$
Xx	$+0.2227$	-0.2521
Yy	-0.4124	$+0.3861$
Σ	-0.1897	$+0.1340$
Table X, D	$+0.2341$	-0.1181
$(x^2+y^2)(1+D)$	$+0.0163$	$+0.0120$

	1	2
$\mu(X-x)$ [ ]	$+0.01174$	-0.00537
$\Delta\mu(X-x)$	-0.00094	$+0.00046$
r_x	$+0.01080$	-0.00491
$\mu(Y-y)$ [ ]	$+0.01782$	-0.00748
$\Delta\mu(Y-y)$	-0.00143	$+0.00064$
r_y	$+0.01639$	-0.00684
$x+r_x$	$+4.08156$	-4.61329
$x+r_x-x'$	$-\ 4781$	$-\ 5021$
$y+r_y$	-5.16981	$+4.84727$
$y+r_y-y'$	$-\ 7256$	$-\ 6982$

The equations of condition are:

$$+4.129a' - 5.097b' + c' = -4781$$
$$-4.563 \quad +4.917 \quad + c' = -5021$$

$$-5.097a' - 4.129b' + d' = -7256$$
$$+4.917 \quad +4.563 \quad + d' = -6982$$

From which:

$$a' = +0.000275$$
$$b' = -0.000001$$
$$c' = -0.04895$$
$$d' = -0.07116$$

and for stars 3 and 4:

	3	4
$x+r_x$	-3.33182	$+3.44347$
$y+r_y$	-4.08275	$+0.85729$

*These values are not sufficiently accurate for this extreme case, and the values used in the following computation were derived by means of the rigorous relations. See p. 60.

The quantities r_x and r_y are computed for these stars as follows:

	3	4
X	+ 3.1340	
Y	+ 4.5571	
μ	+ 0.014063	+ 0.015856
$\Delta\mu$	− 0.001550	+ 0.000243
x	− 0.05816	+ 0.06010
y	− 0.07122	+ 0.01495
$x^2 + y^2$	859	383
$\mu (X-x)$	+ 0.04490	+ 0.04874
$\mu (Y-y)$	+ 0.06509	+ 0.07203
Xx	− 0.18227	+ 0.18835
Yy	− 0.32455	+ 0.06814
Σ	− 0.50682	+ 0.25649
D	+ 1.0275	− 0.2042
$(x^2+y^2)(1+D)$	+ 0.01742	+ 0.00305
$\mu (X-x)$ [ ]	+ 0.04691	− 0.00981
$\Delta\mu (X-x)$	− 0.00495	+ 0.00075
1st approx. r_x	+ 0.04196	− 0.00906
$\mu (Y-y)$ [ ]	+ 0.06801	− 0.01449
$\Delta\mu (Y-y)$	− 0.00717	+ 0.00110
1st approx. r_y	+ 0.06087	− 0.01339
$x - r_x$	− 0.05889	+ 0.06026
$y - r_y$	− 0.07228	+ 0.01519
$(x^2 + y^2)$	87	38
Xx	− 0.1846	+ 0.1888
Yy	− 0.3294	+ 0.0692
Σ	− 0.5140	+ 0.2580
D	+ 1.0575	− 0.2051
$(x^2 + y^2)(1 + D)$	+ 179	+ 34
$\mu (X-x)$ [ ]	+ 0.04828	− 0.00983
$\Delta\mu (X-x)$	− 0.00495	+ 75
2d approx. r_x	+ 0.04333	− 0.00908
$\mu (Y-y)$ [ ]	+ 0.07000	− 0.01453
$\Delta\mu (Y-y)$	− 0.00717	+ 110
2d approx. r_y	+ 0.06283	− 0.01343

Another approximation gives:

	3	4
r_x	+ 0.04340	− 0.00908
r_y	+ 0.06293	− 0.01343
x	− 3.37522	+ 3.45255
y	− 4.14568	+ 0.87072

Transforming these standard coördinates into intervals of right ascension and declination, the center of the plate being at

$$\alpha = 15^\circ \quad 3' \quad 25''.2$$
$$\delta\ -29 \quad 55 \quad 41\ .7,$$

we reproduce the coördinates of the originally assumed points within a tenth of a second of arc:

	α	δ
3 . . .	10° 59′ 59″.9	−34° 0′ 0″.1
4 . . .	19 0 0 .0	−28 59 59 .9

TABLE I

GIVING

$$A = k\,\eta - (d - \delta_\circ)^\circ,$$

as function of $(d - \delta_\circ)^\circ$ or $k\,\eta$.

TABLE I.

$$A = k\eta - (d - \delta_0)^\circ$$

$k\eta$ is always numerically larger than $(d-\delta_0)^\circ$ $0^\circ.00001 = 0''.036 =$ unit.

$d-\delta_0$	A		$k\eta$
0.0	.0		0.0
.1	.0	.1	.1
.2	.1	.2	.2
.3	.3	.4	.3
.4	.7	.6	.4
.5	1.3	.9	.5
.6	2.2	1.3	.6
.7	3.5	1.7	.7
.8	5.2	2.2	.8
.9	7.4	2.7	.9
1.0	10.1	3	1.0
.1	13	4	.1
.2	17	5	.2
.3	22	6	.3
.4	28	7	.4
.5	35	7	.5
.6	42	8	.6
.7	50	9	.7
.8	59	11	.8
.9	70	11	.9
2.0	81	12	2.0
.1	93	14	.1
.2	107	16	.2
.3	123	17	.3
.4	140	19	.4
.5	159	20	.5
.6	179	21	.6
.7	200	23	.7
.8	223	25	.802
.9	248	26	.902
3.0	274		3.003
$d-\delta_0$	A		$k\eta$

$d-\delta_0$	A		$k\eta$	$d-\delta_0$	A		$k\eta$	$d-\delta_0$	A		$k\eta$
3.00	274	3	3.003	3.40	399	4	3.404	3.80	558	5	3.806
1	277	3	013	41	403	4	414	81	563	4	816
2	280	3	023	42	407	4	424	82	567	4	826
3	283	3	033	43	411	3	434	83	571	5	836
4	286	3	043	44	414	3	444	84	576	5	846
5	289	2	053	45	417	4	454	85	581	4	856
6	291	3	063	46	421	4	464	86	585	4	866
7	294	3	073	47	425	4	474	87	589	5	876
8	297	3	083	48	429	3	484	88	594	5	886
9	300	3	093	49	432	4	494	89	599	4	896
3.10	303	3	3.103	3.50	436	4	3.504	3.90	603	5	3.906
11	306	3	113	51	440	3	514	91	608	5	916
12	309	3	123	52	443	4	524	92	613	4	926
13	312	3	133	53	447	4	534	93	617	5	936
14	315	3	143	54	451	4	545	94	622	5	946
15	318	3	153	55	455	4	555	95	627	5	956
16	321	3	163	56	459	4	565	96	632	5	966
17	324	3	173	57	463	4	575	97	637	4	976
18	327	3	183	58	467	3	585	98	641	5	986
19	330	3	193	59	470	4	595	99	646	5	996
3.20	333	4	3.203	3.60	474	4	3.605	4.00	651	5	4.007
21	337	3	213	61	478	4	615	1	656	5	017
22	340	3	223	62	482	4	625	2	661	5	027
23	343	3	233	63	486	4	635	3	666	5	037
24	346	3	243	64	490	5	645	4	671	5	047
25	349	3	253	65	495	4	655	5	676	5	057
26	352	3	264	66	499	4	665	6	681	5	067
27	355	4	274	67	503	4	675	7	686	5	077
28	359	3	284	68	507	4	685	8	691	5	087
29	362	3	294	69	511	4	695	9	696	5	097
3.30	365	4	3.304	3.70	515	4	3.705	4.10	701	5	4.107
31	369	3	314	71	519	4	715	11	706	5	117
32	372	3	324	72	523	5	725	12	711	6	127
33	375	4	334	73	528	4	735	13	717	5	137
34	379	4	344	74	532	4	745	14	722	5	147
35	383	3	354	75	536	5	755	15	727	5	157
36	386	3	364	76	541	4	765	16	732	5	167
37	389	4	374	77	545	4	775	17	737	6	177
38	393	3	384	78	549	4	785	18	743	5	187
39	396	3	394	79	553	5	796	19	748	5	197
3.40	399		3.404	3.80	558		3.806	4.20	753		4.208
$d-\delta_0$	A		$k\eta$	$d-\delta_0$	A		$k\eta$	$d-\delta^\circ$	A		$k\eta$

$d-\delta_0$	A		$k\eta$	$d-\delta_0$	A		$k\eta$
4.20	753	6	4.208	4.60	990	7	4.610
21	759	6	218	61	997	6	620
22	765	5	228	62	1003	7	630
23	770	5	238	63	1010	7	640
24	775	6	248	64	1017	6	650
25	781	6	258	65	1023	6	660
26	787	5	268	66	1029	7	670
27	792	5	278	67	1036	7	680
28	797	6	288	68	1043	6	690
29	803	6	298	69	1049	7	700
4.30	809	6	4.308	4.70	1056	7	4.711
31	815	5	318	71	1063	7	721
32	820	6	328	72	1070	7	731
33	826	6	338	73	1077	7	741
34	832	5	348	74	1084	7	751
35	837	6	358	75	1091	7	761
36	843	6	368	76	1098	7	771
37	849	6	378	77	1105	7	781
38	855	6	389	78	1112	7	791
39	861	6	399	79	1119	7	801
4.40	867	6	4.409	4.80	1126	7	4.811
41	873	6	419	81	1133	7	821
42	879	6	429	82	1140	7	831
43	885	6	439	83	1147	7	841
44	891	6	449	84	1154	7	852
45	897	6	459	85	1161	8	862
46	903	6	469	86	1169	7	872
47	909	6	479	87	1176	7	882
48	915	6	489	88	1183	7	892
49	921	6	499	89	1190	7	902
4.50	927	6	4.509	4.90	1197	8	4.912
51	933	6	519	91	1205	7	922
52	939	6	529	92	1212	8	932
53	945	7	539	93	1220	7	942
54	952	6	550	94	1227	8	952
55	958	7	560	95	1235	7	962
56	965	6	570	96	1242	7	972
57	971	6	580	97	1249	8	982
58	977	7	590	98	1257	8	993
59	984	6	600	99	1265	7	5.003
4.60	990		4.610	5.00	1272		5.013
$d-\delta_0$	A		$k\eta$	$d-\delta_0$	A		$k\eta$

TABLE II

GIVING

$B = (d - \delta)^\circ$, as function of $(\alpha - \alpha_0)$ and δ or d.

TABLE II.

$$B = (d - \delta)^\circ .$$

d is always numerically larger than δ. 0°.000001 = 0″.0036 = unit.

δ \ $a-a_0$	$\frac{h}{2}$	0°.0	hor. diff.	0°.1		hor. diff.	0°.2		hor. diff.	0°.3		hor. diff.	0°.4		hor. diff.	0°.5		$a-a_0$ / d
0°	0	0	0	0		0	0		0	0		0	0		0	0		90
1	2		2	2	2	4	6	6	8	14	14	10	24	24	14	38	38	89
2	3		3	3	1	9	12	6	15	27	13	22	49	25	27	76	38	88
3	5		5	5	2	13	18	6	23	41	14	32	73	24	41	114	38	87
4	6		6	6	1	18	24	6	31	55	14	42	97	24	55	152	38	86
5	8		8	8	2	22	30	6	38	68	13	53	121	24	68	189	37	85
6	9		9	9	1	27	36	6	46	82	14	63	145	24	82	227	38	84
7	11		11	11	2	31	42	6	53	95	13	74	169	24	95	264	37	83
8	12		12	12	1	36	48	6	60	108	13	84	192	23	109	301	37	82
9	14		13	13	1	41	54	6	67	121	13	95	216	24	121	337	36	81
10	15		15	15	2	45	60	6	74	134	13	105	239	23	134	373	36	80
11	16		16	16	1	49	65	5	82	147	13	115	262	23	147	409	36	79
12	18		18	18	2	53	71	6	89	160	13	124	284	22	160	444	35	78
13	19		19	19	1	58	77	6	95	172	12	134	306	22	172	478	34	77
14	20		20	20	1	62	82	5	102	184	12	144	328	22	184	512	34	76
15	22		22	22	2	65	87	5	109	196	12	153	349	21	196	545	33	75
16	23		23	23	1	69	92	5	116	208	12	162	370	21	208	578	33	74
17	24		24	24	1	74	98	6	122	220	12	170	390	20	220	610	32	73
18	26		26	26	2	77	103	5	128	231	11	179	410	20	231	641	31	72
19	27		27	27	1	80	107	4	135	242	11	188	430	20	242	672	31	71
20	28		28	28	1	84	112	5	140	252	10	197	449	19	252	701	29	70
21	29		29	29	1	88	117	5	146	263	11	204	467	18	263	730	29	69
22	30		30	30	1	91	121	4	152	273	10	212	485	18	273	758	28	68
23	31		31	31	1	95	126	5	157	283	10	219	502	17	283	785	27	67
24	32		32	32	1	98	130	4	162	292	9	227	519	17	292	811	26	66
25	33		33	33	1	101	134	4	167	301	9	234	535	16	301	836	25	65
26	34		34	34	1	104	138	4	171	309	8	241	550	15	310	860	24	64
27	35		35	35	1	106	141	3	177	318	9	247	565	15	318	883	23	63
28	36		36	36	1	109	145	4	181	326	8	253	579	14	325	904	21	62
29	37		37	37	1	111	148	3	185	333	7	259	592	13	333	925	21	61
30	38		38	38	1	113	151	3	189	340	7	265	605	13	340	945	20	60
31	39		39	39	1	115	154	3	193	347	7	269	616	11	347	963	18	59
32	39		39	39	0	118	157	3	196	353	6	275	628	12	353	981	18	58
33	40		40	40	1	119	159	2	200	359	6	279	638	10	359	997	16	57
34	40		40	40	0	122	162	3	202	364	5	283	647	9	364	1011	14	56
35	41		41	41	1	123	164	2	205	369	5	287	656	9	369	1025	14	55
36	42		42	42	1	124	166	2	208	374	5	290	664	8	374	1038	13	54
37	42		42	42	0	126	168	2	210	378	4	293	671	7	378	1049	11	53
38	42		42	42	0	127	169	1	212	381	3	296	677	6	381	1058	9	52
39	43		43	43	1	128	171	2	213	384	3	299	683	6	384	1067	9	51
40	43		43	43	0	129	172	1	215	387	3	301	688	5	386	1074	7	50
41	43		43	43	0	130	173	1	216	389	2	302	691	3	389	1080	6	49
42	43		44	43	0	131	174	1	217	391	2	303	694	3	391	1085	5	48
43	44		44	44	1	130	174	0	218	392	1	304	696	2	392	1088	3	47
44	44		44	44	0	130	174	0	218	392	0	306	698	2	392	1090	2	46
45	44		44	44	0	131	175	1	218	393	1	305	698	0	393	1091	1	45
δ / $a-a_0$	$\frac{h}{2}$	0°.0	hor. diff.	0°.1		hor. diff.	0°.2		hor. diff.	0°.3		hor. diff.	0°.4		hor. diff.	0°.5		d / $a-a_0$

$$f(a \pm n w) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \qquad \frac{f''(a)}{2} = \frac{h}{2}.$$

See note on interpolation, p. 64.

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ. **0°.000001 = 0″.0036 = unit.**

δ (a − a₀)	$\frac{h}{2}$	0°.5		hor. diff.	0°.6		hor. diff.	0°.7		hor. diff.	0°.8		hor. diff.	0°.9		hor. diff.	1°.0		d (a − a₀)
0°	0	0	38		00	55	00	00	75	00	00	97	00	000	123	00	000	152	90
1	2	38	38	17	55	55	20	75	74	22	97	98	26	123	124	29	152	152	89
2	3	76	38	34	110	54	39	149	74	46	195	97	52	247	122	57	304	152	88
3	5	114	38	50	164	55	59	223	75	69	292	97	77	369	123	87	456	151	87
4	6	152	37	67	219	54	79	298	73	91	389	96	103	492	122	115	607	151	86
5	8	189	38	84	273	54	98	371	74	114	485	96	129	614	121	144	758	149	85
6	9	227	37	100	327	53	118	445	72	136	581	95	154	735	120	172	907	149	84
7	11	264	37	116	380	53	137	517	72	159	676	94	179	855	119	201	1056	147	83
8	12	301	36	132	433	52	156	589	72	181	770	93	204	974	118	229	1203	145	82
9	14	337	36	148	485	52	176	661	70	202	863	92	229	1092	117	256	1348	144	81
10	15	373	36	164	537	51	194	731	70	224	955	91	254	1209	115	283	1492	143	**80**
11	16	409	35	179	588	51	213	801	69	245	1046	90	278	1324	114	311	1635	140	79
12	18	444	34	195	639	50	231	870	67	266	1136	88	302	1438	112	337	1775	138	78
13	19	478	34	211	689	49	248	937	67	287	1224	87	326	1550	109	363	1913	136	77
14	20	512	33	226	738	47	266	1004	65	307	1311	85	348	1659	108	390	2049	133	76
15	22	545	33	240	785	47	284	1069	64	327	1396	84	371	1767	106	415	2182	130	75
16	23	578	32	254	832	46	301	1133	63	347	1480	82	393	1873	104	439	2312	128	74
17	24	610	31	268	878	45	318	1196	61	366	1562	80	415	1977	101	463	2440	125	73
18	26	641	31	282	923	44	334	1257	59	385	1642	77	436	2078	98	487	2565	122	72
19	27	672	29	295	967	43	349	1316	58	403	1719	76	457	2176	96	511	2687	118	71
20	28	701	29	309	1010	41	364	1374	57	421	1795	74	477	2272	93	533	2805	115	**70**
21	29	730	28	321	1051	40	380	1431	54	438	1869	71	496	2365	90	555	2920	111	69
22	30	758	27	333	1091	39	394	1485	53	455	1940	69	515	2455	88	576	3031	108	68
23	31	785	26	345	1130	37	408	1538	51	471	2009	66	534	2543	84	596	3139	104	67
24	32	811	25	356	1167	36	422	1589	49	486	2075	64	552	2627	81	616	3243	100	66
25	33	836	24	367	1203	35	435	1638	47	501	2139	62	569	2708	77	635	3343	96	65
26	34	860	23	378	1238	33	447	1685	45	516	2201	58	584	2785	74	654	3439	92	64
27	35	883	21	388	1271	31	459	1730	43	529	2259	56	600	2859	71	672	3531	87	63
28	36	904	21	398	1302	30	471	1773	40	542	2315	53	615	2930	67	688	3618	83	62
29	37	925	20	407	1332	28	481	1813	39	555	2368	51	629	2997	64	704	3701	78	61
30	38	945	18	415	1360	27	492	1852	36	567	2419	47	642	3061	60	718	3779	74	**60**
31	39	963	18	424	1387	25	501	1888	34	578	2466	44	655	3121	56	732	3853	69	59
32	39	981	16	431	1412	23	510	1922	31	588	2510	41	667	3177	52	745	3922	64	58
33	40	997	14	438	1435	22	518	1953	29	598	2551	38	678	3229	48	757	3986	60	57
34	40	1011	14	446	1457	19	525	1982	27	607	2589	35	688	3277	44	769	4046	54	56
35	41	1025	13	451	1476	18	533	2009	25	615	2624	32	697	3321	40	779	4100	50	55
36	42	1038	11	456	1494	16	540	2034	22	622	2656	29	705	3361	36	789	4150	45	54
37	42	1049	9	461	1510	14	546	2056	19	629	2685	25	712	3397	32	798	4195	39	53
38	42	1058	9	466	1524	13	551	2075	16	635	2710	22	719	3429	28	805	4234	34	52
39	43	1067	7	470	1537	10	554	2091	14	641	2732	18	725	3457	24	811	4268	29	51
40	43	1074	6	473	1547	9	558	2105	12	645	2750	15	731	3481	19	816	4297	24	**50**
41	43	1080	5	476	1556	6	561	2117	9	648	2765	12	735	3500	15	821	4321	19	49
42	43	1085	3	477	1562	5	564	2126	7	651	2777	9	738	3515	11	825	4340	13	48
43	44	1088	2	479	1567	3	566	2133	4	653	2786	5	740	3526	6	827	4353	8	47
44	44	1090	1	480	1570	1	567	2137	1	654	2791	1	741	3532	2	829	4361	3	46
45	44	1091		480	1571		567	2138		654	2792		742	3534		830	4364		45
δ (a − a₀)	$\frac{h}{2}$	0°.5		hor. diff.	0°.6		hor. diff.	0°.7		hor. diff.	0°.8		hor. diff.	0°.9		hor. diff.	1°.0		d (a − a₀)

$$f(a \pm n\,w) = f(a) \pm n\left[f'(a) \pm n\,\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{h}{2}.$$

TABLE II.

$B = (d - \delta)^\circ$.

d is always numerically larger than δ. $0^\circ.000001 - 0^\circ.0036$ – unit.

$a-a_0$ / δ	$\frac{h}{2}$	1°.0		hor. diff.	1°.1		hor. diff.	1°.2		hor diff.	1°.3		hor diff.	1°.4		hor. diff.	1°.5		$a-a_0$ / d
0	0	000	152	00	000	184	00	000	219	00	000	257	00	000	299	00	000	343	90
1	2	152	152	32	184	184	35	219	219	38	257	258	42	299	298	44	343	342	89
2	3	304	152	64	368	184	70	438	219	77	515	256	82	597	297	88	685	342	88
3	5	456	151	96	552	183	105	657	218	114	771	256	123	894	297	133	1027	340	87
4	6	607	151	128	735	182	140	875	216	152	1027	254	164	1191	295	176	1367	338	86
5	[redacted]	758	149	159	917	181	174	1091	216	190	1281	253	205	1486	293	219	1705	337	85
6	9	907	149	191	1098	179	209	1307	213	227	1534	250	245	1779	291	263	2042	334	84
7	11	1056	147	221	1277	179	243	1520	212	264	1784	249	286	2070	288	306	2376	331	83
8	12	1203	145	253	1456	176	276	1732	210	301	2033	246	325	2358	286	349	2707	328	82
9	14	1348	144	284	1632	174	310	1942	207	337	2279	244	365	2644	282	391	3035	324	81
10	15	1492	143	314	1806	172	343	2149	205	374	2523	240	403	2926	279	433	3359	320	[redacted]
11	16	1635	140	343	1978	170	376	2354	202	409	2763	237	442	3205	275	474	3679	316	79
12	18	1775	138	373	2148	167	408	2556	199	444	3000	234	480	3480	270	515	3995	310	78
13	19	1913	136	402	2315	164	440	2755	195	479	3234	229	516	3750	266	555	4305	305	77
14	21	2049	133	430	2479	161	471	2950	192	513	3463	225	553	4016	261	594	4610	300	76
15	22	2182	130	458	2640	158	502	3142	188	546	3688	221	589	4277	256	633	4910	294	75
16	23	2312	128	486	2798	155	532	3330	184	579	3909	216	624	4533	251	671	5204	288	74
17	24	2440	125	513	2953	151	561	3514	180	611	4125	211	659	4784	245	708	5492	281	73
18	26	2565	122	539	3104	147	590	3694	175	642	4336	205	693	5029	238	744	5773	273	72
19	27	2687	118	564	3251	143	618	3869	171	672	4541	200	726	5267	232	779	6046	267	71
20	28	2805	115	589	3394	139	646	4040	165	701	4741	194	758	5499	225	814	6313	258	70
21	29	2920	111	613	3533	135	672	4205	161	730	4935	189	789	5724	218	847	6571	251	69
22	30	3031	108	637	3668	130	698	4366	155	758	5124	182	818	5942	212	880	6822	243	68
23	31	3139	104	659	3798	126	723	4521	149	785	5306	175	848	6154	203	911	7065	233	67
24	32	3243	100	681	3924	121	746	4670	144	811	5481	169	876	6357	196	941	7298	225	66
25	33	3343	96	702	4045	116	769	4814	138	836	5650	162	903	6553	188	970	7523	216	65
26	34	3439	92	722	4161	111	791	4952	132	860	5812	155	929	6741	180	998	7739	206	64
27	35	3531	87	741	4272	106	812	5084	126	883	5967	148	954	6921	171	1024	7945	196	63
28	36	3618	83	760	4378	100	832	5210	119	905	6115	140	977	7092	162	1049	8141	187	62
29	37	3701	78	777	4478	95	851	5329	113	926	6255	132	999	7254	154	1074	8328	176	61
[redacted]	38	3779	74	794	4573	89	869	5442	107	945	6387	125	1021	7408	145	1096	8504	167	60
31	38	3853	69	809	4662	84	887	5549	99	963	6512	117	1041	7553	135	1118	8671	155	59
32	39	3922	64	824	4746	78	902	5648	93	981	6629	109	1059	7688	126	1138	8826	145	58
33	40	3986	60	838	4824	72	917	5741	86	997	6738	100	1076	7814	117	1157	8971	134	57
34	40	4046	54	850	4896	66	931	5827	78	1011	6838	92	1093	7931	107	1174	9105	123	56
35	41	4100	50	862	4962	60	943	5905	72	1025	6930	84	1108	8038	97	1190	9228	111	55
36	42	4150	45	872	5022	54	955	5977	64	1037	7014	75	1121	8135	87	1204	9339	100	54
37	42	4195	39	881	5076	47	965	6041	56	1048	7089	67	1133	8222	78	1217	9439	89	53
38	42	4234	34	889	5123	42	974	6097	50	1059	7156	58	1144	8300	67	1228	9528	77	52
39	43	4268	29	897	5165	35	982	6147	42	1067	7214	49	1153	8367	57	1238	9605	65	51
40	43	4297	24	903	5200	29	989	6189	34	1074	7263	40	1161	8424	46	1246	9670	54	50
41	43	4321	19	908	5229	22	994	6223	27	1080	7303	32	1167	8470	37	1254	9724	41	49
42	43	4340	13	911	5251	16	999	6250	19	1085	7335	22	1172	8507	26	1258	9765	30	48
43	44	4353	8	914	5267	10	1002	6269	11	1088	7357	13	1176	8533	15	1262	9795	18	47
44	44	4361	3	916	5277	3	1003	6280	4	1090	7370	5	1178	8548	5	1265	9813	6	46
45	44	4364		916	5280		1004	6284		1091	7375		1178	8553		1266	9819		45
δ / $a-a_0$	$\frac{h}{2}$	1°.0		hor. diff.	1° 1		hor. diff.	1°.2		hor. diff	1°.3		hor. diff.	1° 4		hor. diff.	1°.5		d / $a-a_0$

$$f(a \pm n\,w) = f(a) \pm n\left[\, f'(a) \pm n\,\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{h}{2}$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ. $0^\circ.000001 = 0''.0036 =$ unit.

δ \ $a-a_0$	$\frac{h}{2}$	$\frac{v}{2}$	1°.5		*hor. diff.*	1°.6		*hor. diff.*	1°.7		*hor. diff.*	1°.8		*hor. diff.*	1°.9		*hor. diff.*	2°.0		$a-a_0$ / d
0	0		000	343	00	000	390	00	000	440	00	000	494	00	000	550	00	000	609	90
1	2		343	342	47	390	390	50	440	440	54	494	493	56	550	549	59	609	609	89
2	3		685	342	95	780	388	100	880	439	107	987	491	112	1099	548	119	1218	607	88
3	5		1027	340	141	1168	387	151	1319	437	159	1478	490	169	1647	546	178	1825	605	87
4	6		1367	338	188	1555	385	201	1756	435	212	1968	488	225	2193	544	237	2430	602	86
5	8		1705	337	235	1940	383	251	2191	432	265	2456	485	281	2737	540	295	3032	598	85
6	9		2042	334	281	2323	380	300	2623	429	318	2941	481	336	3277	536	353	3630	594	84
7	11		2376	331	327	2703	377	349	3052	425	370	3422	477	391	3813	531	411	4224	589	83
8	12		2707	328	373	3080	373	397	3477	421	422	3899	472	445	4344	526	469	4813	583	82
9	14	3	3035	324	418	3453	369	445	3898	417	473	4371	466	499	4870	520	526	5396	576	81
10	15	3	3359	320	463	3822	364	493	4315	411	522	4837	461	553	5390	513	582	5972	569	**80**
11	16	3	3679	316	507	4186	359	540	4726	405	572	5298	455	605	5903	507	638	6541	561	79
12	18	3	3995	310	550	4545	354	586	5131	399	622	5753	447	657	6410	498	692	7102	553	78
13	19	4	4305	305	594	4899	347	631	5530	392	670	6200	440	708	6908	490	747	7655	543	77
14	21	4	4610	300	636	5246	341	676	5922	386	718	6640	432	758	7398	481	800	8198	533	76
15	22	4	4910	294	677	5587	335	721	6308	377	764	7072	423	807	7879	472	852	8731	522	75
16	23	4	5204	288	718	5922	327	763	6685	369	810	7495	414	856	8351	461	902	9253	511	74
17	24	4	5492	281	757	6249	319	805	7054	361	855	7909	404	903	8812	451	952	9764	499	73
18	26	5	5773	273	795	6568	312	847	7415	352	898	8313	394	950	9263	439	1000	10263	487	72
19	27	5	6046	267	834	6880	303	887	7767	342	940	8707	384	995	9702	427	1048	10750	474	71
20	28	5	6313	258	870	7183	294	926	8109	332	982	9091	372	1038	10129	415	1095	11224	460	**70**
21	29	6	6571	251	906	7477	285	964	8441	322	1022	9463	361	1081	10544	402	1140	11684	445	69
22	30	6	6822	243	940	7762	276	1001	8763	311	1061	9824	349	1122	10946	389	1183	12129	431	68
23	31	6	7065	233	973	8038	266	1036	9074	300	1099	10173	337	1162	11335	375	1225	12560	416	67
24	32	6	7298	225	1006	8304	256	1070	9374	289	1136	10510	324	1200	11710	361	1266	12976	399	66
25	33	6	7523	216	1037	8560	245	1103	9663	277	1171	10834	310	1237	12071	346	1304	13375	384	65
26	34	7	7739	206	1066	8805	235	1135	9940	265	1204	11144	297	1273	12417	331	1342	13759	366	64
27	35	7	7945	196	1095	9040	223	1165	10205	253	1236	11441	283	1307	12748	315	1377	14125	350	63
28	36	7	8141	187	1122	9263	213	1195	10458	239	1266	11724	269	1339	13063	300	1412	14475	332	62
29	37	7	8328	176	1148	9476	200	1221	10697	227	1296	11993	254	1370	13363	283	1444	14807	314	61
30	38	7	8504	167	1172	9676	189	1248	10924	213	1323	12247	239	1399	13646	266	1475	15121	295	[illegible]
31	38	8	8671	155	1194	9865	177	1272	11137	200	1349	12486	224	1426	13912	250	1504	15416	276	59
32	39	8	8826	145	1216	10042	165	1295	11337	186	1373	12710	209	1452	14162	232	1530	15692	258	58
33	40	8	8971	134	1236	10207	152	1316	11523	172	1396	12919	193	1475	14394	215	1556	15950	238	57
34	40	8	9105	123	1254	10359	140	1336	11695	158	1417	13112	176	1497	14609	197	1579	16188	218	56
35	41	8	9228	111	1271	10499	127	1354	11853	143	1435	13288	161	1518	14806	179	1600	16406	198	55
36	42	8	9339	100	1287	10626	114	1370	11996	129	1453	13449	144	1536	14985	161	1619	16604	178	54
37	42	8	9439	89	1301	10740	101	1385	12125	113	1468	13593	128	1553	15146	142	1636	16782	157	53
38	42	8	9528	77	1313	10841	87	1397	12238	99	1483	13721	111	1567	15288	123	1651	16939	137	52
39	43	8	9605	65	1323	10928	75	1409	12337	84	1495	13832	94	1579	15411	105	1665	17076	116	51
40	43	8	9670	54	1333	11003	61	1418	12421	69	1505	13926	77	1590	15516	86	1676	17192	96	**50**
41	43	8	9724	41	1340	11064	47	1426	12490	53	1513	14003	60	1599	15602	67	1686	17288	74	49
42	43	8	9765	30	1346	11111	34	1432	12543	39	1520	14063	43	1606	15669	48	1693	17362	53	48
43	44	8	9795	18	1350	11145	20	1437	12582	23	1524	14106	25	1611	15717	28	1698	17415	31	47
44	44	8	9813	6	1352	11165	7	1440	12605	7	1526	14131	9	1614	15745	10	1701	17446	11	46
45	44	8	9819		1353	11172		1440	12612		1528	14140		1615	15755		1702	17457		45
δ / $a-a_0$	$\frac{h}{2}$	$\frac{v}{2}$	1°.5		*hor. diff.*	1°.6		*hor. diff.*	1°.7		*hor. diff.*	1°.8		*hor. diff.*	1°.9		*hor. diff.*	2°.0		d / $a-a_0$

$$f(a \pm n\,w) = f(a) \pm n\left[f'(a) \pm n\,\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{h}{2} \text{ or } \frac{v}{2}$$

TABLE II.

$B = (d - \delta)^\circ$.

d is always numerically larger than δ. $0^\circ.00001 = 0''.036 =$ unit.

δ \ $a - a_0$	2°.0		hor. diff.	2°.1		hor. diff.	2°.2		hor. diff.	2°.3		hor. diff.	2°.4		hor. diff.	2°.5		$a - a_0$ / d
0	00	61	0	00	67	0	00	74	0	00	81	0	00	88	0	00	95	90
1	61	61	6	67	67	7	74	73	7	81	80	7	88	87	7	95	95	89
2	122	60	12	134	67	13	147	74	14	161	80	14	175	88	15	190	95	88
3	182	61	19	201	67	20	221	73	20	241	80	22	263	87	22	285	95	87
4	243	60	25	268	66	26	294	73	27	321	80	29	350	87	30	380	94	86
[illegible]	303	60	31	334	66	33	367	72	34	401	79	36	437	86	37	474	93	85
6	363	59	37	400	66	39	439	72	41	480	79	43	523	85	44	567	93	84
7	422	59	44	466	65	45	511	71	48	559	78	49	608	85	52	660	92	83
8	481	59	50	531	64	51	582	71	55	637	77	56	693	84	59	752	91	82
9	540	57	55	595	63	58	653	70	61	714	76	63	777	83	66	843	90	81
10	597	57	61	658	63	65	723	69	67	790	75	70	860	82	73	933	89	**80**
11	654	56	67	721	62	71	792	68	73	865	74	77	942	81	80	1022	88	79
12	710	56	73	783	61	77	860	66	79	939	74	84	1023	80	87	1110	86	78
13	766	54	78	844	60	82	926	66	87	1013	71	90	1103	78	93	1196	85	77
14	820	53	84	904	59	88	992	65	92	1084	71	97	1181	77	100	1281	84	76
15	873	52	90	963	57	94	1057	63	98	1155	69	103	1258	75	107	1365	81	75
16	925	51	95	1020	57	100	1120	62	104	1224	68	109	1333	73	113	1446	80	74
17	976	50	101	1077	55	105	1182	60	110	1292	66	114	1406	72	120	1526	78	73
18	1026	49	106	1132	53	110	1242	59	116	1358	64	120	1478	70	126	1604	76	72
19	1075	47	110	1185	52	116	1301	57	121	1422	63	126	1548	69	132	1680	74	71
20	1122	46	115	1237	51	121	1358	56	127	1485	60	132	1617	66	137	1754	72	**70**
21	1168	45	120	1288	49	126	1414	54	131	1545	59	138	1683	64	143	1826	70	69
22	1213	43	124	1337	48	131	1468	52	136	1604	57	143	1747	62	149	1896	67	68
23	1256	42	129	1385	46	135	1520	50	141	1661	55	148	1809	60	154	1963	65	67
24	1298	40	133	1431	44	139	1570	49	146	1716	53	153	1869	57	159	2028	62	66
25	1338	38	137	1475	42	144	1619	46	150	1769	51	157	1926	56	164	2090	60	65
26	1376	36	141	1517	40	148	1665	44	155	1820	48	162	1982	52	168	2150	58	64
27	1412	36	145	1557	39	152	1709	43	159	1868	47	166	2034	51	174	2208	54	63
28	1448	33	148	1596	37	156	1752	40	163	1915	43	170	2085	48	177	2262	52	62
29	1481	31	152	1633	34	159	1792	38	166	1958	42	175	2133	45	181	2314	49	61
30	1512	30	155	1667	33	163	1830	35	170	2000	39	178	2178	42	185	2363	46	**60**
31	1542	27	158	1700	30	165	1865	34	174	2039	37	181	2220	40	189	2409	43	59
32	1569	26	161	1730	29	169	1899	31	177	2076	34	184	2260	37	192	2452	41	58
33	1595	24	164	1759	26	171	1930	29	180	2110	31	187	2297	34	196	2493	37	57
34	1619	22	166	1785	24	174	1959	26	182	2141	29	190	2331	32	199	2530	34	56
35	1641	19	168	1809	22	176	1985	24	185	2170	26	193	2363	28	201	2564	31	55
36	1660	18	171	1831	19	178	2009	22	187	2196	24	195	2391	26	204	2595	28	54
37	1678	16	172	1850	18	181	2031	19	189	2220	20	197	2417	23	206	2623	24	53
38	1694	14	174	1868	15	182	2050	16	190	2240	19	200	2440	19	207	2647	22	52
[illegible]	1708	11	175	1883	13	183	2066	14	193	2259	15	200	2459	17	210	2669	18	51
40	1719	10	177	1896	10	184	2080	12	194	2274	13	202	2476	14	211	2687	15	[illegible]
41	1729	7	177	1906	8	186	2092	9	195	2287	9	203	2490	10	212	2702	11	49
42	1736	6	178	1914	6	187	2101	6	195	2296	7	204	2500	8	213	2713	8	48
43	1742	3	178	1920	4	187	2107	4	196	2303	4	205	2508	5	213	2721	5	47
44	1745	1	179	1924	1	187	2111	1	196	2307	2	206	2513	1	213	2726	2	46
45	1746		179	1925		187	2112		197	2309		205	2514		214	2728		45
δ / $a - a_0$	2°.0		hor. diff.	2°.1		hor. diff.	2°.2		hor. diff.	2°.3		hor. diff.	2°.4		hor. diff.	2°.5		d / $a - a_0$

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\,\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)°.$$

d is always numerically larger than δ. $0°.00001 = 0''.036 =$ unit.

$a-a_0$ / δ	2°.5		hor. diff.	2°.6		hor. diff.	2°.7		hor. diff.	2°.8		hor. diff.	2°.9		hor. diff.	3°.0		$a-a_0$ / d
0°	00	95	0	000	103	0	000	111	0	000	120	0	000	128	0	000	137	90
1	95	95	8	103	103	8	111	111	9	120	119	8	128	128	9	137	137	89
2	190	95	16	206	103	16	222	111	17	239	119	17	256	128	18	274	137	88
3	285	95	24	309	102	24	333	110	25	358	119	26	384	127	27	411	136	87
4	380	94	31	411	102	32	443	110	34	477	118	34	511	127	36	547	136	86
5	474	93	39	513	101	40	553	109	42	595	117	43	638	126	45	683	134	85
6	567	93	47	614	100	48	662	108	50	712	116	52	764	125	53	817	134	84
7	660	92	54	714	100	56	770	108	58	828	116	61	889	124	62	951	133	83
8	752	91	62	814	98	64	878	106	66	944	114	69	1013	122	71	1084	131	82
9	843	90	69	912	98	72	984	105	74	1058	113	77	1135	121	80	1215	130	81
10	933	89	77	1010	96	79	1089	104	82	1171	112	85	1256	120	87	1345	128	**80**
11	1022	88	84	1106	95	87	1193	102	90	1283	110	93	1376	118	97	1473	126	79
12	1110	86	91	1201	93	97	1295	101	98	1393	108	101	1494	116	105	1599	124	78
13	1196	85	98	1294	92	102	1396	99	105	1501	107	109	1610	114	113	1723	123	77
14	1281	84	105	1386	90	109	1495	97	113	1608	104	116	1724	113	122	1846	120	76
15	1365	81	111	1476	88	116	1592	95	120	1712	102	125	1837	109	129	1966	117	75
16	1446	80	118	1564	87	123	1687	93	127	1814	101	132	1946	108	137	2083	115	74
17	1526	78	125	1651	84	129	1780	91	135	1915	98	139	2054	105	144	2198	113	73
18	1604	76	131	1735	82	136	1871	89	142	2013	95	146	2159	102	152	2311	109	72
19	1680	74	137	1817	80	143	1960	86	148	2108	93	153	2261	100	159	2420	107	71
20	1754	72	143	1897	78	149	2046	84	155	2201	90	160	2361	97	166	2527	103	**70**
21	1826	70	149	1975	76	155	2130	81	161	2291	87	167	2458	93	172	2630	100	69
22	1896	67	155	2051	72	160	2211	79	167	2378	85	173	2551	91	179	2730	97	68
23	1963	65	160	2123	71	167	2290	76	173	2463	81	179	2642	87	185	2827	94	67
24	2028	62	166	2194	67	172	2366	73	178	2544	79	185	2729	84	192	2921	90	66
25	2090	60	171	2261	65	178	2439	69	184	2623	75	190	2813	81	198	3011	86	65
26	2150	58	176	2326	62	182	2508	67	190	2698	72	196	2894	77	203	3097	83	64
27	2208	54	180	2388	59	187	2575	64	195	2770	68	201	2971	74	209	3180	78	63
28	2262	52	185	2447	56	192	2639	60	199	2838	65	207	3045	69	213	3258	75	62
29	2314	49	189	2503	53	196	2699	58	204	2903	62	211	3114	66	219	3333	71	61
30	2363	46	193	2556	50	201	2757	53	208	2965	58	215	3180	62	224	3404	66	**60**
31	2409	43	197	2606	47	204	2810	51	212	3023	54	219	3242	59	228	3470	62	59
32	2452	41	201	2653	43	208	2861	47	216	3077	50	224	3301	54	231	3532	58	58
33	2493	37	203	2696	40	212	2908	43	219	3127	47	228	3355	50	235	3590	54	57
34	2530	34	206	2736	37	215	2951	40	223	3174	43	231	3405	46	239	3644	49	56
35	2564	31	209	2773	34	218	2991	36	226	3217	38	234	3451	41	242	3693	44	55
36	2595	28	212	2807	30	220	3027	32	228	3255	35	237	3492	38	245	3737	40	54
37	2623	24	214	2837	27	222	3059	29	231	3290	31	240	3530	33	247	3777	36	53
38	2647	22	217	2864	23	224	3088	25	233	3321	27	242	3563	28	250	3813	30	52
39	2669	18	218	2887	19	226	3113	21	235	3348	23	243	3591	25	252	3843	27	51
40	2687	15	219	2906	16	228	3134	17	237	3371	18	245	3616	20	254	3870	21	**50**
41	2702	11	220	2922	13	229	3151	14	238	3389	15	247	3636	15	255	3891	17	49
42	2713	8	222	2935	9	230	3165	9	239	3404	10	247	3651	11	257	3908	11	48
43	2721	5	223	2944	5	230	3174	6	240	3414	6	248	3662	7	257	3919	7	47
44	2726	2	223	2949	2	231	3180	2	240	3420	2	249	3669	2	257	3926	3	46
45	2728		223	2951		231	3182		240	3422		249	3671		258	3929		45
δ / $a-a_0$	2°.5		hor. diff.	2°.6		hor. diff.	2°.7		hor. diff.	2°.8		hor. diff.	2°.9		hor. diff.	3°.0		d / $a-a_0$

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^2.$$

d is always numerically larger than δ. 0°.00001 = 0".036 = unit.

δ \ $a - a_0$	$\frac{v}{2}$	3°.0		hor. diff.	3°.1		hor. diff.	3°.2		hor. diff.	3°.3		hor. diff.	3°.4		hor. diff.	3°.5		$a - a_0$ / d
0		000	137	0	000	146	00	000	156	00	000	166	00	000	176	00	000	187	**90**
1		137	137	9	146	147	10	156	156	10	166	166	10	176	176	11	187	186	89
2		274	137	19	293	146	19	312	156	20	332	165	20	352	176	21	373	187	88
3		411	136	28	439	145	29	468	155	29	497	165	31	528	175	32	560	185	87
4		547	136	37	584	145	39	623	154	39	662	164	41	703	174	42	745	185	86
5		683	134	46	729	144	48	777	153	49	826	163	51	877	173	53	930	183	85
6		817	134	56	873	143	57	930	152	59	989	162	61	1050	172	63	1113	182	84
7		951	133	65	1016	141	66	1082	151	69	1151	161	71	1222	170	73	1295	181	83
8		1084	131	73	1157	140	76	1233	149	79	1312	158	80	1392	169	84	1476	178	82
9		1215	130	82	1297	139	85	1382	148	[illegible]	1470	157	91	1561	167	93	1654	177	81
10		1345	128	91	1436	137	[illegible]	1530	146	97	1627	155	101	1728	164	103	1831	174	[illegible]
11		1473	126	100	1573	135	103	1676	144	106	1782	153	110	1892	162	113	2005	172	79
12		1599	124	109	1708	132	112	1820	141	115	1935	151	119	2054	160	123	2177	170	78
13		1723	123	117	1840	131	121	1961	139	125	2086	148	128	2214	157	133	2347	166	77
14		1846	120	125	1971	128	129	2100	137	134	2234	145	137	2371	154	142	2513	163	76
15		1966	117	133	2099	126	138	2237	134	142	2379	142	146	2525	152	151	2676	161	75
16		2083	115	142	2225	122	146	2371	130	150	2521	139	156	2677	147	160	2837	156	74
17		2198	113	149	2347	120	154	2501	128	159	2660	136	164	2824	145	169	2993	153	73
18		2311	109	156	2467	117	162	2629	125	167	2796	133	173	2969	140	177	3146	149	72
19		2420	107	164	2584	114	170	2754	121	175	2929	129	180	3109	137	186	3295	145	71
20		2527	103	171	2698	111	177	2875	118	183	3058	125	188	3246	133	194	3440	141	**70**
21		2630	100	179	2809	107	184	2993	114	190	3183	122	196	3379	129	202	3581	137	69
22		2730	97	186	2916	103	191	3107	111	198	3305	117	203	3508	125	210	3718	132	68
23		2827	94	192	3019	100	199	3218	106	204	3422	113	211	3633	120	217	3850	127	67
24		2921	90	198	3119	96	205	3324	102	211	3535	109	218	3753	115	224	3977	123	66
25		3011	86	204	3215	92	211	3426	98	218	3644	104	224	3868	111	232	4100	117	65
26		3097	83	210	3307	88	217	3524	94	224	3748	100	231	3979	106	238	4217	112	64
27		3180	78	215	3395	84	223	3618	90	230	3848	95	237	4085	101	244	4329	107	63
28		3258	75	221	3479	80	229	3708	85	235	3943	91	243	4186	96	250	4436	102	62
29		3333	71	226	3559	75	234	3793	80	241	4034	85	248	4282	91	256	4538	96	61
30		3404	66	230	3634	71	239	3873	76	246	4119	81	254	4373	85	261	4634	91	[illegible]
31		3470	62	235	3705	67	244	3949	70	251	4200	75	258	4458	80	267	4725	84	59
32		3532	58	240	3772	62	247	4019	66	256	4275	70	263	4538	74	271	4809	79	58
33		3590	54	244	3834	57	251	4085	61	260	4345	65	267	4612	69	276	4888	73	57
34		3644	49	247	3891	52	255	4146	56	264	4410	59	271	4681	63	280	4961	67	56
35		3693	44	250	3943	48	259	4202	51	267	4469	54	275	4744	57	284	5028	60	55
36		3737	40	254	3991	43	262	4253	45	270	4523	48	278	4801	52	287	5088	55	54
37	3	3777	36	257	4034	37	264	4298	40	273	4571	43	282	4853	45	290	5143	48	53
38	3	3813	30	258	4071	33	267	4338	35	276	4614	37	284	4898	40	293	5191	42	52
39	3	3843	27	261	4104	28	269	4373	30	278	4651	32	287	4938	33	295	5233	35	51
40	3	3870	21	262	4132	23	271	4403	24	280	4683	26	288	4971	27	297	5268	29	**50**
41	3	3891	17	264	4155	18	272	4427	19	282	4709	20	289	4998	22	299	5297	23	49
42	3	3908	11	265	4173	12	273	4446	14	263	4729	14	291	5020	15	300	5320	16	48
43	3	3919	7	266	4185	8	275	4460	8	283	4743	9	292	5035	9	301	5336	9	47
44	3	3926	3	267	4193	2	275	4468	2	284	4752	2	292	5044	3	301	5345	3	46
45	3	3929		266	4195		275	4470		284	4754		293	5047		301	5348		45
δ / $a - a_0$	$\frac{v}{2}$	3°.0		hor. diff.	3°.1		hor. diff.	3°.2		hor. diff.	3°.3		hor. diff.	3°.4		hor. diff.	3°.5		d \ $a - a_0$

$$f(a + nw) = f(a) \pm n\left[f'(a) \pm n\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^{\circ}.$$

***d* is always numerically larger than δ.** **0°.00001 = 0".036 = unit.**

δ \ a−a₀	$\frac{v}{2}$	3°.5		*hor. diff.*	3°.6		*hor. diff.*	3°.7		*hor. diff.*	3°.8		*hor. diff.*	3°.9		*hor. diff.*	4°.0		a−a₀ / d
0		000		00	000		00	000		00	000		00	000		00	000		90
1		187	187	11	198	198	11	209	209	11	220	220	12	232	232	12	244	244	89
2		373	186	22	395	197	22	417	208	23	440	220	24	464	232	24	488	244	88
3		560	187	32	592	197	33	625	208	35	660	220	35	695	231	36	731	243	87
4		745	185	43	788	196	45	833	208	46	879	219	47	926	231	48	974	243	86
5		930	185	54	984	196	55	1039	206	57	1096	217	59	1155	229	60	1215	241	85
6		1113	183	65	1178	194	66	1244	205	68	1312	216	71	1383	228	71	1454	239	84
7		1295	182	75	1370	192	78	1448	204	79	1527	215	82	1609	226	83	1692	238	83
8		1476	181	85	1561	191	88	1649	201	91	1740	213	93	1833	224	95	1928	236	82
9		1654	178	96	1750	189	99	1849	200	102	1951	211	104	2055	222	107	2162	234	81
10		1831	177	106	1937	187	110	2047	198	112	2159	208	115	2274	219	118	2392	230	80
11		2005	174	117	2122	185	119	2241	194	123	2364	205	127	2491	217	129	2620	228	79
12		2177	172	127	2304	182	130	2434	193	133	2567	203	137	2704	213	141	2845	225	78
13		2347	170	136	2483	179	140	2623	189	144	2767	200	148	2915	211	151	3066	221	77
14		2513	166	146	2659	176	150	2809	186	154	2963	196	158	3121	206	163	3284	218	76
15		2676	163	156	2832	173	160	2992	183	164	3156	193	168	3324	203	173	3497	213	75
16		2837	161	164	3001	169	170	3171	179	173	3344	188	179	3523	199	183	3706	209	74
17		2993	156	174	3167	166	179	3346	175	183	3529	185	189	3718	195	193	3911	205	73
18		3146	153	183	3329	162	188	3517	171	193	3710	181	198	3908	190	203	4111	200	72
19		3295	149	192	3487	158	196	3683	166	202	3885	175	208	4093	185	213	4306	195	71
20		3440	145	200	3640	153	205	3845	162	211	4056	171	217	4273	180	222	4495	189	70
21		3581	141	208	3789	149	214	4003	158	220	4223	167	225	4448	175	231	4679	184	69
22		3718	137	216	3934	145	222	4156	153	228	4384	161	234	4618	170	240	4858	179	68
23		3850	132	223	4073	139	230	4303	147	236	4539	155	243	4782	164	248	5030	172	67
24	3	3977	127	231	4208	135	237	4445	142	244	4689	150	251	4940	158	257	5197	167	66
25	3	4100	123	238	4338	130	244	4582	137	252	4834	145	258	5092	152	265	5357	160	65
26	3	4217	117	245	4462	124	251	4713	131	259	4972	138	265	5237	145	273	5510	153	64
27	3	4329	112	252	4581	119	258	4839	126	265	5104	132	273	5377	140	280	5657	147	63
28	3	4436	107	258	4694	113	265	4959	120	272	5231	127	279	5510	133	286	5796	139	62
29	3	4538	102	263	4801	107	271	5072	113	278	5350	119	286	5636	126	293	5929	133	61
30	3	4634	96	269	4903	102	277	5180	108	284	5464	114	291	5755	119	300	6055	126	60
31	3	4725	91	274	4999	96	282	5281	101	289	5570	106	298	5868	113	305	6173	118	59
32	3	4809	84	279	5088	89	287	5375	94	295	5670	100	303	5973	105	310	6283	110	58
33	3	4888	79	284	5172	84	291	5463	88	300	5763	93	307	6070	97	316	6386	103	57
34	3	4961	73	288	5249	77	296	5545	82	304	5849	86	312	6161	91	320	6481	95	56
35	3	5028	67	291	5319	70	300	5619	74	308	5927	78	317	6244	83	324	6568	87	55
36	3	5088	60	295	5383	64	304	5687	68	312	5999	72	320	6319	75	329	6648	80	54
37	3	5143	55	298	5441	58	307	5748	61	315	6063	64	324	6387	68	332	6719	71	53
38	3	5191	48	301	5492	51	310	5802	54	318	6120	57	326	6446	59	335	6781	62	52
39	3	5233	42	303	5536	44	312	5848	46	321	6169	49	329	6498	52	338	6836	55	51
40	3	5268	35	306	5574	38	314	5888	40	323	6211	42	331	6542	44	340	6882	46	50
41	3	5297	29	307	5604	30	316	5920	32	325	6245	34	333	6578	36	342	6920	38	49
42	4	5320	23	308	5628	24	318	5946	26	326	6272	27	334	6606	28	344	6950	30	48
43	4	5336	16	309	5645	17	318	5963	17	327	6290	18	336	6626	20	345	6971	21	47
44	4	5345	9	310	5655	10	319	5974	11	328	6302	12	336	6638	12	345	6983	12	46
45	4	5348	3	311	5659	4	319	5978	4	327	6305	3	337	6642	4	345	6987	4	45
δ / a−a₀	$\frac{v}{2}$	3°.5		*hor. diff.*	3°.6		*hor. diff.*	3°.7		*hor. diff.*	3°.8		*hor. diff.*	3°.9		*hor. diff.*	4°.0		d / a−a₀

$$f(a \pm nw) = f(a) \pm n \left[f'(a) \pm n \frac{f''(a)}{2} \right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

B = $(d - \delta)^{\circ}$.

d is always numerically larger than δ. $0^{\circ}.00001 = 0''.036$ = unit.

δ \ $a - a_0$	$\frac{v}{2}$	4°.0		hor. diff.	4°.1		hor. diff.	4°.2		hor. diff.	4°.3		hor. diff.	4°.4		hor. diff.	4°.5		$a - a_0$ / d
0		000	244	00	000	256	00	000	269	00	000	282	00	000	296	00	000	309	90
1		244	244	12	256	257	13	269	269	13	282	282	14	296	295	13	309	309	89
2		488	243	25	513	255	25	538	268	26	564	281	27	591	294	27	618	308	88
3		731	243	37	768	255	38	806	268	39	845	280	40	885	294	41	926	307	87
4		974	241	49	1023	253	51	1074	265	51	1125	279	54	1179	291	54	1233	305	86
5		1215	239	61	1276	252	63	1339	265	65	1404	277	66	1470	291	68	1538	304	85
6		1454	238	74	1528	250	76	1604	262	77	1681	275	80	1761	288	81	1842	301	84
7		1692	236	86	1778	248	[redacted]	1866	260	90	1956	273	93	2049	285	94	2143	299	83
8		1928	234	98	2026	245	100	2126	258	103	2229	270	105	2334	283	108	2442	295	82
9		2162	230	109	2271	243	113	2384	254	115	2499	267	118	2617	279	120	2737	293	81
10		2392	228	122	2514	239	124	2638	252	128	2766	263	130	2896	276	134	3030	288	[redacted]
11		2620	225	133	2753	236	137	2890	247	139	3029	260	143	3172	272	146	3318	285	79
12		2845	221	144	2989	233	148	3137	244	152	3289	256	155	3444	268	159	3603	280	78
13		3066	218	156	3222	228	159	3381	240	164	3545	251	167	3712	263	171	3883	275	77
14		3284	213	166	3450	225	171	3621	235	175	3796	247	179	3975	258	183	4158	271	76
15		3497	209	178	3675	219	181	3856	231	187	4043	242	190	4233	253	196	4429	264	75
16	3	3706	205	188	3894	215	193	4087	226	198	4285	236	201	4486	248	207	4693	259	74
17	3	3911	200	198	4109	210	204	4313	220	208	4521	231	213	4734	242	218	4952	254	73
18	3	4111	195	208	4319	205	214	4533	215	219	4752	225	224	4976	236	230	5206	246	72
19	3	4306	189	218	4524	199	224	4748	209	229	4977	219	235	5212	230	240	5452	240	71
20	3	4495	184	228	4723	194	234	4957	203	239	5196	213	246	5442	222	250	5692	233	70
21	3	4679	179	238	4917	187	243	5160	197	249	5409	206	255	5664	216	261	5925	226	69
22	4	4858	172	246	5104	181	253	5357	190	258	5615	200	265	5880	209	271	6151	219	68
23	4	5030	167	255	5285	175	262	5547	183	268	5815	192	274	6089	201	281	6370	210	67
24	4	5197	160	263	5460	168	270	5730	177	277	6007	185	283	6290	194	290	6580	202	66
25	4	5357	153	271	5628	161	279	5907	169	285	6192	177	292	6484	185	298	6782	195	65
26	4	5510	147	279	5789	154	287	6076	162	293	6369	170	300	6669	178	308	6977	185	64
27	4	5657	139	286	5943	147	295	6238	154	301	6539	161	308	6847	169	315	7162	177	63
28	4	5796	133	294	6090	140	302	6392	146	308	6700	154	316	7016	161	323	7339	168	62
29	4	5929	126	301	6230	132	308	6538	138	316	6854	145	323	7177	151	330	7507	159	61
[redacted]	4	6055	118	307	6362	124	314	6676	130	323	6999	136	329	7328	143	338	7666	149	60
31	4	6173	110	313	6486	116	320	6806	122	329	7135	128	336	7471	134	344	7815	140	59
32	4	6283	103	319	6602	108	326	6928	114	335	7263	119	342	7605	125	350	7955	130	58
33	5	6386	95	324	6710	100	332	7042	105	340	7382	109	348	7730	115	355	8085	121	57
34	5	6481	87	329	6810	91	337	7147	96	344	7491	101	354	7845	105	361	8206	110	56
35	5	6568	80	333	6901	83	342	7243	87	349	7592	92	358	7950	96	366	8316	100	55
36	5	6648	71	336	6984	75	346	7330	78	354	7684	82	362	8046	86	370	8416	90	54
37	5	6719	62	340	7059	66	349	7408	69	358	7766	72	366	8132	76	374	8506	79	53
38	5	6781	55	344	7125	58	352	7477	61	361	7838	63	370	8208	66	377	8585	69	52
39	5	6836	46	347	7183	48	355	7538	51	363	7901	54	373	8274	56	380	8654	59	51
40	5	6882	38	349	7231	40	358	7589	41	366	7955	43	375	8330	45	383	8713	48	50
41	5	6920	30	351	7271	31	359	7630	33	368	7998	34	377	8375	36	386	8761	37	49
42	5	6950	21	352	7302	22	361	7663	23	369	8032	25	379	8411	25	387	8798	26	48
43	5	6971	12	353	7324	13	362	7686	14	371	8057	14	379	8436	15	388	8824	16	47
44	5	6983	4	354	7337	4	363	7700	4	371	8071	4	380	8451	5	389	8840	5	46
45	5	6987		354	7341		363	7704		371	8075		381	8456		389	8845		45
δ / $a - a_0$	$\frac{v}{2}$	4°.0		hor. diff.	4°.1		hor. diff.	4°.2		hor. diff.	4°.3		hor. diff.	4°.4		hor. diff.	4°.5		d \ $a - a_0$

$$f(a + nw) = f(a) \pm n\left[f'(a) \pm n\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

***d* is always numerically larger than δ.** 0°.00001 = 0″.036 = unit.

$a-a_0$ / δ	$\frac{v}{2}$	4°.5		hor. diff.	4°.6		hor. diff.	4°.7		hor. diff.	4°.8		hor. diff.	4°.9		hor. diff.	5°.0		$a-a_0$ / d
0		000		00	000		00	000		00	000		00	000		00	000		[illegible]
1		309	309	14	323	323	14	337	337	15	352	352	15	367	367	15	382	382	89
2		618	309	28	646	323	28	674	337	29	703	351	30	733	366	30	763	381	88
3		926	308	42	968	322	42	1010	336	44	1054	351	44	1098	365	46	1144	381	87
4		1233	307	55	1288	320	57	1345	335	58	1403	349	[illegible]	1462	364	61	1523	379	[illegible]
5		1538	305	70	1608	320	70	1678	333	73	1751	348	74	1825	363	75	1900	377	85
6		1842	304	83	1925	317	85	2010	332	86	2096	345	89	2185	360	90	2275	375	84
7		2143	301	97	2240	315	98	2338	328	101	2439	343	103	2542	357	105	2647	372	83
8		2442	299	110	2552	312	112	2664	326	115	2779	340	117	2896	354	120	3016	369	82
9		2737	295	124	2861	309	126	2987	323	128	3115	336	132	3247	351	134	3381	365	81
10		3030	293	136	3166	305	140	3306	319	142	3448	333	146	3594	347	148	3742	361	[illegible]
11	3	3318	288	150	3468	302	153	3621	315	156	3777	329	159	3936	342	163	4099	357	79
12	3	3603	285	162	3765	297	166	3931	310	169	4100	323	174	4274	338	176	4450	351	78
13	3	3883	280	175	4058	293	178	4236	305	183	4419	319	187	4606	332	190	4796	346	77
14	3	4158	275	188	4346	288	191	4537	301	196	4733	314	199	4932	326	204	5136	340	76
15	[illegible]	4429	271	199	4628	282	204	4832	295	208	5040	307	213	5253	321	217	5470	334	75
16	3	4693	264	212	4905	277	216	5121	289	221	5342	302	225	5567	314	230	5797	327	74
17	3	4952	259	224	5176	271	228	5404	283	233	5637	295	237	5874	307	243	6117	320	73
18	4	5206	254	234	5440	264	240	5680	276	245	5925	288	250	6175	301	255	6430	313	72
19	4	5452	246	246	5698	258	251	5949	269	256	6205	280	262	6467	292	267	6734	304	71
[illegible]	4	5692	240	257	5949	251	262	6211	262	[illegible]	6479	274	273	6752	285	279	7031	297	**70**
21	4	5925	233	267	6192	243	273	6465	254	279	6744	265	284	7028	276	291	7319	288	69
22	4	6151	226	277	6428	236	283	6711	246	290	7001	257	295	7296	268	302	7598	279	68
23	4	6370	219	286	6656	228	294	6950	239	299	7249	248	306	7555	259	312	7867	269	67
24	[illegible]	6580	210	296	6876	220	303	7179	229	310	7489	240	316	7805	250	322	8127	260	66
25	5	6782	202	306	7088	212	312	7400	221	319	7719	230	326	8045	240	332	8377	250	65
26	5	6977	195	314	7291	203	321	7612	212	328	7940	221	335	8275	230	342	8617	240	64
27	5	7162	185	323	7485	194	329	7814	202	337	8151	211	344	8495	220	351	8846	229	63
28	5	7339	177	331	7670	185	337	8007	193	346	8353	202	352	8705	210	360	9065	219	62
29	5	7507	168	338	7845	175	346	8191	184	353	8544	191	360	8904	199	368	9272	207	61
30	5	7666	159	345	8011	166	353	8364	173	360	8724	180	368	9092	188	376	9468	196	**60**
31	5	7815	149	352	8167	156	360	8527	163	367	8894	170	375	9269	177	383	9652	184	59
32	5	7955	140	358	8313	146	366	8679	152	374	9053	159	382	9435	166	390	9825	173	58
33	6	8085	130	364	8449	136	372	8821	142	380	9201	148	389	9590	155	396	9986	161	57
34	6	8206	121	369	8575	126	378	8953	132	385	9338	137	394	9732	142	402	10134	148	56
35	6	8316	110	374	8690	115	383	9073	120	391	9464	126	399	9863	131	407	10270	136	55
36	6	8416	100	379	8795	105	387	9182	109	396	9578	114	403	9981	118	413	10394	124	54
37	6	8506	90	383	8889	94	391	9280	98	400	9680	102	408	10088	107	417	10505	111	53
38	6	8585	79	387	8972	83	395	9367	87	403	9770	90	412	10182	94	421	10603	98	52
39	6	8654	69	390	9044	72	398	9442	75	407	9849	79	415	10264	82	424	10688	85	51
40	6	8713	59	392	9105	61	401	9506	64	409	9915	66	418	10333	69	427	10760	72	[illegible]
41	6	8761	48	394	9155	50	403	9558	52	412	9970	55	420	10390	57	429	10819	59	49
42	6	8798	37	396	9194	39	404	9598	40	414	10012	42	422	10434	44	430	10864	45	48
43	6	8824	26	397	9221	27	406	9627	29	415	10042	30	423	10465	31	432	10897	33	47
44	6	8840	16	398	9238	17	406	9644	17	416	10060	18	424	10484	19	432	10916	19	46
[illegible]	6	8845	5	398	9243	5	407	9650	6	415	10065	5	424	10489	5	433	10922	6	45
δ / $a-a_0$	$\frac{v}{2}$	4°.5		hor. diff.	4°.6		hor. diff.	4°.7		hor. diff.	4°.8		hor. diff.	4°.9		hor. diff.	5°.0		d / $a-a_0$

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$B = (d - \delta)^{\circ}$.

d is always numerically larger than δ. **0°.00001 = 0".036 = unit.**

δ \ $a-a_0$	$\frac{v}{2}$	5°.0		hor. diff.	5°.1		hor. diff.	5°.2		hor. diff.	5°.3		hor. diff.	5°.4		hor. diff.	5°.5		$a-a_0$ / d
0		000	382	00	000	397	00	000	413	00	000	429	00	000	446	00	000	462	90
1		382	381	15	397	397	16	413	413	16	429	429	17	446	445	16	462	462	89
2		763	381	31	794	396	32	826	411	32	858	428	33	891	444	33	924	461	88
3		1144	379	46	1190	395	47	1237	411	49	1286	426	49	1335	442	50	1385	459	87
4		1523	377	62	1585	392	63	1648	408	64	1712	424	65	1777	441	67	1844	457	86
5		1900	375	77	1977	390	79	2056	405	80	2136	421	82	2218	437	83	2301	454	85
6		2275	372	92	2367	388	94	2461	403	96	2557	419	98	2655	434	100	2755	450	84
7		2647	369	108	2755	383	109	2864	399	112	2976	414	113	3089	431	116	3205	447	83
8		3016	365	122	3138	380	125	3263	395	127	3390	411	130	3520	426	132	3652	442	82
9		3381	361	137	3518	376	140	3658	391	143	3801	406	145	3946	421	148	4094	437	81
10	3	3742	357	152	3894	371	155	4049	385	158	4207	400	160	4367	416	164	4531	432	80
11	3	4099	351	166	4265	366	169	4434	381	173	4607	395	176	4783	410	180	4963	425	79
12	3	4450	346	181	4631	360	184	4815	374	187	5002	389	191	5193	404	195	5388	419	78
13	3	4796	340	195	4991	354	198	5189	368	202	5391	382	206	5597	397	210	5807	412	77
14	3	5136	334	209	5345	347	212	5557	361	216	5773	376	221	5994	390	225	6219	404	76
15	4	5470	327	222	5692	340	226	5918	354	231	6149	367	235	6384	381	239	6623	397	75
16	4	5797	320	235	6032	333	240	6272	346	244	6516	360	249	6765	374	255	7020	387	74
17	4	6117	313	248	6365	325	253	6618	338	258	6876	351	263	7139	365	268	7407	378	73
18	4	6430	304	260	6690	317	266	6956	330	271	7227	343	277	7504	355	281	7785	369	72
19	4	6734	297	273	7007	309	279	7286	320	284	7570	333	289	7859	346	295	8154	359	71
20	5	7031	288	285	7316	299	290	7606	312	297	7903	323	302	8205	336	308	8513	348	70
21	5	7319	279	296	7615	290	303	7918	301	308	8226	314	315	8541	325	320	8861	337	69
22	5	7598	269	307	7905	281	314	8219	292	321	8540	303	326	8866	314	332	9198	327	68
23	5	7867	260	319	8186	270	325	8511	281	332	8843	292	337	9180	304	345	9525	314	67
24	5	8127	250	329	8456	261	336	8792	271	343	9135	281	349	9484	291	355	9839	303	66
25	5	8377	240	340	8717	249	346	9063	259	353	9416	269	359	9775	280	367	10142	290	65
26	5	8617	229	349	8966	239	356	9322	248	363	9685	258	370	10055	268	377	10432	278	64
27	6	8846	219	359	9205	227	365	9570	236	373	9943	245	380	10323	254	387	10710	264	63
28	6	9065	207	367	9432	215	374	9806	224	382	10188	233	389	10577	242	397	10974	251	62
29	6	9272	196	375	9647	204	383	10030	212	391	10421	220	398	10819	229	406	11225	237	61
30	6	9468	184	383	9851	192	391	10242	200	399	10641	208	407	11048	215	414	11462	223	60
31	6	9652	173	391	10043	180	399	10442	187	407	10849	193	414	11263	201	422	11685	209	59
32	6	9825	161	398	10223	167	406	10629	173	413	11042	181	422	11464	188	430	11894	194	58
33	7	9986	148	404	10390	154	412	10802	161	421	11223	167	429	11652	173	436	12088	180	57
34	7	10134	136	410	10544	142	419	10963	147	427	11390	153	435	11825	158	443	12268	164	56
35	7	10270	124	416	10686	128	424	11110	134	433	11543	138	440	11983	144	449	12432	150	55
36	7	10394	111	420	10814	116	430	11244	120	437	11681	125	446	12127	129	455	12582	134	54
37	7	10505	98	425	10930	102	434	11364	106	442	11806	110	450	12256	115	460	12716	118	53
38	7	10603	85	429	11032	88	438	11470	92	446	11916	95	455	12371	99	463	12834	103	52
39	7	10688	72	432	11120	75	442	11562	77	449	12011	81	459	12470	84	467	12937	87	51
40	7	10760	59	435	11195	62	444	11639	64	453	12092	66	462	12554	68	470	13024	71	50
41	7	10819	45	438	11257	47	446	11703	50	455	12158	52	464	12622	54	473	13095	55	49
42	7	10864	33	440	11304	34	449	11753	35	457	12210	36	466	12676	37	474	13150	39	48
43	7	10897	19	441	11338	20	450	11788	21	458	12246	22	467	12713	23	476	13189	24	47
44	8	10916	6	442	11358	6	451	11809	6	459	12268	6	468	12736	6	477	13213	7	46
45	8	10922		442	11364		451	11815		459	12274		468	12742		478	13220		45
δ / $a-a_0$	$\frac{v}{2}$	5°.0		hor. diff.	5°.1		hor. diff.	5°.2		hor. diff.	5°.3		hor. diff.	5°.4		hor. diff.	5°.5		d / $a-a_0$

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^{\circ}.$$

d **is always numerically larger than** δ. $0^{\circ}.00001 = 0''.036 =$ **unit.**

δ \ $a-a_0$	$\frac{v}{2}$	5°.5		*hor. diff.*	5°.6		*hor. diff.*	5°.7		*hor. diff.*	5°.8		*hor. diff.*	5°.9		*hor. diff.*	6°.0		$a-a_0$ / d
0		000	462	00	000	479	00	000	497	00	000	514	00	000	532	00	000	551	**90**
1		462	462	17	479	479	18	497	496	17	514	514	18	532	532	19	551	550	89
2		924	461	34	958	478	35	993	495	35	1028	513	36	1064	531	37	1101	548	88
3		1385	459	51	1436	476	52	1488	493	53	1541	511	54	1595	528	54	1649	547	87
4		1844	457	68	1912	474	69	1981	491	71	2052	508	71	2123	526	73	2196	544	86
5		2301	454	85	2386	470	86	2472	488	88	2560	505	89	2649	523	91	2740	541	85
6		2755	450	101	2856	467	104	2960	484	105	3065	501	107	3172	519	109	3281	536	84
7		3205	447	118	3323	463	121	3444	479	122	3566	497	125	3691	514	126	3817	532	83
8	3	3652	442	134	3786	459	137	3923	475	140	4063	492	142	4205	509	144	4349	527	82
9	3	4094	437	151	4245	453	153	4398	470	157	4555	486	159	4714	503	162	4876	520	81
10	3	4531	432	167	4698	448	170	4868	464	173	5041	480	176	5217	497	179	5396	514	**80**
11	4	4963	425	183	5146	441	186	5332	457	189	5521	474	193	5714	490	196	5910	507	79
12	4	5388	419	199	5587	434	202	5789	450	206	5995	465	209	6204	482	213	6417	499	78
13	4	5807	412	214	6021	427	218	6239	442	221	6460	458	226	6686	474	230	6916	490	77
14	4	6219	404	229	6448	419	233	6681	434	237	6918	450	242	7160	466	246	7406	481	76
15	5	6623	397	244	6867	411	248	7115	426	253	7368	441	258	7626	455	261	7887	472	75
16	5	7020	387	258	7278	401	263	7541	416	268	7809	431	272	8081	446	278	8359	461	74
17	5	7407	378	272	7679	393	278	7957	406	283	8240	421	287	8527	436	293	8820	450	73
18	5	7785	369	287	8072	382	291	8363	397	298	8661	410	302	8963	424	307	9270	440	72
19	5	8154	359	300	8454	372	306	8760	385	311	9071	399	316	9387	413	323	9710	427	71
20	6	8513	348	313	8826	361	319	9145	374	325	9470	387	330	9800	401	337	10137	414	**70**
21	6	8861	337	326	9187	350	332	9519	363	338	9857	376	344	10201	389	350	10551	402	69
22	6	9198	327	339	9537	338	345	9882	350	351	10233	363	357	10590	375	363	10953	389	68
23	6	9525	314	350	9875	327	357	10232	338	364	10596	350	369	10965	363	377	11342	374	67
24	7	9839	303	363	10202	313	368	10570	325	376	10946	336	382	11328	348	388	11716	360	66
25	7	10142	290	373	10515	301	380	10895	312	387	11282	323	394	11676	334	400	12076	346	65
26	7	10432	278	384	10816	288	391	11207	298	398	11605	308	405	12010	319	412	12422	330	64
27	7	10710	264	394	11104	274	401	11505	284	408	11913	294	416	12329	304	423	12752	314	63
28	7	10974	251	404	11378	260	411	11789	269	418	12207	279	426	12633	289	433	13066	299	62
29	7	11225	237	413	11638	246	420	12058	255	428	12486	264	436	12922	273	443	13365	282	61
30	8	11462	223	422	11884	231	429	12313	240	437	12750	248	445	13195	257	452	13647	266	**60**
31	8	11685	209	430	12115	216	438	12553	224	445	12998	232	454	13452	240	461	13913	248	59
32	8	11894	194	437	12331	202	446	12777	209	453	13230	216	462	13692	223	469	14161	231	58
33	8	12088	180	445	12533	186	453	12986	192	460	13446	200	469	13915	207	477	14392	214	57
34	8	12268	164	451	12719	171	459	13178	177	468	13646	183	476	14122	189	484	14606	196	56
35	8	12432	150	458	12890	154	465	13355	160	474	13829	166	482	14311	172	491	14802	177	55
36	8	12582	134	462	13044	139	471	13515	144	480	13995	149	488	14483	154	496	14979	159	54
37	8	12716	118	467	13183	123	476	13659	128	485	14144	132	493	14637	136	501	15138	141	53
38	8	12834	103	472	13306	107	481	13787	110	489	14276	114	497	14773	118	506	15279	123	52
39	8	12937	87	476	13413	90	484	13897	93	493	14390	97	501	14891	100	511	15402	103	51
40	8	13024	71	479	13503	74	487	13990	77	497	14487	79	504	14991	82	514	15505	85	**50**
41	8	13095	55	482	13577	57	490	14067	59	499	14566	61	507	15073	63	517	15590	65	49
42	8	13150	39	484	13634	40	492	14126	42	501	14627	43	509	15136	45	519	15655	46	48
43	8	13189	24	485	13674	24	494	14168	25	502	14670	26	511	15181	27	520	15701	28	47
44	9	13213	7	485	13698	7	495	14193	7	503	14696	7	512	15208	8	521	15729	8	46
45	9	13220		485	13705		495	14200		503	14703		513	15216		521	15737		45
δ / $a-a_0$	$\frac{v}{2}$	5°.5		*hor. diff.*	5°.6		*hor. diff.*	5°.7		*hor. diff.*	5°.8		*hor. diff.*	5°.9		*hor. diff.*	6°.0		d \ $a-a_0$

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\,\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

B $(d-\delta)^\circ$.

d is always numerically larger than δ. $0^\circ.00001 - 0^\circ.036$ – unit.

$\alpha-\alpha_0$ / δ	$\frac{v}{2}$	6°.0		hor. diff.	6°.1		hor. diff.	6°.2		hor. diff.	6°.3		hor. diff.	6°.4		hor. diff.	6°.5		$\alpha-\alpha_0$ / d
0°		000	551	00	000	569	00	000	588	00	000	607	00	000	627	00	000	647	90
1		551	550	18	569	569	19	588	588	19	607	607	20	627	626	20	647	646	89
2		1101	548	37	1138	567	38	1176	586	38	1214	605	39	1253	625	40	1293	644	88
3		1649	547	56	1705	565	57	1762	584	57	1819	603	59	1878	622	59	1937	642	87
4		2196	544	74	2270	563	76	2346	581	76	2422	600	78	2500	620	79	2579	639	86
5		2740	541	93	2833	559	94	2927	577	95	3022	597	98	3120	615	98	3218	635	85
6		3281	536	111	3392	554	112	3504	573	115	3619	592	116	3735	611	118	3853	631	84
7	3	3817	532	129	3946	550	131	4077	568	134	4211	586	135	4346	605	138	4484	624	83
8	3	4349	527	147	4496	544	149	4645	563	152	4797	581	154	4951	600	157	5108	619	82
9	4	4876	520	164	5040	538	168	5208	556	170	5378	574	173	5551	593	176	5727	611	81
10	4	5396	514	182	5578	532	186	5764	549	188	5952	567	192	6144	585	194	6338	604	80
11	4	5910	507	200	6110	524	203	6313	541	206	6519	559	210	6729	576	213	6942	595	79
12	4	6417	499	217	6634	515	220	6854	533	224	7078	550	227	7305	568	232	7537	585	78
13	4	6916	490	233	7149	507	238	7387	523	241	7628	541	245	7873	558	249	8122	576	77
14	5	7406	481	250	7656	498	254	7910	514	259	8169	531	262	8431	548	267	8698	565	76
15	5	7887	472	267	8154	487	270	8424	504	276	8700	520	279	8979	537	284	9263	554	75
16	6	8359	461	282	8641	477	287	8928	492	292	9220	508	296	9516	525	301	9817	542	74
17	6	8820	450	298	9118	465	302	9420	482	308	9728	497	313	10041	513	318	10359	529	73
18	6	9270	440	313	9583	454	319	9902	469	323	10225	484	329	10554	500	334	10888	515	72
19	7	9710	427	327	10037	442	334	10371	456	338	10709	471	345	11054	486	349	11403	502	71
20	7	10137	414	342	10479	429	348	10827	443	353	11180	458	360	11540	472	365	11905	487	70
21	7	10551	402	357	10908	415	362	11270	429	368	11638	443	374	12012	457	380	12392	472	69
22	7	10953	389	370	11323	401	377	11699	415	382	12081	428	388	12469	442	395	12864	456	68
23	8	11342	374	382	11724	387	390	12114	399	395	12509	413	402	12911	426	409	13320	439	67
24	8	11716	360	395	12111	373	402	12513	385	409	12922	397	415	13337	410	422	13759	423	66
25	8	12076	346	408	12484	357	414	12898	369	421	13319	381	428	13747	393	435	14182	406	65
26	8	12422	330	419	12841	341	426	13267	353	433	13700	364	440	14140	376	448	14588	387	64
27	9	12752	314	430	13182	325	438	13620	335	444	14064	347	452	14516	358	459	14975	369	63
28	9	13066	299	441	13507	309	448	13955	319	456	14411	329	463	14874	340	470	15344	351	62
29	9	13365	282	451	13816	291	458	14274	301	466	14740	311	474	15214	321	481	15695	331	61
30	9	13647	266	460	14107	275	468	14575	284	476	15051	293	484	15535	302	491	16026	311	60
31	9	13913	248	469	14382	257	477	14859	265	485	15344	274	493	15837	283	500	16337	292	59
32	10	14161	231	478	14639	239	485	15124	247	494	15618	255	502	16120	262	509	16629	271	58
33	10	14392	214	486	14878	220	493	15371	228	502	15873	235	509	16382	243	518	16900	250	57
34	10	14606	196	492	15098	203	501	15599	209	509	16108	216	517	16625	223	525	17150	230	56
35	10	14802	177	499	15301	183	507	15808	190	516	16324	195	524	16848	202	532	17380	208	55
36	10	14979	159	505	15484	165	514	15998	170	521	16519	176	531	17050	181	538	17588	187	54
37	10	15138	141	511	15649	145	519	16168	150	527	16695	155	536	17231	160	544	17775	165	53
38	10	15279	123	515	15794	126	524	16318	130	532	16850	135	541	17391	139	549	17940	143	52
39	10	15402	103	518	15920	107	528	16448	111	537	16985	114	545	17530	117	553	18083	121	51
40	10	15505	85	522	16027	88	532	16559	90	540	17099	93	548	17647	96	557	18204	99	50
41	10	15590	65	525	16115	67	534	16649	69	543	17192	71	551	17743	74	560	18303	77	49
42	10	15655	46	527	16182	48	536	16718	50	545	17263	51	554	17817	53	563	18380	54	48
43	11	15701	28	529	16230	28	538	16768	29	546	17314	30	556	17870	31	564	18434	32	47
44	11	15729	8	529	16258	9	539	16797	8	547	17344	9	557	17901	9	565	18466	9	46
45	11	15737		530	16267		538	16805		548	17353		557	17910		565	18475		45
δ / $\alpha-\alpha_0$	$\frac{v}{2}$	6°.0		hor. diff.	6°.1		hor. diff.	6°.2		hor. diff.	6°.3		hor. diff.	6°.4		hor. diff.	6°.5		d / $\alpha-\alpha_0$

$$f(a \pm nw) = f(a) + n\left[f'(a) \pm n\,\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$B = (d - \delta)^\circ.$

d is always numerically larger than δ. $0^\circ.00001 = 0''.036 =$ unit.

δ \ $a-a_0$	$\frac{v}{2}$	6°.5	diff.	*hor. diff.*	6°.6	diff.	*hor. diff.*	6°.7	diff.	*hor. diff.*	6°.8	diff.	*hor. diff.*	6°.9	diff.	*hor. diff.*	7°.0	diff.	$a-a_0$ / d
0		000	647	00	000	667	00	000	687	00	000	708	00	000	729	00	000	751	90
1		647	646	20	667	666	20	687	687	21	708	708	21	729	729	22	751	750	89
2		1293	644	40	1333	665	41	1374	686	42	1416	706	42	1458	727	43	1501	748	88
3		1937	642	61	1998	662	61	2059	683	63	2122	703	63	2185	724	64	2249	745	87
4		2579	639	81	2660	659	82	2742	679	83	2825	699	84	2909	720	85	2994	742	86
5		3218	635	101	3319	654	102	3421	674	103	3524	695	105	3629	716	107	3736	737	85
6		3853	631	120	3973	650	122	4095	670	124	4219	690	126	4345	711	128	4473	731	84
7	3	4484	624	139	4623	645	142	4765	664	144	4909	684	147	5056	704	148	5204	725	83
8	3	5108	619	160	5268	637	161	5429	657	164	5593	678	167	5760	697	169	5929	718	82
9	4	5727	611	178	5905	631	181	6086	650	185	6271	669	186	6457	690	190	6647	709	81
10	4	6338	604	198	6536	622	200	6736	642	204	6940	661	207	7147	680	209	7356	701	80
11	5	6942	595	216	7158	614	220	7378	632	223	7601	651	226	7827	671	230	8057	691	79
12	5	7537	585	235	7772	604	238	8010	623	242	8252	642	246	8498	661	250	8748	680	78
13	5	8122	576	254	8376	593	257	8633	612	261	8894	630	265	9159	649	269	9428	668	77
14	6	8698	565	271	8969	583	276	9245	600	279	9524	619	284	9808	637	288	10096	656	76
15	6	9263	554	289	9552	571	293	9845	588	298	10143	606	302	10445	624	307	10752	642	75
16	6	9817	542	306	10123	559	310	10433	576	316	10749	593	320	11069	611	325	11394	629	74
17	7	10359	529	323	10682	545	327	11009	563	333	11342	579	338	11680	596	343	12023	614	73
18	7	10888	515	339	11227	532	345	11572	547	349	11921	565	355	12276	582	361	12637	598	72
19	8	11403	502	356	11759	517	360	12119	533	367	12486	549	372	12858	566	377	13235	582	71
20	8	11905	487	371	12276	502	376	1265[illegible]	518	383	13035	533	389	13424	548	393	13817	565	70
21	8	12392	472	386	12778	486	392	13170	501	398	13568	516	404	13972	532	410	14382	547	69
22	8	12864	456	400	13264	470	407	13671	485	413	14084	500	420	14504	514	425	14929	529	68
23	9	13320	439	414	13734	454	422	14156	467	428	14584	481	434	15018	495	440	15458	510	67
24	9	13759	423	429	14188	436	435	14623	449	442	15065	462	448	15513	477	455	15968	491	66
25	9	14182	406	442	14624	418	448	15072	431	455	15527	444	463	15990	457	469	16459	470	65
26	10	14588	387	454	15042	399	461	15503	412	468	15971	424	476	16447	436	482	16929	449	64
27	10	14975	369	466	15441	381	474	15915	392	480	16395	404	488	16883	416	495	17378	428	63
28	10	15344	351	478	15822	361	485	16307	372	492	16799	384	500	17299	395	507	17806	407	62
29	11	15695	331	488	16183	341	496	16679	352	504	17183	362	511	17694	373	519	18213	384	61
30	11	16026	311	498	16524	322	507	17031	331	514	17545	341	522	18067	351	530	18597	361	60
31	11	16337	292	509	16846	300	516	17362	310	524	17886	319	532	18418	329	540	18958	338	59
32	11	16629	271	517	17146	280	526	17672	288	533	18205	297	542	18747	305	549	19296	314	58
33	11	16900	250	526	17426	258	534	17960	266	542	18502	274	550	19052	282	558	19610	291	57
34	11	17150	230	534	17684	237	542	18226	244	550	18776	251	558	19334	259	567	19901	266	56
35	11	17380	208	541	17921	214	549	18470	221	557	19027	228	566	19593	234	574	20167	241	55
36	11	17588	187	545	18135	193	556	18691	198	564	19255	204	572	19827	211	581	20408	216	54
37	12	17775	165	553	18328	170	561	18889	176	570	19459	181	579	20038	185	586	20624	192	53
38	12	17940	143	558	18498	148	567	19065	152	575	19640	156	583	20223	162	593	20816	166	52
39	12	18083	121	563	18646	124	571	19217	128	579	19796	133	589	20385	136	597	20982	140	51
40	12	18204	99	566	18770	102	575	19345	105	584	19929	108	592	20521	111	601	21122	114	50
41	12	18303	77	569	18872	79	578	19450	81	587	20037	83	595	20632	86	604	21236	88	49
42	12	18380	54	571	18951	56	580	19531	58	589	20120	59	598	20718	61	606	21324	63	48
43	12	18434	32	573	19007	33	582	19589	33	590	20179	35	600	20779	35	[illegible]	21387	36	47
44	12	18466	9	574	19040	9	582	19622	10	592	20214	10	600	20814	10	609	21423	11	46
45	12	18475		574	19049		583	19632		592	20224		600	20824		610	21434		45
δ / $a-a_0$	$\frac{v}{2}$	6°.5		*hor. diff.*	6°.6		*hor. diff.*	6°.7		*hor. diff.*	6°.8		*hor. diff.*	6°.9		*hor. diff.*	7°.0		d / $a-a_0$

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^{\circ}.$$

d is always numerically larger than δ. $0^{\circ}.00001 = 0''.036 =$ unit.

δ \ $a-a_0$	$\frac{v}{2}$	7°.0		*hor. diff.*	7°.1		*hor. diff.*	7°.2		*hor. diff.*	7°.3		*hor. diff.*	7°.4		*hor. diff.*	7°.5		$a-a_0$ / d
0		000	751	00	000	773	00	000	795	00	000	817	00	000	840	00	000	863	**90**
1		751	750	22	773	771	22	795	793	22	817	816	23	840	838	23	863	861	89
2		1501	748	43	1544	770	44	1588	792	45	1633	814	45	1678	837	46	1724	860	88
3		2249	745	65	2314	767	66	2380	789	67	2447	811	68	2515	834	69	2584	856	87
4		2994	742	87	3081	763	88	3169	785	89	3258	807	91	3349	829	91	3440	852	86
5		3736	737	108	3844	758	110	3954	780	111	4065	802	113	4178	824	114	4292	847	85
6	3	4473	731	129	4602	753	132	4734	774	133	4867	796	135	5002	818	137	5139	841	84
7	4	5204	725	151	5355	746	153	5508	767	155	5663	789	157	5820	811	160	5980	833	83
8	4	5929	718	172	6101	738	174	6275	760	177	6452	781	179	6631	803	182	6813	824	82
9	4	6647	709	192	9839	731	196	7035	751	198	7233	772	201	7434	793	203	7637	816	81
10	5	7356	701	214	7570	720	216	7786	741	219	8005	762	222	8227	784	226	8453	805	**80**
11	5	8057	691	233	8290	711	237	8527	731	240	8767	752	244	9011	772	247	9258	793	79
12	6	8748	680	253	9001	700	257	9258	720	261	9519	739	264	9783	760	268	10051	781	78
13	6	9428	668	273	9701	687	277	9978	707	280	10258	728	285	10543	748	289	10832	768	77
14	7	10096	656	292	10388	675	297	10685	694	301	10986	713	305	11291	733	309	11600	753	76
15	7	10752	642	311	11063	661	316	11379	680	320	11699	699	325	12024	718	329	12353	738	75
16	7	11394	629	330	11724	647	335	12059	665	339	12398	684	344	12742	703	349	13091	722	74
17	8	12023	614	348	12371	631	353	12724	649	358	13082	668	363	13445	687	368	13813	706	73
18	8	12637	598	365	13002	616	371	13373	634	377	13750	651	382	14132	668	387	14519	687	72
19	9	13235	582	383	13618	599	389	14007	616	394	14401	633	399	14800	651	406	15206	668	71
20	9	13817	565	400	14217	581	406	14623	597	411	15034	614	417	15451	632	423	15874	649	**70**
21	9	14382	547	416	14798	563	422	15220	579	428	15648	596	435	16083	612	440	16523	629	69
22	10	14929	529	432	15361	544	438	15799	560	445	16244	575	451	16695	591	457	17152	607	68
23	10	15458	510	447	15905	525	454	16359	540	460	16819	555	467	17286	570	473	17759	586	67
24	11	15968	491	462	16430	505	469	16899	519	475	17374	534	482	17856	548	489	18345	563	66
25	11	16459	470	476	16935	484	483	17418	497	490	17908	511	496	18404	526	504	18908	540	65
26	11	16929	449	490	17419	462	496	17915	476	504	18419	489	511	18930	502	518	19448	516	64
27	12	17378	428	503	17881	440	510	18391	453	517	18908	466	524	19432	479	532	19964	491	63
28	12	17806	407	515	18321	418	523	18844	430	530	19374	441	537	19911	454	544	20455	467	62
29	12	18213	384	526	18739	395	535	19274	406	541	19815	418	550	20365	429	557	20922	441	61
30	12	18597	361	537	19134	372	546	19680	382	553	20233	393	561	20794	403	569	21363	414	**60**
31	12	18958	338	548	19506	348	556	20062	357	564	20626	367	571	21197	378	580	21777	388	59
32	13	19296	314	558	19854	323	565	20419	333	574	20993	342	582	21575	351	590	22165	361	58
33	13	19610	291	567	20177	299	575	20752	307	583	21335	316	591	21926	325	600	22526	333	57
34	13	19901	266	575	20476	273	583	21059	281	592	21651	289	600	22251	297	608	22859	305	56
35	13	20167	241	582	20749	249	591	21340	256	600	21940	262	608	22548	269	616	23164	277	55
36	14	20408	216	590	20998	222	598	21596	228	606	22202	235	615	22817	242	624	23441	248	54
37	14	20624	192	596	21220	197	604	21824	203	613	22437	208	622	23059	213	630	23689	219	53
38	14	20816	166	601	21417	170	610	22027	175	618	22645	180	627	23272	185	636	23908	190	52
39	14	20982	140	605	21587	145	615	22202	148	623	22825	152	632	23457	157	641	24098	160	51
40	14	21122	114	610	21732	117	618	22350	121	627	22977	125	637	23614	127	644	24258	131	**50**
41	14	21236	88	613	21849	91	622	22471	93	631	23102	96	639	23741	99	648	24389	101	49
42	14	21324	63	616	21940	64	624	22564	66	634	23198	67	642	23840	69	650	24490	72	48
43	14	21387	36	617	22004	38	626	22630	39	635	23265	40	644	23909	40	653	24562	41	47
44	15	21423	11	619	22042	10	627	22669	10	636	23305	10	645	23949	11	654	24603	11	46
45	15	21434		618	22052		627	22679		636	23315		645	23960		654	24614		45
δ / $a-a_0$	$\frac{v}{2}$	7°.0		*hor. diff.*	7°.1		*hor. diff.*	7°.2		*hor. diff.*	7°.3		*hor. diff.*	7°.4		*hor. diff.*	7°.5		d \ $a-a_0$

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\,\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

***d* is always numerically larger than *δ*.** $0^\circ.00001 = 0''.036$ = unit.

$a-a_0$ / δ	$\frac{v}{2}$	7°.5		hor. diff.	7°.6		hor. diff.	7°.7		hor. diff.	7°.8		hor. diff.	7°.9		hor. diff.	8°.0		$a-a_0$ / d
0		000	863	00	000	886	00	000	910	00	000	934	00	000	958	00	000	983	90
1		863	861	23	886	885	24	910	908	24	934	932	24	958	957	25	983	981	89
2		1724	860	47	1771	883	47	1818	907	48	1866	930	49	1915	954	49	1964	979	88
3		2584	856	70	2654	879	71	2725	903	71	2796	927	73	2869	951	74	2943	975	87
4		3440	852	93	3533	875	95	3628	898	95	3723	922	97	3820	946	98	3918	970	86
5	3	4292	847	116	4408	870	118	4526	893	119	4645	917	121	4766	940	122	4888	965	85
6	4	5139	841	139	5278	863	141	5419	886	143	5562	909	144	5706	934	147	5853	957	84
7	4	5980	833	161	6141	856	164	6305	879	166	6471	902	169	6640	925	170	6810	949	83
8	5	6813	824	184	6997	847	187	7184	869	189	7373	892	192	7565	915	194	7759	939	82
9	5	7637	816	207	7844	837	209	8053	860	212	8265	883	215	8480	905	218	8698	928	81
10	5	8453	805	228	8681	827	232	8913	849	235	9148	871	237	9385	894	241	9626	917	80
11	6	9258	793	250	9508	815	254	9762	836	257	10019	858	260	10279	881	264	10543	903	79
12	6	10051	781	272	10323	802	275	10598	824	279	10877	845	283	11160	867	286	11446	890	78
13	7	10832	768	293	11125	788	297	11422	809	300	11722	831	305	12027	852	309	12336	874	77
14	8	11600	753	313	11913	774	318	12231	794	322	12553	815	326	12879	837	331	13210	858	76
15	8	12353	738	334	12687	758	338	13025	779	343	13368	799	348	13716	819	352	14068	840	75
16	9	13091	722	354	13445	742	359	13804	761	363	14167	781	368	14535	802	373	14908	822	74
17	9	13813	706	374	14187	724	378	14565	743	383	14948	763	389	15337	783	393	15730	803	73
18	10	14519	687	392	14911	706	397	15308	725	403	15711	744	409	16120	762	413	16533	782	72
19	10	15206	668	411	15617	686	416	16033	705	422	16455	723	427	16882	742	433	17315	761	71
20	11	15874	649	429	16303	666	435	16738	684	440	17178	702	446	17624	720	452	18076	739	70
21	11	16523	629	446	16969	646	453	17422	662	458	17880	680	464	18344	698	471	18815	715	69
22	11	17152	607	463	17615	624	469	18084	641	476	18560	657	482	19042	674	488	19530	692	68
23	12	17759	586	480	18239	601	486	18725	617	492	19217	634	499	19716	650	506	20222	666	67
24	12	18345	563	495	18840	578	502	19342	594	509	19851	609	515	20366	625	522	20888	641	66
25	12	18908	540	510	19418	555	518	19936	569	524	20460	584	531	20991	599	538	21529	615	65
26	13	19448	516	525	19973	530	532	20505	544	539	21044	558	546	21590	573	554	22144	587	64
27	13	19964	491	539	20503	504	546	21049	518	553	21602	532	561	22163	545	568	22731	559	63
28	13	20455	467	552	21007	479	560	21567	492	567	22134	504	574	22708	518	582	23290	531	62
29	14	20922	441	564	21486	453	573	22059	464	579	22638	477	588	23226	489	595	23821	501	61
30	14	21363	414	576	21939	426	584	22523	437	592	23115	448	600	23715	460	607	24322	472	60
31	14	21777	388	588	22365	398	595	22960	409	603	23563	420	612	24175	430	619	24794	441	59
32	15	22165	361	598	22763	370	606	23369	380	614	23983	390	622	24605	400	630	25235	410	58
33	15	22526	333	607	23133	342	616	23749	351	624	24373	360	632	25005	369	640	25645	379	57
34	15	22859	305	616	23475	313	625	24100	321	633	24733	330	641	25374	339	650	26024	347	56
35	15	23164	277	624	23788	285	633	24421	292	642	25063	299	650	25713	306	658	26371	314	55
36	16	23441	248	632	24073	254	640	24713	261	649	25362	268	657	26019	275	666	26685	282	54
37	16	23689	219	638	24327	225	647	24974	231	656	25630	237	664	26294	243	673	26967	249	53
38	16	23908	190	644	24552	195	653	25205	200	662	25867	205	670	26537	211	679	27216	216	52
39	16	24098	160	649	24747	165	658	25405	169	667	26072	174	676	26748	177	684	27432	182	51
40	16	24258	131	654	24912	134	662	25574	138	672	26246	141	679	26925	145	689	27614	149	50
41	16	24389	101	657	25046	104	666	25712	107	675	26387	109	683	27070	112	693	27763	114	49
42	16	24490	72	660	25150	73	669	25819	74	677	26496	77	[illegible]	27182	79	695	27877	81	48
43	16	24562	41	661	25223	42	670	25893	44	680	26573	44	688	27261	45	697	27958	46	47
44	16	24603	11	662	25265	12	672	25937	11	680	26617	12	689	27306	12	698	28004	12	46
45	16	24614		663	25277		671	25948		681	26629		689	27318		698	28016		45
δ / $a-a_0$	$\frac{v}{2}$	7°.5		hor. diff.	7°.6		hor. diff.	7°.7		hor. diff.	7°.8		hor. diff.	7°.9		hor. diff.	8°.0		d / $a-a_0$

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^{c}.$$

d is always numerically larger than *δ* — 0°.00001 = 0″.036 = unit.

δ \ a − a₀	v/2	8°.0	hor. diff.	8°.1	hor. diff.	8°.2	hor. diff.	8°.3	hor. diff.	8°.4	hor. diff.	8°.5	a − a₀ / d
0		000	00	0000	00	0000	00	0000	00	0000	00	0000	90
1		983	25	1008	25	1033	25	1058	26	1084	26	1110	89
2		1964	50	2014	50	2064	51	2115	52	2167	52	2219	88
3		2943	74	3017	76	3093	77	3170	77	3247	79	3326	87
4	3	3918	100	4018	100	4118	102	4220	103	4323	105	4428	86
5	3	4888	125	5013	125	5138	127	5265	129	5394	131	5525	85
6	4	5853	148	6001	151	6152	152	6304	154	6458	156	6614	84
7	[illegible]	6810	173	6983	175	7158	177	7335	179	7514	182	7696	83
8	5	7759	197	7956	199	8155	202	8357	204	8561	207	8768	82
9	6	8698	221	8919	223	9142	226	9368	229	9597	232	9829	81
10	6	9626	244	9870	248	10118	250	10368	254	10622	256	10878	[illegible]
11	[illegible]	10543	267	10810	271	11081	274	11355	278	11633	281	11914	79
12	8	11446	291	11737	294	12031	297	12328	302	12630	305	12935	78
13	8	12336	313	12649	316	12965	321	13286	325	13611	329	13940	77
14	9	13210	335	13545	339	13884	344	14228	347	14575	353	14928	76
15	9	14068	356	14424	362	14786	365	15151	371	15522	375	15897	75
16	10	14908	378	15286	383	15669	388	16057	392	16449	397	16846	74
17	10	15730	399	16129	404	16533	409	16942	414	17356	419	17775	73
18	11	16533	419	16952	425	17377	429	17806	435	18241	441	18682	72
19	12	17315	439	17754	445	18199	450	18649	455	19104	462	19566	71
20	12	18076	458	18534	464	18998	470	19468	476	19944	481	20425	70
21	13	18815	477	19292	483	19775	488	20263	495	20758	509	21260	69
22	13	19530	495	20025	501	20526	508	21034	514	21548	520	22068	68
23	13	20222	512	20734	519	21253	525	21778	532	22310	539	22849	67
24	14	20888	530	21418	535	21953	543	22496	549	23045	556	23601	66
25	14	21529	546	22075	552	22627	559	23186	566	23752	573	24325	65
26	15	22144	560	22704	568	23272	575	23847	583	24430	589	25019	64
27	15	22731	575	23306	583	23889	590	24479	598	25077	605	25682	63
28	15	23290	590	23880	597	24477	605	25082	612	25694	619	26313	62
29	16	23821	603	24424	610	25034	619	25653	626	26279	633	269[illegible]2	61
30	16	24322	616	24938	623	25561	631	26192	639	26831	647	27478	60
31	16	24794	627	25421	636	26057	643	26700	651	27351	659	28010	59
32	17	25235	638	25873	647	26520	655	27175	662	27837	671	28508	58
33	17	25645	649	26294	657	26951	665	27616	673	28289	682	28971	57
34	17	26024	658	26682	667	27349	674	28023	684	28707	691	29398	56
35	18	26371	667	27038	675	27713	684	28397	692	29089	701	29790	55
36	18	26685	675	27360	683	28043	692	28735	700	29435	709	30144	54
37	18	26967	682	27649	690	28339	699	29038	708	29746	716	30462	53
38	18	27216	688	27904	697	28601	706	29307	713	30020	723	30743	52
39	18	27432	693	28125	702	28827	711	29538	719	30257	728	30985	51
40	18	27614	698	28312	706	29018	716	29734	724	30458	733	31191	50
41	18	27763	701	28464	710	29174	719	29893	728	30621	737	31358	49
42	18	27877	704	28581	713	29294	722	30016	731	30747	739	31486	48
43	18	27958	706	28664	714	29378	724	30102	733	30835	741	31576	47
44	18	28004	707	28711	716	29427	725	30152	733	30885	743	31628	46
45	18	28016	707	28723	717	29440	724	30164	734	30898	743	31641	45
δ / a − a₀	v/2	8°.0	hor. diff.	8°.1	hor. diff.	8°.2	hor. diff.	8°.3	hor. diff.	8°.4	hor. diff.	8°.5	d / a − a₀

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\,\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} \qquad \frac{1}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

d is always numerically larger than δ. $0^\circ.00001 = 0''.036 =$ unit.

$a-a_0$ \ δ	$\frac{v}{2}$	8°.5	Δ	hor. diff.	8°.6	Δ	hor. diff.	8°.7	Δ	hor. diff.	8°.8	Δ	hor. diff.	8°.9	Δ	hor. diff.	9°.0	Δ	$a-a_0$ / d
0°		0000		0	0000		0	0000		0	0000		0	0000		0	0000		90
			1110			1137			1164			1191			1218			1246	
1		1110		27	1137		27	1164		27	1191		27	1218		28	1246		89
			1109			1135			1162			1189			1217			1245	
2		2219		53	2272		54	2326		54	2380		55	2435		56	2491		88
			1107			1133			1160			1187			1214			1242	
3		3326		79	3405		81	3486		81	3567		82	3649		84	3733		87
			1102			1129			1155			1182			1210			1237	
4	3	4428		106	4534		107	4641		108	4749		110	4859		111	4970		86
			1097			1122			1149			1176			1203			1230	
5	3	5525		131	5656		134	5790		135	5925		137	6062		138	6200		85
			1089			1116			1142			1169			1196			1223	
6	4	6614		158	6772		160	6932		162	7094		164	7258		165	7423		84
			1082			1108			1134			1160			1187			1214	
7	5	7696		184	7880		186	8066		188	8254		191	8445		192	8637		83
			1072			1098			1124			1150			1176			1204	
8	6	8768		210	8978		212	9190		214	9404		217	9621		220	9841		82
			1061			1086			1112			1138			1165			1191	
9	7	9829		235	10064		238	10302		240	10542		244	10786		246	11032		81
			1049			1074			1099			1125			1151			1177	
10	7	10878		260	11138		263	11401		266	11667		270	11937		272	12209		**80**
			1036			1061			1086			1111			1136			1162	
11	8	11914		285	12199		288	12487		291	12778		295	13073		298	13371		79
			1021			1045			1070			1095			1120			1146	
12	9	12935		309	13244		313	13557		316	13873		320	14193		324	14517		78
			1005			1029			1053			1078			1102			1127	
13	9	13940		333	14273		337	14610		341	14951		344	15295		349	15644		77
			988			1011			1035			1059			1084			1108	
14	10	14928		356	15284		361	15645		365	16010		369	16379		373	16752		76
			969			992			1015			1039			1063			1088	
15	10	15897		379	16276		384	16660		389	17049		393	17442		398	17840		75
			949			973			996			1018			1042			1065	
16	11	16846		403	17249		407	17656		411	18067		417	18484		421	18905		74
			929			951			973			996			1019			1042	
17	12	17775		425	18200		429	18629		434	19063		440	19503		444	19947		73
			907			928			950			973			995			1018	
18	13	18682		446	19128		451	19579		457	20036		462	20498		467	20965		72
			884			905			926			947			969			991	
19	13	19566		467	20033		472	20505		478	20983		484	21467		489	21956		71
			859			880			901			922			943			964	
20	14	20425		488	20913		493	21406		499	21905		505	22410		510	22920		**70**
			835			854			874			894			915			936	
21	14	21260		507	21767		513	22280		519	22799		526	23325		531	23856		69
			808			827			847			867			886			907	
22	15	22068		526	22594		533	23127		539	23666		545	24211		552	24763		68
			781			800			818			837			857			876	
23	15	22849		545	23394		551	23945		558	24503		565	25068		571	25639		67
			752			770			789			807			825			844	
24	15	23601		563	24164		570	24734		576	25310		583	25893		590	26483		66
			724			741			758			776			794			812	
25	16	24325		580	24905		587	25492		594	26086		601	26687		608	27295		65
			694			710			727			744			761			778	
26	17	25019		596	25615		604	26219		611	26830		618	27448		625	28073		64
			663			679			694			710			726			743	
27	17	25682		612	26294		619	26913		627	27540		634	28174		642	28816		63
			631			646			662			677			693			708	
28	18	26313		627	26940		635	27575		642	28217		650	28867		657	29524		62
			599			613			627			642			656			671	
29	17	26912		641	27553		649	28202		657	28859		664	29523		672	30195		61
			566			580			593			606			621			635	
30	18	27478		655	28133		662	28795		670	29465		679	30144		686	30830		**60**
			532			545			558			571			583			596	
31	19	28010		668	28678		675	29353		683	30036		691	30727		699	31426		59
			498			509			521			533			546			558	
32	19	28508		679	29187		687	29874		695	30569		704	31273		711	31984		58
			463			474			485			496			507			518	
33	19	28971		690	29661		698	30359		706	31065		715	31780		722	32502		57
			427			437			447			458			468			479	
34	19	29398		700	30098		708	30806		717	31523		725	32248		733	32981		56
			392			401			410			419			428			438	
35	20	29790		709	30499		717	31216		726	31942		734	32676		743	33419		55
			354			362			371			380			389			397	
36	20	30144		717	30861		726	31587		735	32322		743	33065		751	33816		54
			318			326			333			340			348			356	
37	20	30462		725	31187		733	31920		742	32662		751	33413		759	34172		53
			281			287			294			301			307			314	
38	20	30743		731	31474		740	32214		749	32963		757	33720		766	34486		52
			242			248			254			260			266			272	
39	20	30985		737	31722		746	32468		755	33223		763	33986		772	34758		51
			206			210			215			219			224			229	
40	21	31191		741	31932		751	32683		759	33442		768	34210		777	34987		**50**
			167			171			174			179			183			187	
41	21	31358		745	32103		754	32857		764	33621		772	34393		781	35174		49
			128			132			135			137			140			143	
42	20	31486		749	32235		757	32992		766	33758		775	34533		784	35317		48
			90			92			94			96			99			101	
43	21	31576		751	32327		759	33086		768	33854		778	34632		786	35418		47
			52			53			54			55			56			57	
44	21	31628		752	32380		760	33140		769	33909		779	34688		787	35475		46
			13			13			14			14			13			14	
45	21	31641		752	32393		761	33154		769	33923		778	34701		788	35489		45
δ / $a-a_0$	$\frac{v}{2}$	8°.5		hor. diff.	8°.6		hor. diff.	8°.7		hor. diff.	8°.8		hor. diff.	8°.9		hor. diff.	9°.0		d \ $a-a_0$

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$B = (d - \delta)^{\circ}$.

d is always numerically larger than δ.

0°.00001 = 0".036 = unit.

$a - a_0$ / δ	$\frac{v}{2}$	9°.0		hor. diff.	9°.1		hor. diff.	9°.2		hor. diff.	9°.3		hor. diff.	9°.4		hor. diff.	9°.5		$a - a_0$ / d
0°		0000	1246	0	0000	1274	0	0000	1303	0	0000	1332	0	0000	1361	0	0000	1390	90
1		1246	1245	28	1274	1273	29	1303	1301	29	1332	1330	29	1361	1359	29	1390	1389	89
2		2491	1242	56	2547	1270	57	2604	1298	58	2662	1326	58	2720	1356	59	2779	1385	88
3		3733	1237	84	3817	1265	85	3902	1293	86	3988	1322	88	4076	1350	88	4164	1380	87
4	3	4970	1230	112	5082	1258	113	5195	1287	115	5310	1315	116	5426	1344	118	5544	1373	86
5	4	6200	1223	140	6340	1251	142	6482	1279	143	6625	1307	145	6770	1336	147	6917	1364	85
6	5	7423	1214	168	7591	1241	170	7761	1269	171	7932	1297	174	8106	1325	175	8281	1354	84
7	6	8637	1204	195	8832	1231	198	9030	1258	199	9229	1286	202	9431	1314	204	9635	1342	83
8	7	9841	1191	222	10063	1218	225	10288	1245	227	10515	1273	230	10745	1300	232	10977	1329	82
9	8	11032	1177	249	11281	1203	252	11533	1230	255	11788	1257	257	12045	1286	261	12306	1313	81
10	8	12209	1162	275	12484	1189	279	12763	1215	282	13045	1242	286	13331	1268	288	13619	1296	**80**
11	9	13371	1146	302	13673	1171	305	13978	1198	309	14287	1224	312	14599	1251	316	14915	1278	79
12	10	14517	1127	327	14844	1153	332	15176	1179	335	15511	1205	339	15850	1231	343	16193	1258	78
13	11	15644	1108	353	15997	1134	358	16355	1158	361	16716	1184	365	17081	1210	370	17451	1236	77
14	11	16752	1088	379	17131	1112	382	17513	1137	387	17900	1162	391	18291	1187	396	18687	1212	76
15	12	17840	1065	403	18243	1089	407	18650	1114	412	19062	1138	416	19478	1163	421	19899	1188	75
16	12	18905	1042	427	19332	1065	432	19764	1089	436	20200	1113	441	20641	1138	446	21087	1162	74
17	13	19947	1018	450	20397	1041	456	20853	1063	460	21313	1087	466	21779	1110	470	22249	1135	73
18	14	20965	991	473	21438	1013	478	21916	1036	484	22400	1059	489	22889	1082	495	23384	1105	72
19	14	21956	964	495	22451	986	501	22952	1008	507	23459	1030	512	23971	1053	518	24489	1075	71
20	15	22920	936	517	23437	957	523	23960	978	529	24489	1000	535	25024	1021	540	25564	1044	**70**
21	16	23856	907	538	24394	927	544	24938	948	551	25489	968	556	26045	990	563	26608	1010	69
22	16	24763	876	558	25321	896	565	25886	915	571	26457	935	578	27035	955	583	27618	977	68
23	17	25639	844	578	26217	863	584	26801	883	591	27392	902	598	27990	922	605	28595	941	67
24	18	26483	812	597	27080	830	604	27684	848	610	28294	867	618	28912	885	624	29536	904	66
25	18	27295	778	615	27910	795	622	28532	813	629	29161	831	636	29797	849	643	30440	867	65
26	18	28073	743	632	28705	760	640	29345	777	647	29992	793	654	30646	811	661	31307	829	64
27	19	28816	708	649	29465	724	657	30122	739	663	30785	756	672	31457	772	679	32136	788	63
28	19	29524	671	665	30189	686	672	30861	702	680	31541	717	688	32229	733	695	32924	749	62
29	20	30195	635	680	30875	649	688	31563	663	695	32258	678	704	32962	691	711	33673	706	61
30	20	30830	596	694	31524	609	702	32226	623	710	32936	636	717	33653	651	726	34379	665	**60**
31	20	31426	558	707	32133	571	716	32849	583	723	33572	596	732	34304	608	740	35044	621	59
32	21	31984	518	720	32704	530	728	33432	541	736	34168	553	744	34912	566	753	35665	577	58
33	21	32502	479	732	33234	489	739	33973	500	748	34721	511	757	35478	521	764	36242	533	57
34	21	32981	438	742	33723	448	750	34473	458	759	35232	468	767	35999	478	776	36775	488	56
35	22	33419	397	752	34171	406	760	34931	415	769	35700	424	777	36477	433	786	37263	442	55
36	22	33816	356	761	34577	364	769	35346	372	778	36124	379	786	36910	387	795	37705	395	54
37	22	34172	314	769	34941	320	777	35718	327	785	36503	335	794	37297	342	803	38100	350	53
38	22	34486	272	775	35261	278	784	36045	284	793	36838	290	801	37639	296	811	38450	302	52
39	23	34758	229	781	35539	234	790	36329	239	799	37128	244	807	37935	250	817	38752	254	51
40	23	34987	187	786	35773	191	795	36568	195	804	37372	199	813	38185	202	821	39006	207	**50**
41	23	35174	143	790	35964	146	799	36763	149	808	37571	152	816	38387	156	826	39213	159	49
42	23	35317	101	793	36110	103	802	36912	105	811	37723	107	820	38543	109	829	39372	111	48
43	23	35418	57	795	36213	58	804	37017	59	813	37830	61	822	38652	62	831	39483	63	47
44	23	35475	14	796	36271	14	805	37076	15	815	37891	14	823	38714	14	832	39546	14	46
45	23	35489		796	36285		806	37091		814	37905		823	38728		832	39560		45
δ / $a - a_0$	$\frac{v}{2}$	9°.0		hor. diff.	9°.1		hor. diff.	9°.2		hor. diff.	9°.3		hor. diff.	9°.4		hor. diff.	9°.5		d / $a - a_0$

$$f(a + nw) = f(a) \pm n\left[f'(a) + n\frac{f''(a)}{2}\right], \quad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE II.

$$B = (d - \delta)^\circ.$$

***d* is always numerically larger than δ.** 0°.00001 = 0″.036 = unit.

$\delta-\delta_0$ / δ	$\frac{v}{2}$	9°.5		*hor. diff.*	9°.6		*hor. diff.*	9°.7		*hor. diff.*	9°.8		*hor. diff.*	9°.9		*hor. diff.*	10°.0		$\delta-\delta_0$ / d
0		0000	1390	0	0000	1420	0	0000	1450	0	0000	1480	0	0000	1511	0	0000	1542	90
1		1390	1389	30	1420	1418	30	1450	1448	30	1480	1479	31	1511	1510	31	1542	1541	89
2		2779	1385	59	2838	1415	60	2898	1445	61	2959	1475	62	3021	1505	62	3083	1536	88
3	3	4164	1380	89	4253	1409	90	4343	1439	91	4434	1470	92	4526	1500	93	4619	1531	87
4	4	5544	1373	118	5662	1403	120	5782	1432	122	5904	1462	122	6026	1493	124	6150	1523	86
5	5	6917	1364	148	7065	1393	149	7214	1424	152	7366	1453	153	7519	1483	154	7673	1514	85
6	6	8281	1354	177	8458	1383	180	8638	1412	181	8819	1442	183	9002	1472	185	9187	1502	84
7	6	9635	1342	206	9841	1371	209	10050	1400	211	10261	1429	213	10474	1459	215	10689	1489	83
8	7	10977	1329	235	11212	1357	238	11450	1385	240	11690	1415	243	11933	1444	245	12178	1474	82
9	8	12306	1313	263	12569	1341	266	12835	1370	270	13105	1398	272	13377	1427	275	13652	1456	81
10	9	13619	1296	291	13910	1324	295	14205	1352	298	14503	1380	301	14804	1409	304	15108	1438	**80**
11	10	14915	1278	319	15234	1306	323	15557	1333	326	15883	1361	330	16213	1389	333	16546	1417	79
12	11	16193	1258	347	16540	1284	350	16890	1312	354	17244	1339	358	17602	1367	361	17963	1395	78
13	11	17451	1236	373	17824	1262	378	18202	1289	381	18583	1316	386	18969	1343	389	19358	1371	77
14	12	18687	1212	399	19086	1239	405	19491	1264	408	19899	1291	413	20312	1318	417	20729	1345	76
15	13	19899	1188	426	20325	1214	430	20755	1240	435	21190	1265	440	21630	1291	444	22074	1318	75
16	14	21087	1162	452	21539	1186	456	21995	1211	460	22455	1238	466	22921	1263	471	23392	1288	74
17	15	22249	1135	476	22725	1159	481	23206	1183	487	23693	1207	491	24184	1233	496	24680	1258	73
18	15	23384	1105	500	23884	1129	505	24389	1153	511	24900	1177	517	25417	1201	521	25938	1226	72
19	16	24489	1075	524	25013	1098	529	25542	1121	535	26077	1145	541	26618	1168	546	27164	1192	71
20	17	25564	1044	547	26111	1066	552	26663	1088	559	27222	1110	564	27786	1134	570	28356	1157	**70**
21	18	26608	1010	569	27177	1032	574	27751	1054	581	28332	1076	588	28920	1098	593	29513	1120	69
22	18	27618	977	591	28209	997	596	28805	1018	603	29408	1040	610	30018	1060	615	30633	1083	68
23	19	28595	941	611	29206	961	617	29823	982	625	30448	1001	630	31078	1023	638	31716	1043	67
24	20	29536	904	631	30167	924	638	30805	943	644	31449	963	652	32101	982	658	32759	1003	66
25	20	30440	867	651	31091	885	657	31748	904	664	32412	923	671	33083	941	679	33762	960	65
26	21	31307	829	669	31976	846	676	32652	863	683	33335	881	690	34025	900	697	34722	918	64
27	21	32136	788	686	32822	805	693	33515	823	701	34216	840	709	34925	856	715	35640	874	63
28	22	32924	749	703	33627	764	711	34338	780	718	35056	796	725	35781	813	733	36514	829	62
29	22	33673	706	718	34391	722	727	35118	736	734	35852	752	742	36594	767	749	37343	783	61
30	23	34379	665	734	35113	678	741	35854	693	750	36604	706	757	37361	721	765	38126	736	**60**
31	23	35044	621	747	35791	634	756	36547	647	763	37310	661	772	38082	674	780	38862	688	59
32	23	35665	577	760	36425	590	769	37194	602	777	37971	615	785	35756	627	794	39550	639	58
33	24	36242	533	773	37015	544	781	37796	555	790	38586	566	797	39383	579	806	40189	590	57
34	24	36775	488	784	37559	498	792	38351	509	801	39152	519	810	39962	529	817	40779	540	56
35	25	37263	442	794	38057	451	803	38860	460	811	39671	470	820	40491	479	828	41319	489	55
36	24	37705	395	803	38508	404	812	39320	413	821	40141	421	829	40970	429	838	41808	438	54
37	25	38100	350	812	38912	357	821	39733	363	829	40562	371	837	41399	379	847	42246	386	53
38	25	38450	302	819	39269	308	827	40096	315	837	40933	321	845	41778	327	854	42632	334	52
39	25	38752	254	825	39577	260	834	40411	265	843	41254	270	851	42105	276	861	42966	281	51
40	25	39006	207	831	39837	211	839	40676	215	848	41524	220	857	42381	224	866	43247	228	**50**
41	26	39213	159	835	40048	162	843	40891	166	853	41744	168	861	42605	172	870	43475	175	49
42	26	39372	111	838	40210	113	847	41057	115	855	41912	117	865	42777	119	873	43650	122	48
43	25	39483	63	840	40323	64	849	41172	65	857	42029	67	867	42896	68	876	43772	69	47
44	26	39546	14	841	40387	15	850	41237	15	859	42096	15	[illegible]	42964	15	877	43841	15	46
45	26	39560		842	40402		850	41252		859	42111		868	42979		877	43856		45
δ / $\delta-\delta_0$	$\frac{v}{2}$	9°.5		*hor. diff.*	9°.6		*hor. diff.*	9°.7		*hor. diff.*	9°.8		*hor. diff.*	9°.9		*hor. diff.*	10°.0		d / $\delta-\delta_0$

$$f(a \pm nw) = f(a) \pm n\left[f'(a) \pm n\,\frac{f''(a)}{2}\right], \qquad \frac{f''(a)}{2} = \frac{v}{2}.$$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

for $d - \delta_0 \leqq 1^\circ$, $0^\circ.000001 = 0''.0036 =$ unit; for $d - \delta_0 > 1^\circ$, $0^\circ.00001 = 0''.036 =$ unit.

$\alpha-\alpha_0$ / $d-\delta_0$	0°.0	*hor. diff.*	0°.1	*hor. diff.*	0°.2	*hor. diff.*	0°.3	*hor. diff.*	0°.4	*hor. diff.*	0°.5	$\alpha-\alpha_0$ / $d-\delta_0$
0°.0	0	0	0	1	1	2	3	4	7	6	13	0.0
			0		0		0		0		0	
.1		0	0	1	1	2	3	4	7	6	13	.1
			1		1		2		2		3	
.2		1	1	1	2	3	5	4	9	7	16	.2
			0		2		2		3		4	
.3		1	1	3	4	3	7	5	12	8	20	.3
			2		2		3		4		5	
.4		3	3	3	6	4	10	6	16	9	25	.4
			1		2		4		6		7	
0.5		4	4	4	8	6	14	8	22	10	32	0.5
			2		4		5		6		8	
.6		6	6	6	12	7	19	9	28	12	40	.6
			2		4		6		8		10	
.7		8	8	8	16	9	25	11	36	14	50	.7
			2		4		7		9		11	
.8		10	10	10	20	12	32	13	45	16	61	.8
			2		5		8		11		13	
.9		12	12	13	25	15	40	16	56	18	74	.9
			3		6		8		11		15	
1.0		15	15	16	31	17	48	19	67	22	89	1.0
			0		1		1		1		1	
1.1		2	2	2	4	2	6	2	8	2	10	1.1
			0		0		1		1		2	
1.2		2	2	2	4	3	7	2	9	3	12	1.2
			1		1		1		2		2	
1.3		3	3	2	5	3	8	3	11	3	14	1.3
			0		1		1		2		2	
1.4		3	3	2	6	3	9	4	13	3	16	1.4
			0		1		2		1		2	
1.5		3	3	4	7	4	11	3	14	4	18	1.5
			1		1		1		2		3	
1.6		4	4	4	8	4	12	4	16	5	21	1.6
			0		1		2		2		2	
1.7		4	4	5	9	5	14	4	18	5	23	1.7
			1		1		1		2		3	
1.8		5	5	5	10	5	15	5	20	6	26	1.8
			1		1		2		3		3	
1.9		6	6	5	11	6	17	6	23	6	29	1.9
			0		1		1		2		3	
2.0		6	6	6	12	6	18	7	25	7	32	2.0
			1		2		2		3		3	
2.1		7	7	7	14	6	20	8	28	7	35	2.1
			0		1		2		2		3	
2.2		7	7	8	15	7	22	8	30	8	38	2.2
			1		1		2		3		4	
2.3		8	8	8	16	8	24	9	33	9	42	2.3
			1		2		3		3		3	
2.4		9	9	9	18	9	27	9	36	9	45	2.4
			1		1		2		3		4	
2.5		10	10	9	19	10	29	10	39	10	49	2.5
			0		2		2		3		4	
2.6		10	10	11	21	10	31	11	42	11	53	2.6
			1		1		3		3		4	
2.7		11	11	11	22	12	34	11	45	12	57	2.7
			1		2		2		3		4	
2.8		12	12	12	24	12	36	12	48	13	61	2.8
			1		2		3		4		4	
2.9		13	13	13	26	13	39	13	52	13	65	2.9
			1		2		2		4		5	
3.0		14	14	14	28	13	41	15	56	14	70	3.0
			1		1		3		3		4	
3.1		15	15	14	29	15	44	15	59	15	74	3.1
			1		2		3		4		5	
3.2		16	16	15	31	16	47	16	63	16	79	3.2
			1		2		3		4		5	
3.3		17	17	16	33	17	50	17	67	17	84	3.3
			1		2		3		4		5	
3.4		18	18	17	35	18	53	18	71	18	89	3.4
			1		2		3		4		6	
3.5		19	19	18	37	19	56	19	75	20	95	3.5
			1		3		4		5		5	
3.6		20	20	20	40	20	60	20	80	20	100	3.6
			1		2		3		4		6	
3.7		21	21	21	42	21	63	21	84	22	106	3.7
			1		2		3		5		5	
3.8		22	22	22	44	22	66	23	89	22	111	3.8
			1		2		4		4		6	
3.9		23	23	23	46	24	70	23	93	24	117	3.9
			1		3		3		5		6	
4.0		24	24	25	49	24	73	25	98	25	123	4.0
			2		2		4		5		6	
4.1		26	26	25	51	26	77	26	103	26	129	4.1
			1		3		4		5		7	
4.2		27	27	27	54	27	81	27	108	28	136	4.2
			1		2		4		5		6	
4.3		28	28	28	56	29	85	28	113	29	142	4.3
			2		3		4		6		7	
4.4		30	30	29	59	30	89	30	119	30	149	4.4
			1		3		4		5		6	
4.5		31	31	31	62	31	93	31	124	31	155	4.5
			1		3		4		6		7	
4.6		32	32	33	65	32	97	33	130	32	162	4.6
			2		2		4		5		7	
4.7		34	34	33	67	34	101	34	135	34	169	4.7
			1		3		5		6		8	
4.8		35	35	35	70	36	106	35	141	36	177	4.8
			2		3		4		6		7	
4.9		37	37	36	73	37	110	37	147	37	184	4.9
			1		3		5		6		8	
5.0		38	38	38	76	39	115	38	153	39	192	5.0
$d-\delta_0$ / $\alpha-\alpha_0$	0°.0	*hor. diff.*	0°.1	*hor. diff.*	0°.2	*hor. diff.*	0°.3	*hor. diff.*	0°.4	*hor. diff.*	0°.5	$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

for $d - \delta_0 \leqq 1^\circ$, $0^\circ.000001 = 0''.0036$ = unit; for $d - \delta_0 > 1^\circ$, $0^\circ.00001 = 0''.036$ = unit.

$d-\delta_0$ \ $\alpha-\alpha_0$	0°.5		*hor. diff.*	0°.6		*hor. diff.*	0°.7		*hor. diff.*	0°.8		*hor. diff.*	0°.9		*hor. diff.*	1°.0		$\alpha-\alpha_0$ / $d-\delta_0$
0.0	13		9	22		13	35		17	52		22	74		28	102		0.0
.1	13	0	10	23	1	13	36	1	17	53	1	22	75	1	28	103	1	.1
.2	16	3	10	26	3	13	39	3	18	57	4	22	79	4	29	108	5	.2
.3	20	4	10	30	4	14	44	5	19	63	6	23	86	7	29	115	7	.3
.4	25	5	12	37	7	15	52	8	20	72	9	24	96	10	30	126	11	.4
0.5	32	7	13	45	8	16	61	9	21	82	10	26	108	12	32	140	14	0.5
.6	40	8	15	55	10	18	73	12	23	96	14	27	123	15	33	156	16	.6
.7	50	10	17	67	12	20	87	14	25	112	16	29	141	18	35	176	20	.7
.8	61	11	19	80	13	23	103	16	27	130	18	32	162	21	37	199	23	.8
.9	74	13	22	96	16	25	121	18	30	151	21	34	185	23	40	225	26	.9
1.0	89	15	24	113	17	28	141	20	33	174	23	37	211	26	43	254	29	**1.0**
1.1	10	1	3	13	2	3	16	2	4	20	3	4	24	3	5	29	4	1.1
1.2	12	2	3	15	2	4	19	3	4	23	3	4	27	3	5	32	3	1.2
1.3	14	2	4	18	3	4	22	3	4	26	3	5	31	4	5	36	4	1.3
1.4	16	2	4	20	2	4	24	2	5	29	3	5	34	3	6	40	4	1.4
1.5	18	2	5	23	3	5	28	4	5	33	4	5	38	4	6	44	4	1.5
1.6	21	3	5	26	3	5	31	3	5	36	3	6	42	4	7	49	5	1.6
1.7	23	2	6	29	3	5	34	3	6	40	4	7	47	5	7	54	5	1.7
1.8	26	3	6	32	3	6	38	4	7	45	5	7	52	5	8	60	6	1.8
1.9	29	3	6	35	3	7	42	4	7	49	4	8	57	5	8	65	5	1.9
2.0	32	3	7	39	4	7	46	4	8	54	5	8	62	5	9	71	6	**2.0**
2.1	35	3	7	42	3	8	50	4	9	59	5	9	68	6	9	77	6	2.1
2.2	38	3	8	46	4	9	55	5	9	64	5	10	74	6	10	84	7	2.2
2.3	42	4	9	51	5	9	60	5	10	70	6	10	80	6	11	91	7	2.3
2.4	45	3	10	55	4	10	65	5	11	76	6	10	86	6	12	98	7	2.4
2.5	49	4	10	59	4	11	70	5	12	82	6	11	93	7	12	105	7	2.5
2.6	53	4	11	64	5	12	76	6	12	88	6	12	100	7	13	113	8	2.6
2.7	57	4	12	69	5	12	81	5	13	94	6	13	107	7	14	121	8	2.7
2.8	61	4	13	74	5	13	87	6	14	101	7	14	115	8	15	130	9	2.8
2.9	65	4	14	79	5	14	93	6	15	108	7	15	123	8	15	138	8	2.9
3.0	70	5	14	84	5	15	99	6	16	115	7	16	131	8	16	147	9	**3.0**
3.1	74	4	16	90	6	16	106	7	16	122	7	17	139	8	18	157	10	3.1
3.2	79	5	17	96	6	17	113	7	17	130	8	18	148	9	18	166	9	3.2
3.3	84	5	18	102	6	18	120	7	18	138	8	19	157	9	19	176	10	3.3
3.4	89	5	19	108	6	19	127	7	19	146	8	20	166	9	20	186	10	3.4
3.5	95	6	19	114	6	20	134	7	21	155	9	21	176	10	21	197	11	3.5
3.6	100	5	21	121	7	21	142	8	21	163	8	22	185	9	23	208	11	3.6
3.7	106	6	21	127	6	23	150	8	22	172	9	23	195	10	24	219	11	3.7
3.8	111	5	23	134	7	24	158	8	23	181	9	25	206	11	25	231	12	3.8
3.9	117	6	24	141	7	25	166	8	25	191	10	25	216	10	26	242	11	3.9
4.0	123	6	26	149	8	25	174	8	26	200	9	27	227	11	27	254	12	**4.0**
4.1	129	6	27	156	7	27	183	9	27	210	10	28	238	11	29	267	13	4.1
4.2	136	7	28	164	8	28	192	9	29	221	11	29	250	12	29	279	12	4.2
4.3	142	6	29	171	7	30	201	9	30	231	10	30	261	11	31	292	13	4.3
4.4	149	7	30	179	8	32	211	10	30	241	10	32	273	12	33	306	14	4.4
4.5	155	6	33	188	9	32	220	9	32	252	11	34	286	13	33	319	13	4.5
4.6	162	7	34	196	8	34	230	10	34	264	12	34	298	12	35	333	14	4.6
4.7	169	7	35	204	8	36	240	10	35	275	11	36	311	13	36	347	14	4.7
4.8	177	8	36	213	9	37	250	10	37	287	12	37	324	13	38	362	15	4.8
4.9	184	7	38	222	9	38	260	10	39	299	12	38	337	13	40	377	15	4.9
5.0	192	8	39	231	9	39	270	10	41	311	12	40	351	14	41	392	15	**5.0**
$d-\delta_0$ / $\alpha-\alpha_0$	0°.5		*hor. diff.*	0°.6		*hor. diff.*	0°.7		*hor. diff.*	0°.8		*hor. diff.*	0°.9		*hor. diff.*	1°.0		$d-\delta_0$ \ $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$.

for $d - \delta_0 \leqq 1^\circ$, $0^\circ.000001 = 0''.0036 =$ unit; for $d - \delta_0 > 1^\circ$, $0^\circ.00001 = 0''.036 =$ unit.

$\alpha-\alpha_0$ / $d-\delta_0$	1°.0	*hor. diff.*	1°.1	*hor. diff.*	1°.2	*hor. diff.*	1°.3	*hor. diff.*	1°.4	*hor. diff.*	1°.5	$\alpha-\alpha_0$ / $d-\delta_0$
0°.0	102	33	135	40	175	48	223	56	279	64	343	0°.0
	1		2		2		2		2		2	
.1	103	34	137	40	177	48	225	56	281	64	345	.1
	5		5		6		6		6		7	
.2	108	34	142	41	183	48	231	56	287	65	352	.2
	7		8		9		10		11		11	
.3	115	35	150	42	192	49	241	57	298	65	363	.3
	11		12		13		14		15		17	
.4	126	36	162	43	205	50	255	58	313	67	380	.4
	14		15		16		18		19		20	
0.5	140	37	177	44	221	52	273	59	332	68	400	0.5
	16		18		20		22		24		25	
.6	156	39	195	46	241	54	295	61	356	69	425	.6
	20		22		24		25		27		30	
.7	176	41	217	48	265	55	320	63	383	72	455	.7
	23		25		28		30		32		34	
.8	199	43	242	51	293	57	350	65	415	74	489	.8
	26		29		31		34		37		39	
.9	225	46	271	53	324	60	384	68	452	76	528	.9
	29		32		34		37		40		44	
1.0	254	49	303	55	358	63	421	71	492	80	572	**1.0**
	4		4		4		4		5		5	
1.1	29	5	34	6	40	6	46	8	54	8	62	1.1
	3		4		4		5		5		5	
1.2	32	6	38	6	44	7	51	8	59	8	67	1.2
	4		4		4		5		5		6	
1.3	36	6	42	6	48	8	56	8	64	9	73	1.3
	4		4		5		5		6		6	
1.4	40	6	46	7	53	8	61	9	70	9	79	1.4
	4		5		6		6		6		7	
1.5	44	7	51	8	59	8	67	9	76	10	86	1.5
	5		5		5		6		6		7	
1.6	49	7	56	8	64	9	73	9	82	11	93	1.6
	5		6		6		6		8		7	
1.7	54	8	62	8	70	9	79	11	90	10	100	1.7
	6		6		7		7		7		8	
1.8	60	8	68	9	77	9	86	11	97	11	108	1.8
	5		6		7		8		8		9	
1.9	65	9	74	10	84	10	94	11	105	12	117	1.9
	6		7		7		8		8		9	
2.0	71	10	81	10	91	11	102	11	113	13	126	**2.0**
	6		6		7		8		9		9	
2.1	77	10	87	11	98	12	110	12	122	13	135	2.1
	7		8		8		8		9		10	
2.2	84	11	95	11	106	12	118	13	131	14	145	2.2
	7		7		8		9		10		10	
2.3	91	11	102	12	114	13	127	14	141	14	155	2.3
	7		8		9		9		10		11	
2.4	98	12	110	13	123	13	136	15	151	15	166	2.4
	7		9		9		10		10		11	
2.5	105	14	119	13	132	14	146	15	161	16	177	2.5
	8		8		9		10		10		12	
2.6	113	14	127	14	141	15	156	15	171	18	189	2.6
	8		9		10		11		12		12	
2.7	121	15	136	15	151	16	167	16	183	18	201	2.7
	9		9		10		11		12		13	
2.8	130	15	145	16	161	17	178	17	195	19	214	2.8
	8		10		10		11		12		13	
2.9	138	17	155	16	171	18	189	18	207	20	227	2.9
	9		10		11		12		13		13	
3.0	147	18	165	17	182	19	201	19	220	20	240	**3.0**
	10		10		11		12		13		14	
3.1	157	18	175	18	193	20	213	20	233	21	254	3.1
	9		10		12		12		14		15	
3.2	166	19	185	20	205	20	225	22	247	22	269	3.2
	10		11		12		13		13		15	
3.3	176	20	196	21	217	21	238	22	260	24	284	3.3
	10		11		12		14		15		15	
3.4	186	21	207	22	229	23	252	23	275	24	299	3.4
	11		12		13		13		15		16	
3.5	197	22	219	23	242	23	265	25	290	25	315	3.5
	11		12		13		14		15		16	
3.6	208	23	231	24	255	24	279	26	305	26	331	3.6
	11		12		13		15		15		17	
3.7	219	24	243	25	268	26	294	26	320	28	348	3.7
	12		13		14		15		16		17	
3.8	231	25	256	26	282	27	309	27	336	29	365	3.8
	11		13		14		15		17		18	
3.9	242	27	269	27	296	28	324	29	353	30	383	3.9
	12		13		15		16		17		18	
4.0	254	28	282	29	311	29	340	30	370	31	401	**4.0**
	13		14		15		16		17		18	
4.1	267	29	296	30	326	30	356	31	387	32	419	4.1
	12		14		15		16		18		19	
4.2	279	31	310	31	341	31	372	33	405	33	438	4.2
	13		15		15		17		18		20	
4.3	292	33	325	31	356	33	389	34	423	35	458	4.3
	14		14		16		18		19		20	
4.4	306	33	339	33	372	35	407	35	442	36	478	4.4
	13		15		16		18		19		20	
4.5	319	35	354	34	388	37	425	36	461	37	498	4.5
	14		15		17		18		19		21	
4.6	333	36	369	36	405	38	443	37	480	39	519	4.6
	14		16		18		18		20		22	
4.7	347	38	385	38	423	38	461	39	500	41	541	4.7
	15		16		17		19		21		21	
4.8	362	39	401	39	440	40	480	41	521	41	562	4.8
	15		17		18		19		21		23	
4.9	377	41	418	40	458	41	499	43	542	43	585	4.9
	15		16		18		20		21		22	
5.0	392	42	434	42	476	43	519	44	563	44	607	**5.0**
$d-\delta_0$ / $\alpha-\alpha_0$	1°.0	*hor. diff.*	1°.1	*hor. diff.*	1°.2	*hor. diff.*	1°.3	*hor. diff.*	1°.4	*hor. diff.*	1°.5	$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^{\circ}.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^{\circ}$.

for $d - \delta_0 \leqq 1^{\circ}$, $0^{\circ}.000001 = 0''.0036 =$ unit; for $d - \delta_0 > 1^{\circ}$, $0^{\circ}.00001 = 0''.036 =$ unit.

$\alpha-\alpha_0$ / $d-\delta_0$	1°.5		hor. diff.	1°.6		hor. diff.	1°.7		hor. diff.	1°.8		hor. diff.	1°.9		hor. diff.	2°.0		$\alpha-\alpha_0$ / $d-\delta_0$
0.0°	343		73	416		83	499		94	593		104	697		116	813		0.0°
.1	345	2	74	419	3	83	502	3	94	596	3	104	700	3	116	816	3	.1
.2	352	7	74	426	7	84	510	8	[illegible]	604	8	104	708	8	117	825	9	.2
.3	363	11	75	438	12	85	523	13	94	617	13	105	722	14	118	840	15	.3
.4	380	17	75	455	17	86	541	18	96	637	20	106	743	21	118	861	21	.4
0.5	400	20	77	477	22	87	564	23	97	661	24	108	769	26	120	889	28	0.5
.6	425	25	79	504	27	89	593	29	98	691	30	110	801	32	122	923	34	.6
.7	455	30	81	536	32	90	626	33	101	727	36	111	838	37	124	962	39	.7
.8	489	34	83	572	36	93	665	39	103	768	41	114	882	44	126	1008	46	.8
.9	528	39	86	614	42	95	709	44	106	815	47	116	931	49	129	1060	52	.9
1.0	572	44	88	660	46	98	758	49	109	867	52	119	986	55	131	1117	57	**1.0**
1.1	62	5	9	71	5	10	81	5	11	92	5	13	105	6	13	118	6	1.1
1.2	67	5	10	77	6	10	87	6	12	99	7	12	111	6	14	125	7	1.2
1.3	73	6	10	83	6	11	94	7	12	106	7	13	119	8	14	133	8	1.3
1.4	79	6	10	89	6	12	101	7	12	113	7	13	126	7	15	141	8	1.4
1.5	86	7	10	96	7	12	108	7	13	121	8	14	135	9	15	150	9	1.5
1.6	93	7	11	104	8	12	116	8	13	129	8	15	144	9	15	159	9	1.6
1.7	100	7	12	112	8	13	125	9	13	138	9	15	153	9	16	169	10	1.7
1.8	108	8	13	121	9	13	134	9	14	148	10	16	164	11	16	180	11	1.8
1.9	117	9	13	130	9	14	144	10	14	158	10	16	174	10	17	191	11	1.9
2.0	126	9	13	139	9	15	154	10	15	169	11	17	186	12	17	203	12	**2.0**
2.1	135	9	14	149	10	15	164	10	16	180	11	17	197	11	19	216	13	2.1
2.2	145	10	15	160	11	15	175	11	16	191	11	19	210	13	19	229	13	2.2
2.3	155	10	16	171	11	16	187	12	17	204	13	19	223	13	20	243	14	2.3
2.4	166	11	16	182	11	17	199	12	18	217	13	20	237	14	20	257	14	2.4
2.5	177	11	17	194	12	18	212	13	19	231	14	21	252	15	20	272	15	2.5
2.6	189	12	18	207	13	18	225	13	20	245	14	21	266	14	21	287	15	2.6
2.7	201	12	19	220	13	19	239	14	20	259	14	22	281	15	22	303	16	2.7
2.8	214	13	19	233	13	20	253	14	21	274	15	23	297	16	23	320	17	2.8
2.9	227	13	20	247	14	21	268	15	22	290	16	23	313	16	25	338	18	2.9
3.0	240	13	21	261	14	22	283	15	23	306	16	24	330	17	26	356	18	**3.0**
3.1	254	14	22	276	15	23	299	16	24	323	17	25	348	18	26	374	18	3.1
3.2	269	15	22	291	15	24	315	16	25	340	17	26	366	18	28	394	20	3.2
3.3	284	15	23	307	16	25	332	17	26	358	18	27	385	19	29	414	20	3.3
3.4	299	15	25	324	17	26	350	18	27	377	19	28	405	20	29	434	20	3.4
3.5	315	16	26	341	17	27	368	18	28	396	19	29	425	20	30	455	21	3.5
3.6	331	16	27	358	17	28	386	18	29	415	19	30	445	20	32	477	22	3.6
3.7	348	17	28	376	18	29	405	19	30	435	20	32	467	22	32	499	22	3.7
3.8	365	17	29	394	18	31	425	20	31	456	21	33	489	22	33	522	23	3.8
3.9	383	18	30	413	19	32	445	20	32	477	21	34	511	22	35	546	24	3.9
4.0	401	18	31	432	19	33	465	20	34	499	22	35	534	23	36	570	24	**4.0**
4.1	419	18	33	452	20	34	486	21	35	521	22	36	557	23	38	595	25	4.1
4.2	438	19	35	473	21	35	508	22	36	544	23	38	582	25	38	620	25	4.2
4.3	458	20	35	493	20	37	530	22	38	568	24	38	606	24	40	646	26	4.3
4.4	478	20	37	515	22	38	553	23	39	592	24	40	632	26	41	673	27	4.4
4.5	498	20	39	537	22	39	576	23	40	616	24	41	657	25	43	700	27	4.5
4.6	519	21	40	559	22	41	600	24	41	641	25	43	684	27	44	728	28	4.6
4.7	541	22	41	582	23	42	624	24	43	667	26	44	711	27	45	756	28	4.7
4.8	562	21	43	605	23	43	648	24	45	693	26	46	739	28	46	785	29	4.8
4.9	585	23	44	629	24	45	674	26	46	720	27	47	767	28	48	815	30	4.9
5.0	607	22	46	653	24	46	699	25	48	747	27	49	796	29	50	846	31	[illegible]
$d-\delta_0$ / $\alpha-\alpha_0$	1°.5		hor. diff.	1°.6		hor. diff.	1°.7		hor. diff.	1°.8		hor. diff.	1°.9		hor. diff.	2°.0		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi\,\sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi\,\sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$. $0^\circ.00001 = 0''.036 =$ unit.

$\alpha-\alpha_0$ / $d-\delta_0$	2°.0		*hor. diff.*	2°.1		*hor. diff.*	2°.2		*hor. diff.*	2°.3		*hor. diff.*	2°.4		*hor. diff.*	2°.5		$\alpha-\alpha_0$ / $d-\delta_0$
0°.0	00081		13	00094		14	00108		16	00124		16	00140		19	00159		0°.0
.1	82	1	12	94	0	15	109	1	15	124	0	17	141	1	18	159	0	.1
.2	82	0	13	95	1	15	110	1	15	125	1	17	142	1	18	160	1	.2
.3	84	2	13	97	2	15	112	2	15	127	2	17	144	2	18	162	2	.3
.4	86	2	13	99	2	15	114	2	15	129	2	17	146	2	19	165	3	.4
0.5	89	3	13	102	3	15	117	3	15	132	3	17	149	3	19	168	3	0.5
.6	92	3	13	105	3	15	120	3	16	136	4	18	154	5	19	173	5	.6
.7	96	4	14	110	5	15	125	5	16	141	5	18	159	5	19	178	5	.7
.8	101	5	14	115	5	15	130	5	16	146	5	18	164	5	19	183	5	.8
.9	106	5	14	120	5	15	135	5	17	152	6	18	170	6	20	190	7	.9
1.0	112	6	14	126	6	16	142	7	17	159	7	18	177	7	20	197	7	1.0
1.1	118	6	15	133	7	16	149	7	17	166	7	19	185	8	20	205	8	1.1
1.2	125	7	15	140	7	17	157	8	17	174	8	19	193	8	21	214	9	1.2
1.3	133	8	15	148	8	17	165	8	18	183	9	19	202	9	21	223	9	1.3
1.4	141	8	16	157	9	17	174	9	18	192	9	20	212	10	21	233	10	1.4
1.5	150	9	16	166	9	18	184	10	19	203	11	20	223	11	22	245	12	1.5
1.6	159	9	17	176	10	18	194	10	19	213	10	21	234	11	22	256	11	1.6
1.7	169	10	18	187	11	18	205	11	20	225	12	21	246	12	23	269	13	1.7
1.8	180	11	18	198	11	19	217	12	20	237	12	22	259	13	23	282	13	1.8
1.9	191	11	19	210	12	19	229	12	21	250	13	22	272	13	24	296	14	1.9
2.0	203	12	19	222	12	20	242	13	22	264	14	23	287	15	24	311	15	2.0
2.1	216	13	19	235	13	21	256	13	22	278	14	24	302	15	25	327	16	2.1
2.2	229	13	20	249	14	22	271	15	22	293	15	24	317	15	26	343	16	2.2
2.3	243	14	20	263	14	23	286	15	23	309	16	25	334	17	26	360	17	2.3
2.4	257	14	22	279	16	23	302	16	24	326	17	25	351	17	27	378	18	2.4
2.5	272	15	22	294	15	24	318	16	25	343	17	26	369	18	28	397	19	2.5
2.6	287	15	24	311	17	24	335	17	26	361	18	27	388	19	29	417	20	2.6
2.7	303	16	25	328	17	25	353	18	26	379	18	28	407	19	30	437	20	2.7
2.8	320	17	25	345	17	26	371	18	28	399	20	28	427	20	31	458	21	2.8
2.9	338	18	26	364	19	26	390	19	29	419	20	29	448	21	32	480	22	2.9
3.0	356	18	26	382	18	28	410	20	29	439	20	31	470	22	32	502	22	3.0
3.1	374	18	28	402	20	29	431	21	30	461	22	31	492	22	33	525	23	3.1
3.2	394	20	28	422	20	30	452	21	31	483	22	32	515	23	35	550	25	3.2
3.3	414	20	29	443	21	31	474	22	32	506	23	33	539	24	35	574	24	3.3
3.4	434	20	31	465	22	31	496	22	33	529	23	34	563	24	37	600	26	3.4
3.5	455	21	32	487	22	32	519	23	35	554	25	35	589	26	37	626	26	3.5
3.6	477	22	33	510	23	33	543	24	36	579	25	36	615	26	38	653	27	3.6
3.7	499	22	34	533	23	35	568	25	36	604	25	38	642	27	39	681	28	3.7
3.8	522	23	35	557	24	36	593	25	38	631	27	39	670	28	40	710	29	3.8
3.9	546	24	36	582	25	37	619	26	39	658	27	39	697	27	43	740	30	3.9
4.0	570	24	37	607	25	39	646	27	40	686	28	40	726	29	44	770	30	4.0
4.1	595	25	37	632	25	41	673	27	41	714	28	42	756	30	45	801	31	4.1
4.2	620	25	40	660	28	41	701	28	42	743	29	44	787	31	46	833	32	4.2
4.3	646	26	41	687	27	43	730	29	43	773	30	45	818	31	47	865	32	4.3
4.4	673	27	42	715	28	44	759	29	45	804	31	46	850	32	48	898	33	4.4
4.5	700	27	44	744	29	45	789	30	46	835	31	47	882	32	50	932	34	4.5
4.6	728	28	45	773	29	47	820	31	47	867	32	49	916	34	51	967	35	4.6
4.7	756	28	47	803	30	48	851	31	49	900	33	50	950	34	53	1003	36	4.7
4.8	785	29	49	834	31	49	883	32	51	934	34	52	986	36	53	1039	36	4.8
4.9	815	30	50	865	31	51	916	33	52	968	34	53	1021	35	55	1076	37	4.9
5.0	846	31	51	897	32	52	949	33	54	1003	35	54	1057	36	57	1114	38	5.0
$d-\delta_0$ / $\alpha-\alpha_0$	2°.0		*hor. diff.*	2°.1		*hor. diff.*	2°.2		*hor. diff.*	2°.3		*hor. diff.*	2°.4		*hor. diff.*	2°.5		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ **is always numerically larger than** $(\alpha - \alpha_0)^\circ$. $0^\circ.00001 = 0''.036 =$ unit.

$\alpha-\alpha_0$ / $d-\delta_0$	2°.5	hor. diff.	2°.6	hor. diff.	2°.7	hor. diff.	2°.8	hor. diff.	2°.9	hor. diff.	3°.0	$\alpha-\alpha_0$ / $d-\delta_0$
0°.0	00159	20	00179	21	00200	23	00223	25	00248	[illegible]	00274	0°.0
.1	159	20	179	21	200	24	224	24	248	27	275	.1
.2	160	20	180	22	202	23	225	25	250	26	276	.2
.3	162	20	182	[illegible]	204	23	227	25	252	[illegible]	278	.3
.4	165	20	185	22	207	23	230	25	255	27	282	.4
0.5	168	21	189	21	210	24	234	25	259	27	286	0.5
.6	173	20	193	22	215	24	239	[illegible]	264	27	291	.6
.7	178	20	198	22	220	24	244	26	270	27	297	.7
.8	183	21	204	22	226	24	250	[illegible]	276	28	304	.8
.9	190	21	211	22	233	25	258	26	284	27	311	.9
1.0	197	21	218	23	241	25	266	26	292	[illegible]	320	1.0
1.1	205	22	227	23	250	25	275	[illegible]	301	29	330	1.1
1.2	214	22	236	23	259	26	285	27	312	28	340	1.2
1.3	223	23	246	24	270	25	295	28	323	29	352	1.3
1.4	233	23	256	25	281	26	307	[illegible]	335	29	364	1.4
1.5	245	23	268	25	293	26	319	28	347	30	377	1.5
1.6	256	24	280	25	305	28	333	28	361	31	392	1.6
1.7	269	24	293	26	319	28	347	29	376	31	407	1.7
1.8	282	25	307	26	333	29	362	29	391	32	423	1.8
1.9	296	26	322	27	349	[illegible]	377	31	408	32	440	1.9
2.0	311	26	337	28	365	29	394	31	425	32	457	2.0
2.1	327	27	354	28	382	30	412	31	443	33	476	2.1
2.2	343	28	371	[illegible]	399	31	430	32	462	34	496	2.2
2.3	360	[illegible]	388	30	418	31	[illegible]	33	482	[illegible]	517	2.3
2.4	378	[illegible]	407	30	437	32	469	[illegible]	503	35	538	2.4
2.5	397	30	427	30	457	33	490	34	524	36	560	2.5
2.6	417	30	447	32	479	33	512	35	547	37	584	2.6
2.7	437	31	468	32	500	35	535	36	571	37	608	2.7
2.8	458	32	490	33	523	[illegible]	558	37	595	38	633	2.8
2.9	480	32	512	35	547	36	583	37	620	39	659	2.9
3.0	502	34	536	35	571	37	608	38	646	40	686	3.0
3.1	525	35	560	36	596	38	634	39	673	41	714	3.1
3.2	550	35	585	37	622	39	661	40	701	42	743	3.2
3.3	574	37	611	38	649	39	688	42	730	42	772	3.3
3.4	600	37	637	[illegible]	677	40	717	43	760	44	804	3.4
3.5	626	39	665	40	705	42	747	43	790	46	836	3.5
3.6	653	40	693	41	734	43	777	45	822	46	868	3.6
3.7	681	41	722	[illegible]	765	[illegible]	809	[illegible]	854	48	902	3.7
3.8	710	42	752	43	795	46	841	46	887	[illegible]	936	3.8
3.9	740	43	783	[illegible]	827	47	874	48	922	[illegible]	971	3.9
4.0	770	44	814	46	860	48	908	49	957	51	1008	4.0
4.1	801	[illegible]	846	47	893	[illegible]	942	51	993	52	1045	4.1
4.2	833	46	879	49	928	50	978	52	1030	53	1083	4.2
4.3	865	48	913	50	963	51	1014	53	1067	55	1122	4.3
4.4	898	50	948	51	999	53	1052	54	1106	56	1162	4.4
4.5	932	51	983	[illegible]	1036	54	1090	55	1145	58	1203	4.5
4.6	967	52	1019	54	1073	56	1129	57	1186	59	1245	4.6
4.7	1003	53	1056	56	1112	57	1169	58	1227	60	1287	4.7
4.8	1039	55	1094	57	1151	58	1209	[illegible]	1269	62	1331	4.8
4.9	1076	57	1133	58	1191	60	1251	61	1312	64	1376	4.9
5.0	1114	58	1172	60	1232	62	1294	63	1357	64	1421	5.0
$d-\delta_0$ / $\alpha-\alpha_0$	2°.5	hor. diff.	2°.6	hor. diff.	2°.7	hor. diff.	2°.8	hor. diff.	2°.9	hor. diff.	3°.0	$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$. **$0^\circ.00001 = 0''.036 =$ unit.**

$\alpha-\alpha_0$ / $d-\delta_0$	3°.0		hor. diff.	3°.1		hor. diff.	3°.2		hor. diff.	3°.3		hor. diff.	3°.4		hor. diff.	3°.5		$\alpha-\alpha_0$ / $d-\delta_0$
0°.0	00274	1	29	00303	0	30	00333	0	32	00365	1	34	00399	1	37	00436	0	**0°.0**
.1	275	1	28	303	2	30	333	2	33	366	1	34	400	2	36	436	2	.1
.2	276	2	29	305	2	30	335	2	32	367	3	35	402	2	36	438	3	.2
.3	278	4	29	307	3	30	337	4	33	370	3	34	404	4	37	441	3	.3
.4	282	4	28	310	4	31	341	4	32	373	5	35	408	4	36	444	5	.4
0.5	286	5	28	314	6	31	345	6	33	378	5	34	412	6	37	449	6	0.5
.6	291	6	29	320	6	31	351	6	32	383	7	35	418	7	37	455	7	.6
.7	297	7	29	326	7	31	357	7	33	390	7	35	425	8	37	462	8	.7
.8	304	7	29	333	8	31	364	9	33	397	9	36	433	9	37	470	9	.8
.9	311	9	30	341	9	32	373	9	33	406	10	36	442	9	37	479	10	.9
1.0	320	10	30	350	10	32	382	10	34	416	10	35	451	11	38	489	11	**1.0**
1.1	330	10	30	360	11	32	392	11	34	426	12	36	462	12	38	500	13	1.1
1.2	340	12	31	371	12	32	403	13	35	438	12	36	474	13	39	513	13	1.2
1.3	352	12	31	383	12	33	416	13	34	450	14	37	487	14	39	526	14	1.3
1.4	364	13	31	395	14	34	429	14	35	464	15	37	501	15	39	540	16	1.4
1.5	377	15	32	409	15	34	443	15	36	479	15	37	516	16	40	556	17	1.5
1.6	392	15	32	424	15	34	458	16	36	494	17	38	532	17	41	573	17	1.6
1.7	407	16	32	439	17	35	474	17	37	511	17	38	549	18	41	590	19	1.7
1.8	423	17	33	456	17	35	491	18	37	528	19	40	568	19	41	609	20	1.8
1.9	440	17	33	473	19	36	509	18	38	547	20	40	587	20	42	629	20	1.9
2.0	457	19	35	492	19	36	528	20	39	567	20	40	607	21	42	649	22	**2.0**
2.1	476	20	35	511	21	37	548	21	39	587	22	41	628	23	43	671	23	2.1
2.2	496	21	36	532	21	37	569	22	40	609	23	42	651	23	43	694	24	2.2
2.3	517	21	36	553	22	38	591	23	41	632	23	42	674	24	44	718	26	2.3
2.4	538	22	37	575	23	39	614	24	41	655	25	43	698	26	46	744	26	2.4
2.5	560	24	38	598	25	40	638	25	42	680	26	44	724	27	46	770	27	2.5
2.6	584	24	39	623	25	40	663	26	43	706	26	45	751	27	46	797	28	2.6
2.7	608	25	40	648	26	41	689	27	43	732	28	46	778	28	47	825	29	2.7
2.8	633	26	41	674	27	42	716	28	44	760	29	46	806	30	48	854	31	2.8
2.9	659	27	42	701	28	43	744	29	45	789	30	47	836	31	49	885	32	2.9
3.0	686	28	43	729	29	44	773	29	46	819	30	48	867	31	50	917	32	**3.0**
3.1	714	29	44	758	29	44	802	31	47	849	32	49	898	33	51	949	34	3.1
3.2	743	29	44	787	31	46	833	32	48	881	33	50	931	34	52	983	35	3.2
3.3	772	32	46	818	32	47	865	33	49	914	34	51	965	35	53	1018	36	3.3
3.4	804	32	46	850	32	48	898	34	50	948	35	52	1000	35	54	1054	37	3.4
3.5	836	32	46	882	34	50	932	34	51	983	35	52	1035	37	56	1091	38	3.5
3.6	868	34	48	916	35	50	966	36	52	1018	37	54	1072	38	57	1129	39	3.6
3.7	902	34	49	951	36	51	1002	37	53	1055	38	55	1110	40	58	1168	40	3.7
3.8	936	35	51	987	36	52	1039	38	54	1093	39	57	1150	40	58	1208	41	3.8
3.9	971	37	52	1023	37	54	1077	38	55	1132	40	58	1190	41	59	1249	43	3.9
4.0	1008	37	52	1060	39	55	1115	40	57	1172	41	59	1231	42	61	1292	43	**4.0**
4.1	1045	38	54	1099	39	56	1155	41	58	1213	42	60	1273	43	62	1335	45	4.1
4.2	1083	39	55	1138	41	58	1196	41	59	1255	43	61	1316	44	64	1380	45	4.2
4.3	1122	40	57	1179	41	58	1237	43	61	1298	44	62	1360	46	65	1425	47	4.3
4.4	1162	41	58	1220	42	60	1280	43	62	1342	45	64	1406	46	66	1472	47	4.4
4.5	1203	42	59	1262	44	61	1323	45	64	1387	46	65	1452	48	67	1519	49	4.5
4.6	1245	42	61	1306	44	62	1368	46	65	1433	47	67	1500	48	68	1568	50	4.6
4.7	1287	44	63	1350	45	64	1414	46	66	1480	48	68	1548	49	70	1618	51	4.7
4.8	1331	45	64	1395	46	65	1460	48	68	1528	49	69	1597	51	72	1669	52	4.8
4.9	1376	45	65	1441	47	67	1508	49	69	1577	50	71	1648	52	73	1721	53	4.9
5.0	1421		67	1488		69	1557		70	1627		73	1700		74	1774		**5.0**
$d-\delta_0$ / $\alpha-\alpha_0$	3°.0		hor. diff.	3°.1		hor. diff.	3°.2		hor. diff.	3°.3		hor. diff.	3°.4		hor. diff.	3°.5		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$. $0^\circ.00001 = 0''.036 =$ unit.

$\alpha-\alpha_0$ / $d-\delta_0$	3°.5		*hor. diff.*	3°.6		*hor. diff.*	3°.7		*hor. diff.*	3°.8		*hor. diff.*	3°.9		*hor. diff.*	4°.0		$\alpha-\alpha_0$ / $d-\delta_0$
0.0	00436		38	00474		41	00515		42	00557		46	00603		48	00651		0.0
.1	436	0	39	475	1	41	516	1	42	558	1	46	604	1	48	652	1	.1
.2	438	2	39	477	2	40	517	1	43	560	2	46	606	2	48	654	2	.2
.3	441	3	38	479	2	41	520	3	43	563	3	46	609	3	48	657	3	.3
.4	444	3	39	483	4	41	524	4	43	567	4	46	613	4	48	661	4	.4
0.5	449	5	39	488	5	41	529	5	43	572	5	46	618	5	48	666	5	0.5
.6	455	6	39	494	6	41	535	6	44	579	7	46	625	7	48	673	7	.6
.7	462	7	39	501	7	42	543	8	44	587	8	46	633	8	48	681	8	.7
.8	470	8	40	510	9	41	551	8	44	595	8	47	642	9	48	690	9	.8
.9	479	9	40	519	9	42	561	10	44	605	10	47	652	10	49	701	11	.9
1.0	489	10	40	529	10	42	571	10	45	616	11	47	663	11	[illegible]	712	11	1.0
1.1	500	11	41	541	12	42	583	12	45	628	12	47	675	12	50	725	13	1.1
1.2	513	13	41	554	13	43	597	14	45	642	14	47	689	14	50	739	14	1.2
1.3	526	13	41	567	13	44	611	14	45	656	14	48	704	15	50	754	15	1.3
1.4	540	14	42	582	15	44	626	15	46	672	16	48	720	16	51	771	17	1.4
1.5	556	16	42	598	16	44	642	16	47	689	17	48	737	17	51	788	17	1.5
1.6	573	17	42	615	17	45	660	18	47	707	18	49	756	19	51	807	19	1.6
1.7	590	17	43	633	18	45	678	18	48	726	19	49	775	19	52	827	20	1.7
1.8	609	19	43	652	19	46	698	20	48	746	20	50	796	21	53	849	22	1.8
1.9	629	20	44	673	21	46	719	21	48	767	21	51	818	22	53	871	22	1.9
2.0	649	20	45	694	21	47	741	22	49	790	23	51	841	23	54	895	24	2.0
2.1	671	22	46	717	23	47	764	23	50	814	24	52	866	25	54	920	25	2.1
2.2	694	23	46	740	23	48	788	24	51	839	25	53	892	26	55	947	27	2.2
2.3	718	24	47	765	25	49	814	26	51	865	26	53	918	26	56	974	27	2.3
2.4	744	26	47	791	26	49	840	26	52	892	27	54	946	28	57	1003	29	2.4
2.5	770	26	48	818	27	50	868	28	53	921	29	55	976	30	57	1033	30	2.5
2.6	797	27	49	846	28	51	897	29	53	950	29	56	1006	30	58	1064	31	2.6
2.7	825	28	50	875	29	52	927	30	54	981	31	56	1037	31	59	1096	32	2.7
2.8	854	29	51	905	30	53	958	31	55	1013	32	57	1070	33	60	1130	34	2.8
2.9	885	31	52	937	32	53	990	32	56	1046	33	58	1104	34	61	1165	35	2.9
3.0	917	32	52	969	32	55	1024	34	56	1080	34	59	1139	35	62	1201	36	3.0
3.1	949	32	54	1003	34	55	1058	34	58	1116	36	60	1176	37	62	1238	37	3.1
3.2	983	34	54	1037	34	57	1094	36	58	1152	36	61	1213	37	64	1277	39	3.2
3.3	1018	35	55	1073	36	57	1130	36	60	1190	38	62	1252	39	64	1316	39	3.3
3.4	1054	36	56	1110	37	58	1168	38	61	1229	39	63	1292	40	65	1357	41	3.4
3.5	1091	37	57	1148	38	59	1207	39	62	1269	40	64	1333	41	67	1400	43	3.5
3.6	1129	38	58	1187	39	61	1248	41	62	1310	41	65	1375	42	68	1443	43	3.6
3.7	1168	39	59	1227	40	62	1289	41	64	1353	43	66	1419	44	69	1488	45	3.7
3.8	1208	40	61	1269	42	63	1332	43	65	1397	44	67	1464	45	70	1534	46	3.8
3.9	1249	41	63	1312	43	63	1375	43	66	1441	44	69	1510	46	71	1581	47	3.9
4.0	1292	43	63	1355	43	65	1420	45	67	1487	46	70	1557	47	72	1629	48	4.0
4.1	1335	43	64	1399	44	67	1466	46	69	1535	48	71	1606	49	73	1679	50	4.1
4.2	1380	45	65	1445	46	68	1513	47	70	1583	48	72	1655	49	75	1730	51	4.2
4.3	1425	45	67	1492	47	69	1561	48	71	1632	49	74	1706	51	76	1782	52	4.3
4.4	1472	47	68	1540	48	70	1610	49	73	1683	51	75	1758	52	77	1835	53	4.4
4.5	1519	47	70	1589	49	72	1661	51	74	1735	52	76	1811	53	79	1890	55	4.5
4.6	1568	49	71	1639	50	74	1713	52	75	1788	53	78	1866	55	80	1946	56	4.6
4.7	1618	50	73	1691	52	74	1765	52	77	1842	54	79	1921	55	82	2003	57	4.7
4.8	1669	51	74	1743	52	75	1818	53	79	1897	55	81	1978	57	84	2062	59	4.8
4.9	1721	52	76	1797	54	77	1874	56	80	1954	57	82	2036	58	85	2121	59	4.9
5.0	1774	53	78	1852	55	78	1930	56	82	2012	58	83	2095	59	86	2181	60	5.0
$d-\delta_0$ / $\alpha-\alpha_0$	3°.5		*hor. diff.*	3°.6		*hor. diff.*	3°.7		*hor. diff.*	3°.8		*hor. diff.*	3°.9		*hor. diff.*	4°.0		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$. $0^\circ.00001 = 0''.036 =$ unit.

$\alpha-\alpha_0$ / $d-\delta_0$	4°.0		hor. diff.	4°.1		hor. diff.	4°.2		hor. diff.	4°.3		hor. diff.	4°.4		hor. diff.	4°.5		$\alpha-\alpha_0$ / $d-\delta_0$
0.0	00651	1	50	00701	1	53	00754	1	55	00809	1	58	00867	1	61	00928	1	0.0
.1	652	2	50	702	2	53	755	2	55	810	2	58	868	2	61	929	2	.1
.2	654	3	50	704	3	53	757	3	55	812	3	58	870	3	61	931	3	.2
.3	657	4	50	707	4	53	760	4	55	815	5	58	873	5	61	934	5	.3
.4	661	5	50	711	6	53	764	6	56	820	6	58	878	6	61	939	6	.4
0.5	666	7	51	717	7	53	770	7	56	826	7	58	884	7	61	945	8	0.5
.6	673	8	51	724	8	53	777	8	56	833	8	58	891	9	62	953	9	.6
.7	681	9	51	732	9	53	785	10	56	841	10	59	900	10	62	962	10	.7
.8	690	11	51	741	11	54	795	11	56	851	12	59	910	12	62	972	12	.8
.9	701	11	51	752	12	54	806	12	57	863	12	59	922	12	62	984	12	.9
1.0	712	13	52	764	13	54	818	14	57	875	14	59	934	14	62	996	15	1.0
1.1	725	14	52	777	14	55	832	14	57	889	15	59	948	16	63	1011	16	1.1
1.2	739	15	52	791	16	55	846	16	58	904	16	[illegible]	964	17	63	1027	17	1.2
1.3	754	17	53	807	17	55	862	18	58	920	18	61	981	18	63	1044	18	1.3
1.4	771	17	53	824	18	56	880	18	58	938	19	61	999	19	63	1062	20	1.4
1.5	788	19	54	842	20	56	898	20	59	957	20	61	1018	21	64	1082	22	1.5
1.6	807	20	55	862	20	56	918	21	59	977	22	62	1039	22	65	1104	22	1.6
1.7	827	22	55	882	22	57	939	23	60	999	23	62	1061	24	65	1126	24	1.7
1.8	849	22	55	904	23	58	962	24	60	1022	24	63	1085	25	65	1150	26	1.8
1.9	871	24	56	927	25	59	986	25	60	1046	26	64	1110	26	66	1176	27	1.9
2.0	895	25	57	952	26	59	1011	26	61	1072	27	64	1136	27	67	1203	28	2.0
2.1	920	27	58	978	27	59	1037	27	62	1099	28	64	1163	30	68	1231	29	2.1
2.2	947	27	58	1005	28	59	1064	29	63	1127	30	66	1193	30	67	1260	31	2.2
2.3	974	29	59	1033	30	60	1093	30	64	1157	31	[illegible]	1223	31	68	1291	33	2.3
2.4	1003	30	60	1063	30	60	1123	32	65	1188	32	66	1254	33	70	1324	33	2.4
2.5	1033	31	60	1093	32	62	1155	33	65	1220	33	67	1287	34	70	1357	35	2.5
2.6	1064	32	61	1125	33	63	1188	34	65	1253	35	68	1321	36	71	1392	37	2.6
2.7	1096	34	62	1158	34	64	1222	35	66	1288	36	69	1357	37	72	1429	38	2.7
2.8	1130	35	62	1192	36	65	1257	37	67	1324	38	70	1394	38	73	1467	39	2.8
2.9	1165	36	63	1228	37	66	1294	37	68	1362	38	70	1432	40	74	1506	41	2.9
3.0	1201	37	64	1265	37	66	1331	40	69	1400	41	72	1472	41	75	1547	42	3.0
3.1	1238	39	64	1302	40	69	1371	40	70	1441	41	72	1513	43	76	1589	43	3.1
3.2	1277	39	65	1342	40	69	1411	42	71	1482	43	74	1556	43	76	1632	45	3.2
3.3	1316	41	66	1382	43	71	1453	43	72	1525	44	74	1599	45	78	1677	46	3.3
3.4	1357	43	68	1425	44	71	1496	44	73	1569	45	75	1644	47	79	1723	47	3.4
3.5	1400	43	69	1469	44	71	1540	46	74	1614	47	77	1691	48	79	1770	49	3.5
3.6	1443	45	70	1513	46	73	1586	47	75	1661	48	78	1739	49	80	1819	50	3.6
3.7	1488	46	71	1559	47	74	1633	48	76	1709	49	79	1788	51	81	1869	52	3.7
3.8	1534	47	72	1606	49	75	1681	50	77	1758	51	81	1839	52	82	1921	53	3.8
3.9	1581	48	74	1655	49	76	1731	51	78	1809	52	82	1891	53	83	1974	55	3.9
4.0	1629	50	75	1704	51	78	1782	52	79	1861	53	83	1944	54	85	2029	56	4.0
4.1	1679	51	76	1755	52	79	1834	53	80	1914	55	84	1998	57	87	2085	57	4.1
4.2	1730	52	77	1807	54	80	1887	55	[illegible]	1969	56	86	2055	57	87	2142	58	4.2
4.3	1782	53	79	1861	54	81	1942	56	83	2025	57	87	2112	58	88	2200	61	4.3
4.4	1835	55	80	1915	56	83	1998	57	84	2082	59	88	2170	61	91	2261	61	4.4
4.5	1890	56	81	1971	57	84	2055	59	86	2141	60	90	2231	61	91	2322	63	4.5
4.6	1946	57	82	2028	59	[illegible]	2114	60	87	2201	62	91	2292	63	93	2385	65	4.6
4.7	2003	59	84	2087	60	87	2174	61	[illegible]	2262	64	93	2355	64	95	2450	65	4.7
4.8	2062	59	85	2147	61	88	2235	62	91	2326	64	93	2419	65	96	2515	67	4.8
4.9	2121	60	87	2208	62	89	2297	64	93	2390	65	94	2484	67	98	2582	68	4.9
5.0	2181		89	2270		91	2361		94	2455		96	2551		99	2650		5.0
$d-\delta_0$ / $\alpha-\alpha_0$	4°.0		hor. diff.	4°.1		hor. diff.	4°.2		hor. diff.	4°.3		hor. diff.	4°.4		hor. diff.	4°.5		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$. $0^\circ.00001 = 0''.036 =$ unit.

$\alpha-\alpha_0$ / $d-\delta_0$	4°.5		hor. diff.	4°.6		hor. diff.	4°.7		hor. diff.	4°.8		hor. diff.	4°.9		hor. diff.	5°.0		$\alpha-\alpha_0$ / $d-\delta_0$
0.0	00928		63	00991		[illegible]	01057		69	01126		71	01197		75	01272		0.0
.1	929	1	63	992	1	66	1058	1	69	1127	1	71	1198	1	75	1273	1	.1
.2	931	2	63	994	2	66	1060	2	69	1129	2	72	1201	3	75	1276	3	.2
.3	934	3	63	997	3	67	1064	4	69	1133	4	72	1205	4	75	1280	4	.3
.4	939	5	63	1002	5	67	1069	5	69	1138	5	72	1210	5	75	1285	5	.4
0.5	945	6	64	1009	7	[illegible]	1075	6	70	1145	7	72	1217	7	75	1292	7	0.5
.6	953	8	64	1017	8	66	1083	8	70	1153	8	72	1225	8	76	1301	9	.6
.7	962	9	64	1026	9	[illegible]	1092	9	70	1162	9	73	1235	10	76	1311	10	.7
.8	972	10	64	1036	10	67	1103	11	70	1173	11	73	1246	11	76	1322	11	.8
.9	984	12	64	1048	12	67	1115	12	71	1186	13	73	1259	13	76	1335	13	.9
1.0	996	12	65	1061	13	68	1129	14	70	1199	13	74	1273	14	76	1349	14	1.0
1.1	1011	15	65	1076	15	68	1144	15	71	1215	16	74	1289	16	76	1365	16	1.1
1.2	1027	16	65	1092	16	[illegible]	1160	16	72	1232	17	74	1306	17	77	1383	18	1.2
1.3	1044	17	66	1110	18	68	1178	18	72	1250	18	75	1325	19	77	1402	19	1.3
1.4	1062	18	67	1129	19	69	1198	20	72	1270	20	75	1345	20	78	1423	21	1.4
1.5	1082	20	67	1149	20	70	1219	21	72	1291	21	76	1367	22	78	1445	22	1.5
1.6	1104	22	67	1171	22	70	1241	22	73	1314	23	76	1390	23	79	1469	24	1.6
1.7	1126	22	68	1194	23	71	1265	24	73	1338	24	77	1415	25	79	1494	25	1.7
1.8	1150	24	69	1219	25	71	1290	25	74	1364	26	77	1441	26	80	1521	27	1.8
1.9	1176	26	69	1245	26	71	1316	26	75	1391	27	77	1468	27	81	1549	28	1.9
2.0	1203	27	69	1272	27	72	1344	28	75	1419	28	78	1497	29	[illegible]	1579	30	2.0
2.1	1231	28	70	1301	29	73	1374	30	75	1449	30	79	1528	31	82	1610	31	2.1
2.2	1260	29	71	1331	30	74	1405	31	76	1481	32	[illegible]	1561	33	[illegible]	1643	33	2.2
2.3	1291	31	72	1363	32	74	1437	32	77	1514	33	80	1594	33	83	1677	34	2.3
2.4	1324	33	72	1396	33	75	1471	34	78	1549	35	[illegible]	1629	35	84	1713	36	2.4
2.5	1357	33	73	1430	34	76	1506	35	78	1584	35	82	1666	37	85	1751	38	2.5
2.6	1392	35	74	1466	36	77	1543	37	79	1622	38	82	1704	38	[illegible]	1790	39	2.6
2.7	1429	37	74	1503	37	78	1581	38	80	1661	39	83	1744	40	86	1830	40	2.7
2.8	1467	38	75	1542	39	78	1620	39	81	1701	40	84	1785	41	87	1872	42	2.8
2.9	1506	39	76	1582	40	79	1661	41	82	1743	42	85	1828	43	88	1916	44	2.9
3.0	1547	41	77	1624	42	80	1704	43	82	1786	43	86	1872	44	89	1961	45	3.0
3.1	1589	42	77	1666	42	82	1748	44	83	1831	45	87	1918	46	90	2008	47	3.1
3.2	1632	43	79	1711	45	82	1793	45	84	1877	46	[illegible]	1965	47	91	2056	48	3.2
3.3	1677	45	80	1757	46	82	1839	46	86	1925	48	89	2014	49	92	2106	50	3.3
3.4	1723	46	81	1804	47	84	1888	49	86	1974	49	90	2064	50	[illegible]	2157	51	3.4
3.5	1770	47	82	1852	48	85	1937	49	[illegible]	2025	51	91	2116	52	94	2210	53	3.5
3.6	1819	49	84	1903	51	85	1988	51	89	2077	52	92	2169	53	95	2264	54	3.6
3.7	1869	50	85	1954	51	87	2041	53	90	2131	54	93	2224	55	96	2320	56	3.7
3.8	1921	52	86	2007	53	88	2095	54	91	2186	55	94	2280	56	98	2378	58	3.8
3.9	1974	53	87	2061	54	90	2151	56	92	2243	57	95	2338	58	99	2437	59	3.9
4.0	2029	55	88	2117	56	91	2208	57	93	2301	58	97	2398	60	99	2497	60	4.0
4.1	2085	56	89	2174	57	92	2266	58	95	2361	60	98	2459	61	100	2559	62	4.1
4.2	2142	57	90	2232	58	94	2326	60	96	2422	61	99	2521	62	102	2623	64	4.2
4.3	2200	58	92	2292	60	95	2387	61	97	2484	62	101	2585	64	103	2688	65	4.3
4.4	2261	61	92	2353	61	97	2450	63	98	2548	64	102	2650	65	105	2755	67	4.4
4.5	2322	61	94	2416	63	98	2514	64	100	2614	66	103	2717	67	106	2823	68	4.5
4.6	2385	63	96	2481	65	98	2579	65	102	2681	67	105	2786	69	107	2893	70	4.6
4.7	2450	65	96	2546	65	100	2646	67	103	2749	68	107	2856	70	108	2964	71	4.7
4.8	2515	65	98	2613	67	102	2715	69	104	2819	70	108	2927	71	110	3037	73	4.8
4.9	2582	67	100	2682	69	103	2785	70	106	2891	72	109	3000	73	112	3112	75	4.9
5.0	2650	68	102	2752	70	105	2857	72	107	2964	73	111	3075	75	113	3188	76	5.0
$d-\delta_0$ / $\alpha-\alpha_0$	4°.5		hor. diff.	4°.6		hor. diff.	4°.7		hor. diff.	4°.8		hor. diff.	4°.9		hor. diff.	5°.0		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$. $0^\circ.00001 = 0''.036 =$ unit.

$d-\delta_0$ \ $\alpha-\alpha_0$	5°.0		hor. diff.	5°.1		hor. diff.	5°.2		hor. diff.	5°.3		hor. diff.	5°.4		hor. diff.	5°.5		$\alpha-\alpha_0$ / $d-\delta_0$
0.0°	01272	1	79	01351	1	81	01432	1	85	01517	1	88	01605	0	91	01696	0	0.0°
.1	1273	3	79	1352	2	81	1433	3	85	1518	2	87	1605	3	91	1696	3	.1
.2	1276	4	78	1354	4	82	1436	4	84	1520	4	88	1608	4	91	1699	5	.2
.3	1280	5	78	1358	6	82	1440	5	84	1524	6	88	1612	6	92	1704	5	.3
.4	1285	7	79	1364	7	81	1445	7	85	1530	7	88	1618	7	91	1709	8	.4
0.5	1292	9	79	1371	8	81	1452	9	85	1537	9	88	1625	9	92	1717	9	0.5
.6	1301	10	78	1379	10	82	1461	10	85	1546	11	88	1634	11	92	1726	11	.6
.7	1311	11	78	1389	12	82	1471	12	86	1557	12	[illegible]	1645	12	92	1737	12	.7
.8	1322	13	79	1401	13	82	1483	14	86	1569	14	88	1657	14	92	1749	14	.8
.9	1335	14	79	1414	15	83	1497	15	86	1583	15	88	1671	16	92	1763	16	.9
1.0	1349	16	80	1429	17	83	1512	17	86	1598	17	89	1687	17	92	1779	18	1.0
1.1	1365	18	81	1446	18	83	1529	18	86	1615	19	89	1704	19	93	1797	20	1.1
1.2	1383	19	81	1464	19	83	1547	20	87	1634	20	89	1723	21	94	1817	21	1.2
1.3	1402	21	81	1483	21	84	1567	21	87	1654	22	90	1744	22	94	1838	22	1.3
1.4	1423	22	81	1504	23	84	1588	23	88	1676	23	90	1766	24	94	1860	25	1.4
1.5	1445	24	82	1527	24	84	1611	25	88	1699	25	91	1790	26	95	1885	26	1.5
1.6	1469	25	82	1551	26	85	1636	26	88	1724	27	92	1816	27	95	1911	28	1.6
1.7	1494	27	83	1577	27	85	1662	28	89	1751	28	92	1843	29	96	1939	29	1.7
1.8	1521	28	83	1604	29	86	1690	29	89	1779	30	93	1872	30	96	1968	31	1.8
1.9	1549	30	84	1633	30	86	1719	31	90	1809	32	93	1902	33	97	1999	33	1.9
2.0	1579	31	84	1663	32	87	1750	33	91	1841	33	94	1935	34	97	2032	34	2.0
2.1	1610	33	85	1695	34	88	1783	34	91	1874	35	95	1969	35	97	2066	36	2.1
2.2	1643	34	86	1729	35	88	1817	36	92	1909	37	95	2004	37	98	2102	38	2.2
2.3	1677	36	87	1764	36	89	1853	37	93	1946	38	95	2041	39	99	2140	40	2.3
2.4	1713	38	87	1800	38	90	1890	39	94	1984	39	96	2080	41	100	2180	41	2.4
2.5	1751	39	87	1838	40	91	1929	41	94	2023	42	98	2121	42	100	2221	43	2.5
2.6	1790	40	88	1878	42	92	1970	42	95	2065	43	98	2163	43	101	2264	45	2.6
2.7	1830	42	90	1920	43	92	2012	44	96	2108	44	98	2206	46	103	2309	46	2.7
2.8	1872	44	91	1963	44	93	2056	45	96	2152	46	100	2252	47	103	2355	48	2.8
2.9	1916	45	91	2007	46	94	2101	47	97	2198	48	101	2299	48	104	2403	50	2.9
3.0	1961	47	92	2053	47	95	2148	49	98	2246	50	101	2347	51	106	2453	51	3.0
3.1	2008	48	92	2100	49	97	2197	50	99	2296	51	102	2398	52	106	2504	53	3.1
3.2	2056	50	93	2149	51	98	2247	52	100	2347	53	103	2450	54	107	2557	55	3.2
3.3	2106	51	94	2200	53	99	2299	53	101	2400	54	104	2504	55	108	2612	56	3.3
3.4	2157	53	96	2253	54	99	2352	55	102	2454	56	105	2559	57	109	2668	58	3.4
3.5	2210	54	97	2307	55	100	2407	57	103	2510	58	106	2616	59	110	2726	60	3.5
3.6	2264	56	98	2362	57	102	2464	58	104	2568	59	107	2675	61	111	2786	62	3.6
3.7	2320	58	99	2419	58	103	2522	60	105	2627	61	109	2736	62	112	2848	63	3.7
3.8	2378	59	100	2478	60	104	2582	61	106	2688	63	110	2798	64	113	2911	65	3.8
3.9	2437	60	101	2538	62	105	2643	63	108	2751	64	111	2862	65	114	2976	67	3.9
4.0	2497	62	103	2600	63	106	2706	64	109	2815	66	112	2927	67	116	3043	68	4.0
4.1	2559	64	104	2663	65	107	2770	66	111	2881	68	113	2994	69	117	3111	70	4.1
4.2	2623	65	105	2728	67	108	2836	68	113	2949	69	114	3063	70	118	3181	72	4.2
4.3	2688	67	107	2795	68	109	2904	70	114	3018	70	115	3133	72	120	3253	73	4.3
4.4	2755	68	108	2863	69	111	2974	71	114	3088	72	117	3205	74	121	3326	75	4.4
4.5	2823	70	109	2932	72	113	3045	73	115	3160	74	119	3279	76	122	3401	77	4.5
4.6	2893	71	111	3004	73	114	3118	74	116	3234	76	121	3355	77	123	3478	79	4.6
4.7	2964	73	113	3077	75	115	3192	76	118	3310	78	122	3432	79	125	3557	80	4.7
4.8	3037	75	115	3152	76	116	3268	78	120	3388	79	123	3511	80	126	3637	82	4.8
4.9	3112	76	116	3228	77	118	3346	78	121	3467	80	124	3591	82	128	3719	84	4.9
5.0	3188		117	3305		119	3424		123	3547		126	3673		130	3803		5.0
$d-\delta_0$ / $\alpha-\alpha_0$	5°.0		hor. diff.	5°.1		hor. diff.	5°.2		hor. diff.	5°.3		hor. diff.	5°.4		hor. diff.	5°.5		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha)^{\circ}.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^{\circ}$. $0^{\circ}.00001 = 0''.036 =$ unit.

$\alpha-\alpha_0$ / $d-\delta_0$	5°.5		hor. diff.	5°.6		hor. diff.	5°.7		hor. diff.	5°.8		hor. diff.	5°.9		hor. diff.	6°.0		$\alpha-\alpha_0$ / $d-\delta_0$
0.0	01696	0	94	01790	1	98	01888	1	101	01989	1	105	02094	1	109	02203	1	0.0
.1	1696	3	95	1791	3	98	1889	3	101	1990	3	105	2095	3	109	2204	3	.1
.2	1699	5	95	1794	4	98	1892	4	101	1993	5	105	2098	5	109	2207	5	.2
.3	1704	5	94	1798	6	98	1896	6	102	1998	6	105	2103	6	109	2212	6	.3
.4	1709	8	95	1804	8	98	1902	8	102	2004	8	105	2109	8	109	2218	8	.4
0.5	1717	9	95	1812	9	98	1910	10	102	2012	10	105	2117	10	109	2226	10	0.5
.6	1726	11	95	1821	11	99	1920	11	102	2022	11	105	2127	12	109	2236	12	.6
.7	1737	12	95	1832	13	99	1931	13	102	2033	13	106	2139	13	109	2248	14	.7
.8	1749	14	96	1845	15	99	1944	15	102	2046	15	106	2152	15	110	2262	15	.8
.9	1763	16	97	1860	16	99	1959	16	102	2061	17	106	2167	18	110	2277	18	.9
1.0	1779	18	97	1876	18	99	1975	18	103	2078	19	107	2185	19	110	2295	19	1.0
1.1	1797	20	97	1894	20	99	1993	20	104	2097	20	107	2204	21	110	2314	21	1.1
1.2	1817	21	97	1914	21	99	2013	22	104	2117	22	108	2225	22	110	2335	23	1.2
1.3	1838	22	97	1935	23	100	2035	24	104	2139	24	108	2247	24	111	2358	25	1.3
1.4	1860	25	98	1958	25	101	2059	25	104	2163	26	108	2271	26	112	2383	27	1.4
1.5	1885	26	98	1983	26	101	2084	27	105	2189	28	108	2297	28	113	2410	28	1.5
1.6	1911	28	98	2009	28	102	2111	29	106	2217	29	108	2325	30	113	2438	30	1.6
1.7	1939	29	98	2037	30	103	2140	30	106	2246	31	109	2355	32	113	2468	32	1.7
1.8	1968	31	99	2067	32	103	2170	32	107	2277	32	110	2387	33	113	2500	34	1.8
1.9	1999	33	100	2099	33	103	2202	34	107	2309	35	111	2420	35	114	2534	36	1.9
2.0	2032	34	100	2132	35	104	2236	36	108	2344	37	111	2455	37	115	2570	38	2.0
2.1	2066	36	101	2167	37	105	2272	38	109	2381	38	111	2492	39	116	2608	39	2.1
2.2	2102	38	102	2204	39	106	2310	39	109	2419	40	112	2531	41	116	2647	41	2.2
2.3	2140	40	103	2243	40	106	2349	41	110	2459	42	113	2572	42	116	2688	43	2.3
2.4	2180	41	103	2283	42	107	2390	43	111	2501	43	113	2614	44	117	2741	46	2.4
2.5	2221	43	104	2325	44	108	2433	45	111	2544	45	114	2658	46	119	2777	47	2.5
2.6	2264	45	105	2369	46	109	2478	46	111	2589	47	115	2704	48	120	2824	49	2.6
2.7	2309	46	106	2415	47	109	2524	48	112	2636	49	116	2752	50	121	2873	50	2.7
2.8	2355	48	107	2462	49	110	2572	50	113	2685	51	117	2802	51	121	2923	52	2.8
2.9	2403	50	108	2511	50	111	2622	51	114	2736	52	117	2853	54	122	2975	54	2.9
3.0	2453	51	108	2561	52	112	2673	53	115	2788	54	119	2907	55	122	3029	56	3.0
3.1	2504	53	109	2613	54	113	2726	55	116	2842	56	120	2962	57	123	3085	58	3.1
3.2	2557	55	110	2667	56	114	2781	57	117	2898	58	121	3019	58	124	3143	60	3.2
3.3	2612	56	111	2723	57	115	2838	58	118	2956	60	121	3077	61	126	3203	62	3.3
3.4	2668	58	112	2780	60	116	2896	60	120	3016	61	122	3138	63	127	3265	63	3.4
3.5	2726	60	114	2840	61	116	2956	62	121	3077	63	124	3201	64	127	3328	65	3.5
3.6	2786	62	115	2901	62	117	3018	64	122	3140	65	125	3265	66	128	3393	67	3.6
3.7	2848	63	115	2963	65	119	3082	66	123	3205	67	126	3331	68	129	3460	69	3.7
3.8	2911	65	117	3028	66	120	3148	67	124	3272	68	127	3399	70	130	3529	71	3.8
3.9	2976	67	118	3094	68	121	3215	69	125	3340	70	129	3469	71	131	3600	73	3.9
4.0	3043	68	119	3162	69	122	3284	71	126	3410	72	130	3540	73	133	3673	75	4.0
4.1	3111	70	120	3231	72	124	3355	73	127	3482	74	131	3613	76	135	3748	76	4.1
4.2	3181	72	122	3303	73	125	3428	74	128	3556	76	133	3689	77	135	3824	78	4.2
4.3	3253	73	123	3376	75	126	3502	76	130	3632	78	134	3766	79	136	3902	81	4.3
4.4	3326	75	125	3451	77	127	3578	78	132	3710	80	135	3845	80	138	3983	82	4.4
4.5	3401	77	127	3528	78	128	3656	80	134	3790	81	135	3925	82	140	4065	84	4.5
4.6	3478	79	128	3606	80	130	3736	81	135	3871	83	136	4007	85	142	4149	86	4.6
4.7	3557	80	129	3686	81	131	3817	83	137	3954	84	138	4092	86	143	4235	88	4.7
4.8	3637	82	130	3767	84	133	3900	85	138	4038	86	140	4178	88	145	4323	89	4.8
4.9	3779	84	132	3851	85	134	3985	87	139	4124	88	142	4266	90	146	4412	91	4.9
5.0	3803		133	3936		136	4072		140	4212		144	4356		147	4503		5.0
$d-\delta_0$ / $\alpha-\alpha_0$	5°.5		hor. diff.	5°.6		hor. diff.	5°.7		hor. diff.	5°.8		hor. diff.	5°.9		hor. diff.	6°.0		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi\,\sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi\,\sec d$ **is always numerically larger than** $(\alpha-\alpha_0)^\circ$. $0^\circ.00001 = 0''.036 =$ unit.

$\alpha-\alpha_0$ / $d-\delta_0$	6°.0		*hor. diff.*	6°.1		*hor. diff.*	6°.2		*hor. diff.*	6°.3		*hor. diff.*	6°.4		*hor. diff.*	6°.5		$\alpha-\alpha_0$ / $d-\delta_0$
0.0°	02203	1	112	02315	1	116	02431	1	120	02551	1	124	02675	1	128	02803	1	0.0°
.1	2204	3	112	2316	3	116	2432	3	120	2552	3	124	2676	3	128	2804	3	.1
.2	2207	5	112	2319	5	116	2435	5	120	2555	5	124	2679	5	128	2807	5	.2
.3	2212	6	112	2324	7	116	2440	7	120	2560	7	124	2684	7	128	2812	7	.3
.4	2218	8	113	2331	8	116	2447	9	120	2567	9	124	2691	9	128	2819	9	.4
0.5	2226	10	113	2339	10	117	2456	10	120	2576	11	124	2700	11	128	2828	11	0.5
.6	2236	12	113	2349	12	117	2466	12	121	2587	12	124	2711	12	128	2839	13	.6
.7	2248	14	113	2361	14	117	2478	14	121	2599	14	124	2723	15	129	2852	15	.7
.8	2262	15	113	2375	16	117	2492	16	121	2613	16	125	2738	17	129	2867	17	.8
.9	2277	18	114	2391	18	117	2508	18	121	2629	19	126	2755	18	129	2884	18	.9
1.0	2295	19	114	2409	20	117	2526	20	122	2648	20	125	2773	21	129	2902	21	1.0
1.1	2314	21	115	2429	21	117	2546	22	122	2668	22	126	2794	22	129	2923	23	1.1
1.2	2335	23	115	2450	23	118	2568	24	122	2690	24	126	2816	25	130	2946	25	1.2
1.3	2358	25	115	2473	25	119	2592	25	122	2714	26	127	2841	26	130	2971	27	1.3
1.4	2383	27	115	2498	27	119	2617	27	123	2740	28	127	2867	29	131	2998	29	1.4
1.5	2410	28	115	2525	29	119	2644	30	124	2768	30	128	2896	30	131	3027	31	1.5
1.6	2438	30	116	2554	31	120	2674	32	124	2798	32	128	2926	32	132	3058	33	1.6
1.7	2468	32	117	2585	33	121	2706	33	124	2830	34	128	2958	35	133	3091	35	1.7
1.8	2500	34	118	2618	35	121	2739	35	125	2864	35	129	2993	36	133	3126	36	1.8
1.9	2534	36	119	2653	36	121	2774	37	125	2899	38	130	3029	38	133	3162	39	1.9
2.0	2570	38	119	2689	38	122	2811	39	126	2937	40	130	3067	40	134	3201	41	2.0
2.1	2608	39	119	2727	40	123	2850	41	127	2977	41	130	3107	42	135	3242	43	2.1
2.2	2647	41	120	2767	42	124	2891	42	127	3018	44	131	3149	44	136	3285	44	2.2
2.3	2688	43	121	2809	44	124	2933	45	129	3062	45	131	3193	46	136	3329	47	2.3
2.4	2741	46	122	2853	46	125	2978	46	129	3107	47	132	3239	48	137	3376	49	2.4
2.5	2777	47	122	2899	48	125	3024	48	130	3154	49	133	3287	50	138	3425	51	2.5
2.6	2824	49	123	2947	49	125	3072	51	131	3203	51	134	3337	52	139	3476	53	2.6
2.7	2873	50	123	2996	51	127	3123	52	131	3254	53	135	3389	54	140	3529	54	2.7
2.8	2923	52	124	3047	54	128	3175	54	132	3307	55	136	3443	56	140	3583	57	2.8
2.9	2975	54	126	3101	55	128	3229	56	133	3362	57	137	3499	58	141	3640	59	2.9
3.0	3029	56	127	3156	57	129	3285	58	134	3419	59	138	3557	60	142	3699	61	3.0
3.1	3085	58	128	3213	59	130	3343	60	135	3478	61	139	3617	62	143	3760	63	3.1
3.2	3143	60	129	3272	61	131	3403	62	136	3539	63	140	3679	63	144	3823	64	3.2
3.3	3203	62	130	3333	62	132	3465	63	137	3602	64	140	3742	66	145	3887	67	3.3
3.4	3265	63	130	3395	64	133	3528	66	138	3666	67	142	3808	68	146	3954	69	3.4
3.5	3328	65	131	3459	66	135	3594	68	139	3733	69	143	3876	70	147	4023	71	3.5
3.6	3393	67	132	3525	69	137	3662	69	140	3802	70	144	3946	72	148	4094	73	3.6
3.7	3460	69	134	3594	70	137	3731	71	141	3872	73	146	4018	73	149	4167	74	3.7
3.8	3529	71	135	3664	72	138	3802	74	143	3945	75	146	4091	76	150	4241	77	3.8
3.9	3600	73	136	3736	74	140	3876	75	144	4020	76	147	4167	78	151	4318	79	3.9
4.0	3673	75	137	3810	76	141	3951	77	145	4096	78	149	4245	80	152	4397	81	4.0
4.1	3748	76	138	3886	78	142	4028	79	146	4174	81	151	4325	81	153	4478	83	4.1
4.2	3824	78	140	3964	80	143	4107	81	148	4255	82	151	4406	84	155	4561	84	4.2
4.3	3902	81	142	4044	82	144	4188	83	149	4337	84	153	4490	86	155	4645	87	4.3
4.4	3983	82	143	4126	83	145	4271	85	150	4421	86	155	4576	87	156	4732	89	4.4
4.5	4065	84	144	4209	85	147	4356	87	151	4507	88	156	4663	89	158	4821	91	4.5
4.6	4149	86	145	4294	88	149	4443	88	152	4595	90	157	4752	92	160	4912	93	4.6
4.7	4235	88	147	4382	89	149	4531	91	154	4685	93	159	4844	93	161	5005	95	4.7
4.8	4323	89	148	4471	91	151	4622	93	156	4778	94	159	4937	96	163	5100	97	4.8
4.9	4412	91	150	4562	93	153	4715	94	157	4872	96	161	5033	97	164	5197	99	4.9
5.0	4503		152	4655		154	4809		159	4968		162	5130		166	5296		5.0
$d-\delta_0$ / $\alpha-\alpha_0$	6°.0		*hor. diff.*	6°.1		*hor. diff.*	6°.2		*hor. diff.*	6°.3		*hor. diff.*	6°.4		*hor. diff.*	6°.5		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^{\circ}.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^{\circ}$. $0^{\circ}.00001 = 0''.036 =$ unit.

$d - \delta_0$ \ $\alpha - \alpha_0$	6°.5		*hor. diff.*	6°.6		*hor. diff.*	6°.7		*hor. diff.*	6°.8		*hor. diff.*	6°.9		*hor. diff.*	7°.0		$\alpha - \alpha_0$ / $d - \delta_0$
0.0	02803	1	132	02935	1	136	03071	1	140	03211	2	144	03355	2	149	03504	2	0.0
.1	2804	3	132	2936	3	136	3072	3	141	3213	3	144	3357	3	149	3506	3	.1
.2	2807	5	132	2939	5	136	3075	5	141	3216	5	144	3360	5	149	3509	5	.2
.3	2812	7	132	2944	7	136	3080	7	141	3221	6	144	3365	8	149	3514	7	.3
.4	2819	9	132	2951	9	136	3087	9	141	3228	9	145	3373	9	148	3521	9	.4
0.5	2828	11	132	2960	11	136	3096	11	141	3237	11	145	3382	12	148	3530	12	0.5
.6	2839	13	132	2971	13	136	3107	13	141	3248	14	146	3394	13	148	3542	14	.6
.7	2852	15	132	2984	16	136	3120	16	142	3262	16	145	3407	16	149	3556	16	.7
.8	2867	17	133	3000	17	136	3136	17	142	3278	17	145	3423	18	149	3572	18	.8
.9	2884	18	133	3017	19	136	3153	20	142	3295	20	146	3441	20	149	3590	21	.9
1.0	2902	21	134	3036	21	137	3173	22	142	3315	22	146	3461	22	150	3611	23	1.0
1.1	2923	23	134	3057	23	138	3195	23	142	3337	24	146	3483	25	151	3634	24	1.1
1.2	2946	25	134	3080	26	138	3218	26	143	3361	26	147	3508	26	150	3658	27	1.2
1.3	2971	27	135	3106	27	138	3244	28	143	3387	28	147	3534	28	151	3685	29	1.3
1.4	2998	29	135	3133	29	139	3272	30	143	3415	30	147	3562	31	152	3714	31	1.4
1.5	3027	31	135	3162	31	140	3302	32	143	3445	32	148	3593	33	152	3745	33	1.5
1.6	3058	33	135	3193	34	141	3334	34	143	3477	35	149	3626	35	152	3778	36	1.6
1.7	3091	35	136	3227	35	141	3368	35	144	3512	36	149	3661	37	153	3814	37	1.7
1.8	3126	36	136	3262	38	141	3403	38	145	3548	38	150	3698	39	153	3851	39	1.8
1.9	3162	39	138	3300	39	141	3441	40	145	3586	41	151	3737	41	153	3890	42	1.9
2.0	3201	41	138	3339	41	142	3481	42	146	3627	43	151	3778	43	154	3932	44	2.0
2.1	3242	43	138	3380	44	143	3523	44	147	3670	44	151	3821	46	155	3976	46	2.1
2.2	3285	44	139	3424	45	143	3567	46	147	3714	47	153	3867	47	155	4022	49	2.2
2.3	3329	47	140	3469	48	144	3613	49	148	3761	49	153	3914	50	157	4071	50	2.3
2.4	3376	49	141	3517	49	145	3662	50	148	3810	51	154	3964	52	157	4121	53	2.4
2.5	3425	51	141	3566	52	146	3712	52	149	3861	53	155	4016	54	158	4174	55	2.5
2.6	3476	53	142	3618	53	146	3764	55	150	3914	56	156	4070	56	159	4229	57	2.6
2.7	3529	54	142	3671	56	148	3819	56	151	3970	57	156	4126	58	160	4286	59	2.7
2.8	3583	57	144	3727	57	148	3875	59	152	4027	59	157	4184	60	161	4345	61	2.8
2.9	3640	59	144	3784	60	150	3934	60	152	4086	62	158	4244	62	162	4406	63	2.9
3.0	3699	61	145	3844	62	150	3994	63	154	4148	64	158	4306	64	163	4469	65	3.0
3.1	3760	63	146	3906	64	151	4057	64	155	4212	65	158	4370	67	164	4534	68	3.1
3.2	3823	64	147	3970	65	151	4121	67	156	4277	68	160	4437	69	165	4602	70	3.2
3.3	3887	67	148	4035	68	153	4188	69	157	4345	70	161	4506	71	166	4672	72	3.3
3.4	3954	69	149	4103	70	154	4257	71	158	4415	72	162	4577	73	167	4744	74	3.4
3.5	4023	71	150	4173	72	155	4328	73	159	4487	74	163	4650	75	168	4818	76	3.5
3.6	4094	73	151	4245	74	156	4401	75	160	4561	77	164	4725	78	169	4894	79	3.6
3.7	4167	74	152	4319	76	157	4476	77	162	4638	78	165	4803	79	170	4973	80	3.7
3.8	4241	77	154	4395	79	158	4553	80	163	4716	80	166	4882	82	171	5053	83	3.8
3.9	4318	79	156	4474	82	159	4633	81	163	4796	83	168	4964	84	172	5136	85	3.9
4.0	4397	81	157	4554	82	160	4714	83	165	4879	84	169	5048	86	173	5221	87	4.0
4.1	4478	83	158	4636	84	161	4797	86	166	4963	87	171	5134	88	174	5308	90	4.1
4.2	4561	84	159	4720	86	163	4883	87	167	5050	89	172	5222	90	176	5398	91	4.2
4.3	4645	87	161	4806	89	164	4970	90	169	5139	91	173	5312	93	177	5489	94	4.3
4.4	4732	89	163	4895	90	165	5060	92	170	[illegible]	93	175	5405	94	178	5583	96	4.4
4.5	4821	91	164	4985	92	167	5152	94	171	5323	95	176	5499	97	180	5679	98	4.5
4.6	4912	93	165	5077	95	169	5246	96	172	5418	98	178	5596	98	181	5777	100	4.6
4.7	5005	95	167	5172	95	170	5342	98	174	5516	99	178	5694	101	183	5877	103	4.7
4.8	5100	97	168	5268	98	172	5440	100	175	5615	101	180	5795	103	185	5980	104	4.8
4.9	5197	99	169	5366	101	174	5540	102	170	[illegible]	[illegible]	182	5598	105	186	6084	107	4.9
5.0	5296		171	5467		175	5642		178	[illegible]		183	6003		188	6191		5.0
$d - \delta_0$ / $\alpha - \alpha_0$	6°.5		*hor. diff.*	6°.6		*hor. diff.*	6°.7		*hor. diff.*	6°.8		*hor. diff.*	6°.9		*hor. diff.*	7°.0		$d - \delta_0$ / $\alpha - \alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^{\circ}.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^{\circ}$. **$0^{\circ}.00001 = 0''.036 =$ unit.**

$\alpha-\alpha_0$ / $d-\delta_0$	7°.0		hor. diff.	7°.1		hor. diff.	7°.2		hor. diff.	7°.3		hor. diff.	7°.4		hor. diff.	7°.5		$\alpha-\alpha_0$ / $d-\delta_0$
0°.0	03504	2	153	03657	1	157	03814	2	162	03976	2	166	04142	2	171	04313	2	0°.0
.1	3506	3	152	3658	4	158	3816	3	162	3978	3	166	4144	3	171	4315	3	.1
.2	3509	5	153	3662	6	157	3819	5	162	3981	6	166	4147	6	171	4318	6	.2
.3	3514	7	154	3668	7	156	3824	8	163	3987	7	166	4153	8	171	4324	8	.3
.4	3521	9	154	3675	9	157	3832	10	162	3994	10	167	4161	10	171	4332	10	.4
0.5	3530	12	154	3684	12	158	3842	12	162	4004	13	167	4171	13	171	4342	13	0.5
.6	3542	14	154	3696	14	158	3854	14	163	4017	14	167	4184	14	171	4355	15	.6
.7	3556	16	154	3710	16	158	3868	17	163	4031	17	167	4198	17	172	4370	17	.7
.8	3572	18	154	3726	18	159	3885	18	163	4048	19	167	4215	19	172	4387	20	.8
.9	3590	21	154	3744	21	159	3903	21	164	4067	21	167	4234	22	173	4407	21	.9
1.0	3611	23	154	3765	23	159	3924	23	164	4088	24	168	4256	24	172	4428	24	1.0
1.1	3634	24	154	3788	25	159	3947	25	165	4112	25	168	4280	26	172	4452	27	1.1
1.2	3658	27	155	3813	27	159	3972	28	165	4137	28	169	4306	28	173	4479	28	1.2
1.3	3685	29	155	3840	29	160	4000	30	165	4165	31	169	4334	31	173	4507	31	1.3
1.4	3714	31	155	3869	32	161	4030	32	166	4196	32	169	4365	33	173	4538	34	1.4
1.5	3745	33	156	3901	34	161	4062	34	166	4228	35	170	4398	35	174	4572	36	1.5
1.6	3778	36	157	3935	36	161	4096	37	167	4263	36	170	4433	37	175	4608	38	1.6
1.7	3814	37	157	3971	38	162	4133	38	166	4299	39	171	4470	40	176	4646	40	1.7
1.8	3851	39	158	4009	40	162	4171	41	167	4338	42	172	4510	42	176	4686	42	1.8
1.9	3890	42	159	4049	43	163	4212	43	168	4380	43	172	4552	44	176	4728	45	1.9
2.0	3932	44	160	4092	45	163	4255	45	168	4423	46	173	4596	46	177	4773	47	2.0
2.1	3976	46	161	4137	47	163	4300	48	169	4469	48	173	4642	49	178	4820	50	2.1
2.2	4022	49	162	4184	48	164	4348	50	169	4517	50	174	4691	51	179	4870	52	2.2
2.3	4071	50	161	4232	52	166	4398	52	169	4567	53	175	4742	54	180	4922	54	2.3
2.4	4121	53	163	4284	53	166	4450	54	170	4620	55	176	4796	55	180	4976	56	2.4
2.5	4174	55	163	4337	55	167	4504	56	171	4675	57	176	4851	58	181	5032	59	2.5
2.6	4229	57	163	4392	58	168	4560	59	172	4732	60	177	4909	60	182	5091	61	2.6
2.7	4286	59	164	4450	60	169	4619	60	173	4792	61	177	4969	63	183	5152	63	2.7
2.8	4345	61	165	4510	62	169	4679	63	174	4853	64	179	5032	64	183	5215	65	2.8
2.9	4406	63	166	4572	64	170	4742	65	175	4917	66	179	5096	67	184	5280	68	2.9
3.0	4469	65	166	4636	66	171	4807	67	176	4983	68	180	5163	69	185	5348	70	3.0
3.1	4534	68	168	4702	69	172	4874	70	177	5051	71	181	5232	72	186	5418	73	3.1
3.2	4602	70	169	4771	71	173	4944	72	178	5122	73	182	5304	74	187	5491	75	3.2
3.3	4672	72	170	4842	73	174	5016	74	179	5195	75	183	5378	76	188	5566	77	3.3
3.4	4744	74	171	4915	75	175	5090	76	180	5270	77	184	5454	78	189	5643	79	3.4
3.5	4818	76	172	4990	77	176	5166	78	181	5347	80	185	5532	81	190	5722	82	3.5
3.6	4894	79	173	5067	80	177	5244	81	183	5427	82	186	5613	83	191	5804	84	3.6
3.7	4973	80	174	5147	82	178	5325	83	184	5509	84	187	5696	85	192	5888	86	3.7
3.8	5053	83	176	5229	84	179	5408	86	185	5593	86	188	5781	88	193	5974	89	3.8
3.9	5136	85	177	5313	86	181	5494	87	185	5679	89	190	5869	90	194	6063	91	3.9
4.0	5221	87	178	5399	88	182	5581	90	187	5768	91	191	5959	92	195	6154	93	4.0
4.1	5308	90	179	5487	91	184	5671	92	188	5859	93	192	6051	95	196	6247	96	4.1
4.2	5398	91	180	5578	93	185	5763	94	189	5952	96	194	6146	97	197	6343	99	4.2
4.3	5489	94	182	5671	95	186	5857	96	191	6048	97	195	6243	99	199	6442	101	4.3
4.4	5583	96	183	5766	97	187	5953	99	192	6145	100	197	6342	101	201	6543	102	4.4
4.5	5679	98	184	5863	99	189	6052	101	193	6245	102	198	6443	104	202	6645	105	4.5
4.6	5777	100	185	5962	102	191	6153	103	194	6347	105	200	6547	106	203	6750	108	4.6
4.7	5877	103	187	6064	104	192	6256	106	196	6452	107	201	6653	108	205	6858	110	4.7
4.8	5980	104	188	6168	106	194	6362	107	197	6559	109	202	6761	111	207	6968	112	4.8
4.9	6084	107	190	6274	108	195	6469	110	199	6668	111	204	6872	113	208	7080	114	4.9
5.0	6191		191	6382		197	6579		200	6779		206	6985		209	7194		5.0
$d-\delta_0$ / $\alpha-\alpha_0$	7°.0		hor. diff.	7°.1		hor. diff.	7°.2		hor. diff.	7°.3		hor. diff.	7°.4		hor. diff.	7°.5		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi\,\sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi\,\sec d$ is always numerically larger than $(\alpha-\alpha_0)^\circ$. $0^\circ.00001 = 0''.036 = $ unit.

$d-\delta_0$ \ $\alpha-\alpha_0$	7°.5		hor. diff.	7°.6		hor. diff.	7°.7		hor. diff.	7°.8		hor. diff.	7°.9		hor. diff.	8°.0		$\alpha-\alpha_0$ / $d-\delta_0$
0°.0	04313	2	176	04489	2	180	04669	2	185	04854	2	190	05044	2	195	05239	2	0°.0
.1	4315	3	176	4491	3	180	4671	3	185	4856	3	190	5046	4	195	5241	4	.1
.2	4318	6	176	4494	6	180	4674	6	185	4859	6	191	5050	6	195	5245	6	.2
.3	4324	8	176	4500	8	180	4680	9	185	4865	9	191	5056	9	195	5251	9	.3
.4	4332	10	176	4508	10	181	4689	10	185	4874	10	191	5065	10	195	5260	10	.4
0.5	4342	13	176	4518	13	181	4699	13	185	4884	13	191	5075	13	195	5270	14	0.5
.6	4355	15	176	4531	15	181	4712	15	185	4897	16	191	5088	16	196	5284	16	.6
.7	4370	17	176	4546	17	181	4727	18	186	4913	18	191	5104	18	196	5300	18	.7
.8	4387	20	176	4563	20	182	4745	20	186	4931	20	191	5122	21	196	5318	21	.8
.9	4407	21	176	4583	22	182	4765	22	186	4951	23	192	5143	23	196	5339	23	.9
1.0	4428	24	177	4605	24	182	4787	25	187	4974	25	192	5166	25	196	5362	26	**1.0**
1.1	4452	27	177	4629	27	183	4812	27	187	4999	28	192	5191	28	197	5388	28	1.1
1.2	4479	28	177	4656	29	183	4839	30	188	5027	30	192	5219	30	197	5416	30	1.2
1.3	4507	31	178	4685	32	184	4869	32	188	5057	32	192	5249	33	197	5446	33	1.3
1.4	4538	34	179	4717	34	184	4901	34	188	5089	35	193	5282	35	197	5479	36	1.4
1.5	4572	36	179	4751	36	184	4935	37	189	5124	37	193	5317	37	198	5515	38	1.5
1.6	4608	38	179	4787	39	185	4972	39	189	5161	39	193	5354	40	199	5553	40	1.6
1.7	4646	40	180	4826	41	185	5011	41	189	5200	42	194	5394	43	199	5593	43	1.7
1.8	4686	42	181	4867	43	185	5052	43	190	5242	44	195	5437	45	199	5636	46	1.8
1.9	4728	45	182	4910	45	185	5095	46	191	5286	47	196	5482	47	200	5682	48	1.9
2.0	4773	47	182	4955	48	186	5141	48	192	5333	49	196	5529	50	201	5730	50	**2.0**
2.1	4820	50	183	5003	50	186	5189	51	193	5382	51	197	5579	52	201	5780	53	2.1
2.2	4870	52	183	5053	52	187	5240	53	193	5433	54	198	5631	54	202	5833	55	2.2
2.3	4922	54	183	5105	55	188	5293	56	194	5487	56	198	5685	57	203	5888	58	2.3
2.4	4976	56	184	5160	57	189	5349	58	194	5543	59	199	5742	60	204	5946	60	2.4
2.5	5032	59	185	5217	59	190	5407	60	195	5602	61	200	5802	62	204	6006	63	2.5
2.6	5091	61	185	5276	62	191	5467	63	196	5663	64	201	5864	64	205	6069	65	2.6
2.7	5152	63	186	5338	65	192	5530	65	197	5727	66	201	5928	67	206	6134	67	2.7
2.8	5215	65	188	5403	66	192	5595	67	198	5793	68	202	5995	69	206	6201	70	2.8
2.9	5280	68	189	5469	69	193	5662	70	199	5861	71	203	6064	72	207	6271	73	2.9
3.0	5348	70	190	5538	71	194	5732	72	200	5932	73	204	6136	74	208	6344	75	**3.0**
3.1	5418	73	191	5609	74	195	5804	75	201	6005	75	205	6210	76	209	6419	78	3.1
3.2	5491	75	192	5683	76	196	5879	77	201	6080	78	206	6286	79	211	6497	80	3.2
3.3	5566	77	193	5759	78	197	5956	79	202	6158	80	207	6365	81	212	6577	82	3.3
3.4	5643	79	194	5837	80	198	6035	82	203	6238	83	208	6446	84	213	6659	85	3.4
3.5	5722	82	195	5917	83	200	6117	84	204	6321	85	209	6530	86	214	6744	87	3.5
3.6	5804	84	196	6000	85	201	6201	86	205	6406	88	210	6616	89	215	6831	90	3.6
3.7	5888	86	197	6085	88	202	6287	89	207	6494	90	211	6705	91	216	6921	92	3.7
3.8	5974	89	199	6173	90	203	6376	91	208	6584	92	212	6796	94	217	7013	95	3.8
3.9	6063	91	200	6263	93	204	6467	94	209	6676	95	214	6890	96	218	7108	97	3.9
4.0	6154	93	202	6356	95	205	6561	96	210	6771	97	215	6986	98	219	7205	100	**4.0**
4.1	6247	96	204	6451	97	206	6657	98	211	6868	100	216	7084	102	221	7305	102	4.1
4.2	6343	99	205	6548	99	207	6755	101	213	6968	102	218	7186	103	221	7407	105	4.2
4.3	6442	101	205	6647	102	209	6856	103	214	7070	104	219	7289	106	223	7512	107	4.3
4.4	6543	102	206	6749	104	210	6959	106	215	7174	107	221	7395	108	224	7619	110	4.4
4.5	6645	105	208	6853	107	212	7065	108	216	7281	109	222	7503	111	226	7729	113	4.5
4.6	6750	108	210	6960	109	213	7173	110	217	7390	112	224	7614	113	228	7842	115	4.6
4.7	6858	110	211	7069	111	214	7283	113	219	7502	114	225	7727	115	230	7957	117	4.7
4.8	6968	112	212	7180	113	216	7396	115	220	7616	117	226	7842	118	232	8074	120	4.8
4.9	7080	114	213	7293	116	218	7511	117	222	7733	119	227	7960	121	234	8194	122	4.9
5.0	7194		215	7409		219	7628		224	7852		229	8081		235	8316		**5.0**
$d-\delta_0$ / $\alpha-\alpha_0$	7°.5		hor. diff.	7°.6		hor. diff.	7°.7		hor. diff.	7°.8		hor. diff.	7°.9		hor. diff.	8°.0		$d-\delta_0$ \ $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$. $0^\circ.00001 = 0''.036 =$ unit.

$d-\delta_0$ \ $\alpha-\alpha_0$	8°.0		hor. diff.	8°.1		hor. diff.	8°.2		hor. diff.	8°.3		hor. diff.	8°.4		hor. diff.	8°.5		$\alpha-\alpha_0$ / $d-\delta_0$
0°.0	05239	2	200	05439	2	205	05644	2	211	05855	2	215	06070	2	221	06291	2	0°.0
.1	5241	4	200	5441	4	205	5646	4	211	5857	4	215	6072	4	221	6293	4	.1
.2	5245	6	200	5445	6	205	5650	6	211	5861	6	215	6076	6	221	6297	6	.2
.3	5251	9	200	5451	9	205	5656	9	211	5867	9	215	6082	9	[illegible]	6303	9	.3
.4	5260	10	200	5460	11	205	5665	11	211	5876	11	215	6091	12	221	6312	12	.4
0.5	5270	14	201	5471	14	205	5676	14	211	5887	14	216	6103	14	221	6324	14	0.5
.6	5284	16	201	5485	16	205	5690	16	211	5901	16	216	6117	17	221	6338	17	.6
.7	5300	18	201	5501	19	205	5706	19	211	5917	19	217	6134	19	221	6355	20	.7
.8	5318	21	202	5520	21	205	5725	21	211	5936	22	217	6153	22	222	6375	22	.8
.9	5339	23	202	5541	23	205	5746	24	212	5958	24	217	6175	24	222	6397	25	.9
1.0	5362	26	202	5564	26	206	5770	26	212	5982	27	217	6199	27	223	6422	27	1.0
1.1	5388	28	202	5590	29	206	5796	29	213	6009	29	217	6226	30	223	6449	30	1.1
1.2	5416	30	203	5619	31	206	5825	32	213	6038	32	218	6256	32	223	6479	33	1.2
1.3	5446	33	204	5650	33	207	5857	34	213	6070	34	218	6288	35	224	6512	35	1.3
1.4	5479	36	204	5683	36	208	5891	37	213	6104	37	219	6323	37	224	6547	38	1.4
1.5	5515	38	204	5719	38	209	5928	39	213	6141	39	219	6360	40	225	6585	40	1.5
1.6	5553	40	204	5757	41	210	5967	41	213	6180	42	220	6400	43	225	6625	43	1.6
1.7	5593	43	205	5798	44	210	6008	44	214	6222	45	221	6443	45	225	6668	46	1.7
1.8	5636	46	206	5842	46	210	6052	47	215	6267	47	221	6488	48	226	6714	48	1.8
1.9	5682	48	206	5888	48	211	6099	49	215	6314	50	222	6536	50	226	6762	51	1.9
2.0	5730	50	206	5936	51	212	6148	52	216	6364	52	222	6586	53	227	6813	53	2.0
2.1	5780	53	207	5987	54	213	6200	54	216	6416	55	223	6639	55	227	6866	57	2.1
2.2	5833	55	208	6041	56	213	6254	56	217	6471	57	223	6694	58	229	6923	59	2.2
2.3	5888	58	209	6097	58	213	6310	59	218	6528	60	224	6752	61	230	6982	61	2.3
2.4	5946	60	209	6155	61	214	6369	62	219	6588	63	225	6813	63	230	7043	64	2.4
2.5	6006	63	210	6216	63	215	6431	64	220	6651	65	225	6876	66	231	7107	67	2.5
2.6	6069	65	210	6279	66	216	6495	67	221	6716	68	226	6942	68	232	7174	70	2.6
2.7	6134	67	211	6345	69	217	6562	69	222	6784	70	226	7010	71	234	7244	72	2.7
2.8	6201	70	213	6414	71	217	6631	72	223	6854	73	227	7081	74	235	7316	74	2.8
2.9	6271	73	214	6485	73	218	6703	74	224	6927	75	228	7155	76	235	7390	77	2.9
3.0	6344	75	214	6558	76	219	6777	77	225	7002	78	229	7231	79	236	7467	80	3.0
3.1	6419	78	215	6634	79	220	6854	79	226	7080	80	230	7310	82	237	7547	82	3.1
3.2	6497	80	216	6713	81	220	6933	82	227	7160	83	232	7392	84	237	7629	85	3.2
3.3	6577	82	217	6794	83	221	7015	85	228	7243	86	233	7476	87	238	7714	88	3.3
3.4	6659	85	218	6877	86	223	7100	87	229	7329	89	234	7563	89	239	7802	90	3.4
3.5	6744	87	219	6963	89	224	7187	90	231	7418	91	234	7652	92	240	7892	93	3.5
3.6	6831	90	221	7052	91	225	7277	92	232	7509	93	235	7744	94	241	7985	95	3.6
3.7	6921	92	222	7143	93	226	7369	95	233	7602	96	236	7838	97	242	8080	98	3.7
3.8	7013	95	223	7236	96	228	7464	97	234	7698	98	237	7935	100	243	8178	101	3.8
3.9	7108	97	224	7332	99	229	7561	100	235	7796	101	239	8035	102	244	8279	104	3.9
4.0	7205	100	226	7431	101	230	7661	103	236	7897	104	240	8137	105	246	8383	106	4.0
4.1	7305	102	227	7532	104	232	7764	105	237	8001	106	241	8242	107	247	8489	109	4.1
4.2	7407	105	229	7636	106	233	7869	108	238	8107	108	242	8349	110	249	8598	111	4.2
4.3	7512	107	230	7742	108	235	7977	110	238	8215	111	244	8459	113	250	8709	114	4.3
4.4	7619	110	231	7850	111	237	8087	112	239	8326	114	246	8572	115	251	8823	117	4.4
4.5	7729	113	232	7961	114	238	8199	115	241	8440	116	247	8687	118	253	8940	119	4.5
4.6	7842	115	233	8075	116	239	8314	117	242	8556	119	249	8805	120	254	9059	122	4.6
4.7	7957	117	234	8191	118	240	8431	120	244	8675	122	250	8925	123	256	9181	125	4.7
4.8	8074	120	235	8309	121	242	8551	123	246	8797	124	251	9048	126	258	9306	127	4.8
4.9	8194	122	236	8430	124	244	8674	125	247	8921	127	253	9174	128	259	9433	129	4.9
5.0	8316		238	8554		245	8799		249	9048		254	9302		260	9562		5.0
$d-\delta_0$ / $\alpha-\alpha_0$	8°.0		hor. diff.	8°.1		hor. diff.	8°.2		hor. diff.	8°.3		hor. diff.	8°.4		hor. diff.	8°.5		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi\,\sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi\,d$ sec is always numerically larger than $(\alpha - \alpha_0)^\circ$. $0^\circ.00001 = 0^s.036 =$ unit

$d-\delta_0$ \ $\alpha-\alpha_0$	8°.5		hor. diff.	8°.6		hor. diff.	8°.7		hor. diff.	8°.8		hor. diff.	8°.9		hor. diff.	9°.0		$d-\delta_0$
0°.0	06291		226	06517		231	06748		237	06985		242	07227		248	07475		0°.0
.1	6293	2	226	6519	2	231	6750	2	237	6987	2	242	7229	2	248	7477	2	.1
.2	6297	4	226	6523	4	231	6754	4	237	6991	4	242	7233	4	248	7481	4	.2
.3	6303	6	226	6529	6	232	6761	7	237	6998	7	242	7240	7	248	7488	7	.3
.4	6312	9	226	6538	9	232	6770	9	237	7007	9	243	7250	10	248	7498	10	.4
0.5	6324	12	226	6550	12	232	6782	12	237	7019	12	243	7262	12	248	7510	12	0.5
.6	6338	14	226	6564	14	233	6797	15	237	7034	15	243	7277	15	248	7525	15	.6
.7	6355	17	227	6582	18	232	6814	17	237	7051	17	244	7295	18	248	7543	18	.7
.8	6375	20	226	6601	19	233	6834	20	237	7071	20	245	7316	21	248	7564	21	.8
.9	6397	22	227	6624	23	233	6857	23	237	7094	23	245	7339	23	249	7588	24	.9
1.0	6422	25	227	6649	25	233	6882	25	238	7120	26	245	7365	26	249	7614	26	1.0
1.1	6449	27	228	6677	28	233	6910	28	239	7149	29	245	7394	29	249	7643	29	1.1
1.2	6479	30	228	6707	30	234	6941	31	240	7181	32	245	7426	32	249	7675	32	1.2
1.3	6512	33	228	6740	33	234	6974	33	241	7215	34	245	7460	34	250	7710	35	1.3
1.4	6547	35	229	6776	36	234	7010	36	241	7251	36	245	7496	36	251	7747	37	1.4
1.5	6585	38	229	6814	38	235	7049	39	241	7290	39	246	7536	40	251	7787	40	1.5
1.6	6625	40	230	6855	41	236	7091	42	241	7332	42	246	7578	42	252	7830	43	1.6
1.7	6668	43	230	6898	43	238	7136	45	241	7377	45	246	7623	45	252	7875	45	1.7
1.8	6714	46	231	6945	47	238	7183	47	241	7424	47	247	7671	48	253	7924	49	1.8
1.9	6762	48	231	6993	48	239	7232	49	242	7474	50	248	7722	51	253	7975	51	1.9
2.0	6813	51	232	7045	52	239	7284	52	243	7527	53	248	7775	53	254	8029	54	2.0
2.1	6866	53	234	7100	55	239	7339	55	243	7582	55	249	7831	56	255	8086	57	2.1
2.2	6923	57	234	7157	57	239	7396	57	244	7640	58	250	7890	59	255	8145	59	2.2
2.3	6982	59	234	7216	59	240	7456	60	245	7701	61	251	7952	62	255	8207	62	2.3
2.4	7043	61	235	7278	62	241	7519	63	246	7765	64	251	8016	64	256	8272	65	2.4
2.5	7107	64	236	7343	65	242	7585	66	246	7831	66	252	8083	67	257	8340	68	2.5
2.6	7174	67	236	7410	67	243	7653	68	247	7900	69	253	8153	70	258	8411	71	2.6
2.7	7244	70	236	7480	70	244	7724	71	248	7972	72	253	8225	72	259	8484	73	2.7
2.8	7316	72	237	7553	73	244	7797	73	249	8046	74	255	8301	76	259	8560	76	2.8
2.9	7390	74	239	7629	76	244	7873	76	250	8123	77	256	8379	78	260	8639	79	2.9
3.0	7467	77	240	7707	78	245	7952	79	251	8203	80	257	8460	81	261	8721	82	3.0
3.1	7547	80	241	7788	81	246	8034	82	252	8286	83	258	8544	84	262	8806	85	3.1
3.2	7629	82	242	7871	83	247	8118	84	253	8371	85	259	8630	86	263	8893	87	3.2
3.3	7714	85	243	7957	86	248	8205	87	254	8459	88	260	8719	89	264	8983	90	3.3
3.4	7802	88	244	8046	89	249	8295	90	255	8550	91	261	8811	92	265	9076	93	3.4
3.5	7892	90	245	8137	91	250	8387	92	256	8643	93	262	8905	94	267	9172	96	3.5
3.6	7985	93	246	8231	94	251	8482	95	257	8739	96	263	9002	97	269	9271	99	3.6
3.7	8080	95	248	8328	97	252	8580	98	258	8838	99	264	9102	100	270	9372	101	3.7
3.8	8178	98	249	8427	99	254	8681	101	259	8940	102	265	9205	103	271	9476	104	3.8
3.9	8279	101	250	8529	102	255	8784	103	261	9045	105	266	9311	106	272	9583	107	3.9
4.0	8383	104	251	8634	105	256	8890	106	262	9152	107	267	9419	108	273	9692	109	4.0
4.1	8489	106	252	8741	107	258	8999	109	263	9262	110	268	9530	111	274	9804	112	4.1
4.2	8598	109	253	8851	110	259	9110	111	265	9375	113	269	9644	114	276	9920	116	4.2
4.3	8709	111	255	8964	113	260	9324	114	266	9490	115	271	9761	117	277	10038	118	4.3
4.4	8823	114	256	9079	115	262	9441	117	267	9608	118	273	9881	120	277	10158	120	4.4
4.5	8940	117	257	9197	118	263	9460	119	269	9729	121	274	10003	122	279	10282	124	4.5
4.6	9059	119	259	9318	121	264	9582	122	270	9852	123	276	10128	125	280	10408	126	4.6
4.7	9181	122	260	9441	123	266	9707	125	272	9979	127	277	10256	128	282	10538	130	4.7
4.8	9306	125	261	9567	126	268	9835	128	273	10108	129	278	10386	130	284	10670	132	4.8
4.9	9433	127	263	9696	129	269	9965	130	275	10240	132	279	10519	133	285	10804	134	4.9
5.0	9562	129	265	9827	131	271	10098	133	276	10374	134	281	10655	136	287	10942	138	5.0
$d-\delta_0$ / $\alpha-\alpha_0$	8°.5		hor. diff.	8°.6		hor. diff.	8°.7		hor. diff.	8°.8		hor. diff.	8°.9		hor. diff.	9°.0		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d \quad (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$. **$0^\circ.00001 = 0''.036 =$ unit.**

$\alpha-\alpha_0$ / $d-\delta_0$	9°.0		hor. diff.	9°.1		hor. diff.	9°.2		hor. diff.	9°.3		hor. diff.	9°.4		hor. diff.	9°.5		$\alpha-\alpha_0$ / $d-\delta_0$
0.0	07475		255	07730		259	07989		265	08254		272	08526		277	08803		0.0
.1	7477	2	255	7732	2	259	7991	2	265	8256	2	272	8528	2	277	8805	2	.1
.2	7481	4	255	7736	4	259	7995	4	265	8260	4	272	8532	4	277	8809	4	.2
.3	7488	7	255	7743	7	259	8002	7	265	8367	7	272	8539	7	277	8816	7	.3
.4	7498	10	255	7753	10	259	8012	10	265	8377	10	272	8549	10	277	8826	10	.4
0.5	7510	12	255	7765	12	260	8025	13	265	8290	13	272	8562	13	277	8839	13	0.5
.6	7525	15	256	7781	16	260	8041	16	265	8306	16	272	8578	16	277	8855	16	.6
.7	7543	18	256	7799	18	260	8059	18	265	8324	18	273	8597	19	277	8874	19	.7
.8	7564	21	256	7820	21	260	8080	21	266	8346	22	273	8619	22	277	8896	22	.8
.9	7588	24	256	7844	24	260	8104	24	266	8370	24	273	8643	24	278	8921	25	.9
1.0	7614	26	256	7870	26	261	8131	27	266	8397	27	273	8670	27	279	8949	28	1.0
1.1	7643	29	257	7900	30	261	8161	30	266	8427	30	273	8700	30	280	8980	31	1.1
1.2	7675	32	257	7932	32	261	8193	32	267	8460	33	274	8734	34	280	9014	34	1.2
1.3	7710	35	257	7967	35	261	8228	35	268	8496	36	274	8770	36	280	9050	36	1.3
1.4	7747	37	257	8004	37	263	8267	39	268	8535	39	274	8809	39	280	9089	39	1.4
1.5	7787	40	258	8045	41	263	8308	41	268	8576	41	275	8851	42	281	9132	43	1.5
1.6	7830	43	258	8088	43	264	8352	44	268	8620	44	276	8896	45	281	9177	45	1.6
1.7	7875	45	259	8134	46	264	8398	46	269	8667	47	276	8943	47	282	9225	48	1.7
1.8	7924	49	259	8183	49	265	8448	50	269	8717	50	277	8994	51	283	9277	52	1.8
1.9	7975	51	260	8235	52	265	8500	52	270	8770	53	278	9048	54	283	9331	54	1.9
2.0	8029	54	260	8289	54	266	8555	55	271	8826	56	278	9104	56	283	9387	56	2.0
2.1	8086	57	260	8346	57	267	8613	58	272	8885	59	278	9163	59	284	9447	60	2.1
2.2	8145	59	261	8406	60	268	8674	61	272	8946	61	280	9226	63	284	9510	63	2.2
2.3	8207	62	262	8469	63	268	8737	63	273	9010	64	281	9291	65	284	9575	65	2.3
2.4	8272	65	263	8535	66	269	8804	67	274	9078	68	281	9359	68	285	9644	69	2.4
2.5	8340	68	264	8604	69	269	8873	69	275	9148	70	281	9429	70	287	9716	72	2.5
2.6	8411	71	265	8676	72	269	8945	72	276	9221	73	282	9503	74	288	9791	75	2.6
2.7	8484	73	266	8750	74	270	9020	75	277	9297	76	282	9579	76	289	9868	77	2.7
2.8	8560	76	267	8827	77	271	9098	78	278	9376	79	283	9659	80	290	9949	81	2.8
2.9	8619	79	267	8906	79	273	9179	81	279	9458	82	284	9742	83	290	10032	83	2.9
3.0	8721	82	268	8989	83	274	9263	84	279	9542	84	285	9827	85	291	10118	86	3.0
3.1	8806	85	269	9075	86	274	9349	86	280	9629	87	286	9915	88	292	10207	89	3.1
3.2	8893	87	270	9163	88	276	9439	90	281	9720	91	287	10007	92	293	10300	93	3.2
3.3	8983	90	271	9254	91	277	9531	92	282	9813	93	288	10101	94	294	10395	95	3.3
3.4	9076	93	272	9348	94	277	9625	94	283	9908	95	290	10198	97	295	10493	98	3.4
3.5	9172	96	273	9445	97	278	9723	98	284	10007	99	291	10298	100	296	10594	101	3.5
3.6	9271	99	274	9545	100	279	9824	101	285	10109	102	292	10401	103	297	10698	104	3.6
3.7	9372	101	275	9647	102	280	9927	103	286	10213	104	294	10507	106	298	10805	107	3.7
3.8	9476	104	276	9752	105	282	10034	107	287	10321	108	295	10616	109	299	10915	110	3.8
3.9	9583	107	277	9860	106	283	10143	109	289	10432	111	296	10728	112	300	11028	113	3.9
4.0	9692	109	279	9971	111	284	10255	112	290	10545	113	297	10842	114	302	11144	116	4.0
4.1	9804	112	281	10085	114	285	10370	115	291	10661	116	298	10959	117	304	11263	119	4.1
4.2	9920	116	281	10201	116	287	10488	118	293	10781	120	299	11080	121	305	11385	122	4.2
4.3	10038	118	282	10320	119	289	10609	121	294	10903	122	300	11203	123	306	11509	124	4.3
4.4	10158	120	284	10442	122	290	10732	123	296	11028	125	302	11330	127	307	11637	128	4.4
4.5	10282	124	286	10568	126	291	10859	127	296	11155	127	304	11459	129	309	11768	131	4.5
4.6	10408	126	288	10696	128	292	10988	129	298	11286	131	305	11591	132	311	11902	134	4.6
4.7	10538	130	289	10827	131	293	11120	132	299	11419	133	307	11726	135	312	12038	136	4.7
4.8	10670	132	290	10960	133	295	11255	135	301	11556	137	308	11864	138	314	12178	140	4.8
4.9	10804	134	292	11096	136	297	11393	138	302	11695	139	309	12004	140	316	12320	142	4.9
5.0	10942	138	293	11235	139	299	11534	141	304	11838	143	310	12148	144	317	12465	145	5.0
$d-\delta_0$ / $\alpha-\alpha_0$	9°.0		hor. diff.	9°.1		hor. diff.	9°.2		hor. diff.	9°.3		hor. diff.	9°.4		hor. diff.	9°.5		$d-\delta_0$ / $\alpha-\alpha_0$

TABLE III.

$$C = k\,\xi \sec d - (\alpha - \alpha_0)^\circ.$$

$k\,\xi \sec d$ is always numerically larger than $(\alpha - \alpha_0)^\circ$. $0^\circ.00001 = 0''.036 =$ unit.

$\alpha - \alpha_0$ / d δ_0	9°.5		hor. diff.	9°.6		hor. diff.	9°.7		hor. diff.	9°.8		hor. diff.	9°.9		hor. diff.	10°.0		$\alpha - \alpha_0$ / d δ_0
0.0	08803		283	09086		289	09375		295	09670		301	09971		309	10280		0.0
.1	8805	2	283	9088	2	289	9377	2	295	9672	2	301	9973	2	309	10282	2	.1
.2	8809	4	283	9092	4	289	9381	4	295	9676	4	301	9977	4	309	10286	4	.2
.3	8816	7	283	9099	7	289	9388	7	296	9684	8	301	9985	8	309	10294	8	.3
.4	8826	10	284	9110	11	289	9399	11	296	9695	11	301	9996	11	309	10305	11	.4
0.5	8839	13	284	9123	13	289	9412	13	296	9708	13	301	10009	13	309	10318	13	0.5
.6	8855	16	284	9139	16	289	9428	16	297	9725	17	301	10026	17	309	10335	17	.6
.7	8874	19	284	9158	19	289	9447	19	297	9744	19	301	10045	19	310	10355	20	.7
.8	8896	22	284	9180	22	290	9470	23	297	9767	23	301	10068	23	310	10378	23	.8
.9	8921	25	284	9205	25	291	9496	26	297	9793	26	301	10094	26	310	10404	26	.9
1.0	8949	28	284	9233	28	291	9524	28	297	9821	28	302	10123	29	310	10433	29	1.0
1.1	8980	31	284	9264	31	291	9555	31	298	9853	32	303	10156	33	310	10466	33	1.1
1.2	9014	34	284	9298	34	291	9589	34	298	9887	34	304	10191	35	310	10501	35	1.2
1.3	9050	36	285	9335	37	292	9627	38	298	9925	38	304	10229	38	310	10539	38	1.3
1.4	9089	39	286	9375	40	293	9667	40	298	9965	40	305	10270	41	311	10581	42	1.4
1.5	9132	43	286	9418	43	293	9711	44	298	10009	44	305	10314	44	312	10626	45	1.5
1.6	9177	45	287	9464	46	293	9757	46	299	10056	47	305	10361	47	313	10674	48	1.6
1.7	9225	48	287	9512	48	294	9806	49	299	10105	49	307	10412	51	313	10725	51	1.7
1.8	9277	52	287	9564	52	295	9859	53	299	10158	53	307	10465	53	313	10778	53	1.8
1.9	9331	54	288	9619	55	295	9914	55	300	10214	56	307	10521	56	314	10835	57	1.9
2.0	9387	56	289	9676	57	296	9972	58	301	10273	59	308	10581	60	314	10895	60	2.0
2.1	9447	60	290	9737	61	296	10033	61	303	10336	63	308	10644	63	314	10958	63	2.1
2.2	9510	63	291	9801	64	296	10097	64	304	10401	65	308	10709	65	316	11025	67	2.2
2.3	9575	65	292	9867	66	298	10165	68	304	10469	68	309	10778	69	316	11094	69	2.3
2.4	9644	69	292	9936	69	299	10235	70	305	10540	71	310	10850	72	316	11166	72	2.4
2.5	9716	72	293	10009	73	299	10308	73	306	10614	74	310	10924	74	318	11242	76	2.5
2.6	9791	75	294	10085	76	299	10384	76	307	10691	77	311	11002	78	319	11321	79	2.6
2.7	9868	77	295	10163	78	300	10463	79	308	10771	80	312	11083	81	319	11402	81	2.7
2.8	9949	81	295	10244	81	302	10546	83	308	10854	83	313	11167	84	320	11487	85	2.8
2.9	10032	83	296	10328	84	303	10631	85	309	10940	86	314	11254	87	321	11575	88	2.9
3.0	10118	86	297	10415	87	304	10719	88	310	11029	89	315	11344	90	322	11666	91	3.0
3.1	10207	89	298	10505	90	305	10810	91	311	11121	92	316	11437	93	323	11760	94	3.1
3.2	10300	93	299	10599	94	306	10905	95	311	11216	95	317	11533	96	324	11857	97	3.2
3.3	10395	95	300	10695	96	308	11003	98	311	11314	98	318	11632	99	325	11957	100	3.3
3.4	10493	98	301	10794	99	309	11103	100	313	11416	102	319	11735	103	326	12061	104	3.4
3.5	10594	101	302	10896	102	310	11206	103	314	11520	104	321	11841	106	326	12167	106	3.5
3.6	10698	104	303	11001	105	311	11312	106	315	11627	107	323	11950	109	327	12277	110	3.6
3.7	10805	107	304	11109	108	312	11421	109	317	11738	111	323	12061	111	328	12389	112	3.7
3.8	10915	110	306	11221	112	313	11534	113	317	11851	113	325	12176	115	329	12505	116	3.8
3.9	11028	113	307	11335	114	314	11649	115	319	11968	117	325	12293	117	331	12624	119	3.9
4.0	11144	116	308	11452	117	315	11767	118	320	12087	119	327	12414	121	332	12746	122	4.0
4.1	11263	119	310	11573	121	316	11889	122	321	12210	123	328	12538	124	333	12871	125	4.1
4.2	11385	122	311	11696	123	317	12013	124	322	12335	125	330	12665	127	335	13000	129	4.2
4.3	11509	124	313	11822	126	318	12140	127	324	12464	129	331	12795	130	336	13131	131	4.3
4.4	11637	128	314	11951	129	320	12271	131	325	12596	132	332	12928	133	337	13265	134	4.4
4.5	11768	131	315	12083	132	321	12404	133	327	12731	135	333	13064	136	339	13403	138	4.5
4.6	11902	134	316	12218	135	322	12540	136	329	12869	138	334	13203	139	341	13544	141	4.6
4.7	12038	136	318	12356	138	324	12680	140	330	13010	141	335	13345	142	343	13688	144	4.7
4.8	12178	140	319	12497	141	325	12822	142	332	13154	144	337	13491	146	344	13835	147	4.8
4.9	12320	142	321	12641	144	327	12968	146	333	13301	147	338	13639	148	346	13985	150	4.9
5.0	12465	145	323	12788	147	328	13116	148	335	13451	150	340	13791	152	347	14138	153	5.0
d δ_0 / $\alpha - \alpha_0$	9°.5		hor. diff.	9°.6		hor. diff.	9°.7		hor. diff.	9°.8		hor. diff.	9°.9		hor. diff.	10°.0		d δ_0 / $\alpha - \alpha_0$

TABLES IV–XI FOR THE COMPUTATION OF CORRECTIONS FOR REFRACTION.

Table IV	gives	X.	Argument	$(s - \alpha)$ and δ	for $\left\{\begin{array}{l} -45^\circ < (s-\alpha) < +45^\circ \\ -40^\circ < \delta < +90^\circ \end{array}\right\}$
" V	"	Y.	"	$(s - \alpha)$ " δ	
" VI	"	Δ.	"	$(s - \alpha)$	for $-180^\circ < (s - \alpha) < +180^\circ$.
" VII	"	$\left\{\begin{array}{l} \log X \cos(\Delta - \delta) \\ X \cos(\Delta - \delta) \end{array}\right\}$	"	$(s - \alpha)$	

The above tables are computed for the latitude of the Students' Observatory: $37^\circ\ 52'.4$.

Table VIII for conversion of degrees into radians.
" IX giving $x^2 + y^2$. Arguments x and y.
" X " D. Argument $xX + yY$.
" XI " Z. " X and Y.

TABLE IV.

X

X has the same sign as $(s - \alpha_0)$.

$s-\alpha$ / δ	0°	d.	5°	d.	10°	d.	15°	d.	20°	d.	25°	d.	30°	d.	35°	d.	40°	d.	45°	$s-\alpha$ / δ
−40°	0.000	331	0.331	351	0.682	397	1.079	476	1.555	619	2.174	962	3.058							**−40°**
39		306	0.306	323	0.629	362	0.991	429	1.420	547	1.967	756	2.723							39
38		285	0.285	299	.584	333	0.917	390	1.307	489	1.796	660	2.456							38
37		266	.266	279	.545	308	0.853	357	1.210	442	1.652	584	2.236	845	3.081					37
36		250	.250	261	.511	287	0.798	331	1.129	402	1.531	522	2.053	737	2.790					36
35		236	.236	246	.482	268	0.750	307	1.057	369	1.426	473	1.899	651	2.550					35
34		223	.223	232	.455	252	0.707	287	.994	341	1.335	431	1.766	578	2.349	877	3.226			34
33		212	.212	220	.432	238	0.670	269	.939	317	1.256	395	1.651	527	2.178	759	2.937			33
32		201	.201	210	.411	225	0.636	253	.889	297	1.186	365	1.551	479	2.030	676	2.706			32
31		192	.192	199	.391	214	0.605	240	.845	278	1.123	340	1.463	439	1.902	608	2.510			31
−30		184	.184	190	.374	204	0.578	227	.805	262	1.067	318	1.385	404	1.789	552	2.341	823	3.164	**−30**
29		177	.177	182	.359	194	0.553	216	.769	248	1.017	297	1.314	376	1.690	504	2.194	736	2.930	29
28		170	.170	174	.344	187	0.531	205	.736	235	.971	280	1.251	351	1.602	463	2.065	663	2.728	28
27		163	.163	168	.331	179	0.510	196	.706	224	.930	266	1.196	326	1.522	429	1.951	602	2.553	27
26		157	.157	162	.319	172	0.491	188	.679	213	.892	252	1.144	307	1.451	398	1.849	551	2.400	26
25		152	.152	156	.308	165	0.473	181	.654	204	.858	238	1.096	291	1.387	371	1.758	507	2.265	25
24		147	.147	151	.298	159	0.457	174	.631	195	.826	227	1.053	275	1.328	348	1.676	469	2.145	24
23		142	.142	146	.288	154	0.442	168	.610	187	.797	217	1.014	260	1.274	328	1.602	436	2.038	23
22		138	.138	141	.279	149	0.428	162	.590	180	.770	207	0.977	248	1.225	309	1.534	407	1.941	22
21		134	.134	137	.271	144	0.415	156	.571	174	.745	199	0.944	236	1.180	293	1.473	382	1.855	21
−20		130	.130	133	.263	140	0.403	151	.554	168	.722	191	0.913	226	1.139	278	1.417	358	1.775	**−20**
19		127	.127	129	.256	136	0.392	146	.538	162	.700	184	0.884	216	1.100	265	1.365	338	1.703	19
18		123	.123	126	.249	133	0.382	141	.523	157	.680	177	0.857	208	1.065	252	1.317	319	1.636	18
17		120	.120	123	.243	129	0.372	138	.510	151	.661	171	0.832	200	1.032	240	1.272	304	1.576	17
16		117	.117	120	.237	125	0.362	135	.497	146	.643	166	0.809	192	1.001	230	1.231	289	1.520	16
15		115	.115	117	.232	122	0.354	130	.484	143	.627	160	0.787	185	0.972	221	1.193	275	1.468	15
14		112	.112	114	.226	119	0.345	128	.473	138	.611	156	0.767	178	.945	213	1.158	262	1.420	14
13		110	.110	111	.221	117	0.338	124	.462	135	.597	152	0.749	171	.920	205	1.125	251	1.376	13
12		107	.107	110	.217	114	0.331	121	.452	131	.583	148	0.731	166	.897	197	1.094	241	1.335	12
11		105	.105	107	.212	112	0.324	118	.442	128	.570	144	0.714	161	.875	190	1.065	231	1.296	11
−10		103	.103	105	.208	109	0.317	116	.433	125	.558	139	0.697	157	.854	184	1.038	222	1.260	**−10**
9		101	.101	103	.204	107	0.311	113	.424	122	.546	136	.682	152	.834	178	1.012	214	1.226	9
8		99	.099	101	.200	105	0.305	111	.416	119	.535	132	.667	149	.816	172	0.988	206	1.194	8
7		98	.098	99	.197	103	0.300	108	.408	117	.525	129	.654	145	.799	167	.966	199	1.165	7
6		96	.096	97	.193	101	0.294	107	.401	114	.515	126	.641	141	.782	163	.945	192	1.137	6
5		94	.094	96	.190	99	0.289	105	.394	112	.506	123	.629	138	.767	157	.924	187	1.111	5
4		93	.093	94	.187	98	0.285	102	.387	110	.497	121	.618	134	.752	153	.905	181	1.086	4
3		91	.091	93	.184	96	0.280	101	.381	108	.489	118	.607	131	.738	149	.887	176	1.063	3
2		90	.090	91	.181	95	0.276	99	.375	106	.481	116	.597	128	.725	145	.870	171	1.041	2
1		89	.089	90	.179	93	0.272	97	.369	104	.473	114	.587	125	.712	142	.854	166	1.020	1
0		87	.087	89	.176	92	0.268	96	.364	102	.466	111	.577	123	.700	139	.839	161	1.000	**0**
δ / $s-\alpha$	0°	d.	5°	d.	10°	d.	15°	d.	20°	d.	25°	d.	30°	d.	35°	d.	40°	d.	45°	δ / $s-\alpha$

$$X = \tan (s - \alpha) \cos \Delta \sec (\Delta - \delta)$$

TABLE IV.

X

X has the same sign as $(s - \alpha_0)$.

$s-\alpha$ / δ	0°	d.	5°	d	10°	d.	15°	d.	20°	d.	25°	d.	30°	d.	35°	d.	40°	d.	45°	$s-\alpha$ / δ
0°	0.000	87	0.087	89	0.176	92	0.268	96	0.364	102	0.466	111	0.577	123	0.700	139	0.839	161	1.000	0°
1		86	.086	88	.174	90	.264	95	359	100	.459	109	.568	121	.689	136	.825	156	.981	1
2		85	085	87	.172	89	.261	93	.354	99	.453	107	.560	118	.678	133	.811	153	.964	2
3		84	.084	86	.170	87	.257	92	.349	98	.447	105	552	116	.668	130	.798	149	.947	3
4		83	.083	85	.168	86	.254	91	.345	96	441	104	.545	113	.658	128	.786	145	.931	4
5		82	082	84	.166	85	.251	90	.341	94	435	102	.537	112	.649	125	.774	142	.916	5
6		81	.081	83	.164	84	.248	89	.337	93	430	100	.530	110	.640	122	.762	139	.901	6
7		80	.080	82	.162	83	.245	88	333	92	.425	99	.524	108	.632	120	.752	136	.888	7
[illegible]		80	.080	80	.160	83	.243	86	.329	91	.420	98	518	106	624	118	.742	133	875	8
9		79	.079	80	159	82	.241	85	326	90	.416	96	.512	104	.616	116	.732	130	.862	9
+10		78	.078	79	157	81	238	85	323	88	.411	95	.506	103	.609	114	.723	127	.850	+10
11		77	.077	79	.156	80	.236	83	.319	88	.407	94	.501	101	.602	112	.714	125	.839	11
12		77	.077	78	.155	79	.234	82	.316	87	.403	93	.496	99	.595	111	.706	123	.829	12
13		76	.076	77	.153	79	.232	81	.313	86	.399	92	.491	98	.589	109	.698	121	.819	13
14		75	.075	77	.152	78	.230	81	.311	85	.396	90	486	98	.584	106	.690	119	.809	14
15		75	.075	76	.151	77	.228	80	.308	84	.392	90	.482	96	.578	105	.683	117	.800	15
16		74	.074	76	.150	76	226	80	306	83	.389	89	.478	95	.573	103	.676	115	.791	16
17		74	.074	75	.149	76	.225	79	.304	82	.386	88	.474	94	.568	102	.670	113	783	17
18		73	.073	75	.148	75	.223	79	.302	81	.383	87	470	93	563	100	.663	112	.775	18
19		73	.073	74	.147	75	222	78	.300	80	380	86	.466	92	.558	99	.657	110	.767	19
+20		73	.073	73	.146	75	.221	77	.298	80	.378	85	.463	91	.554	98	.652	108	.760	+20
21		72	.072	73	.145	74	219	77	.296	80	.376	84	.460	90	.550	97	.647	106	.753	21
22		72	.072	72	.144	74	.218	76	.294	80	.374	83	.457	89	.546	96	.642	105	.747	22
23		71	.071	73	.144	73	.217	76	.293	78	.371	83	.454	88	.542	95	.637	104	.741	23
24		71	.071	72	.143	73	.216	75	.291	78	.369	82	.451	88	.539	93	.632	103	.735	24
25		71	071	72	.143	72	.215	75	.290	77	.367	82	.449	87	.536	92	.628	101	.729	25
26		71	.071	71	.142	72	.214	74	.288	78	.366	80	.446	87	.533	91	.624	100	.724	26
27		70	.070	71	.141	72	.213	74	.287	77	.364	79	.443	87	.530	91	.621	98	.719	27
28		70	070	71	.141	71	212	74	.286	77	.363	78	.441	86	.527	90	.617	98	.715	28
29		70	.070	70	.140	72	.212	73	285	76	.361	78	.439	86	525	89	.614	96	.710	29
+30		70	.070	70	.140	71	.211	73	284	76	.360	77	.437	85	.522	89	.611	95	.706	+30
31		69	.069	71	.140	71	.211	73	.284	75	359	77	436	84	.520	88	.608	94	.702	31
32		69	.069	70	.139	71	.310	73	.283	75	.358	77	.435	83	.518	87	.605	94	.699	32
33		69	.069	70	.139	71	.210	72	.282	75	.357	77	.434	82	.516	87	.603	93	.696	33
34		69	.069	70	.139	70	.209	73	.282	74	.356	77	433	82	.515	86	.601	92	.693	34
35		69	.069	70	.139	70	.209	72	.281	75	.356	76	.432	81	.513	86	.599	91	.690	35
36		69	.069	69	.138	71	.209	72	.281	74	355	76	.431	81	512	85	.597	90	.687	36
37		69	.069	69	.138	71	.209	72	.281	74	.355	76	431	80	.511	84	.595	90	.685	37
38		69	069	69	.138	71	.209	71	280	74	.354	77	.431	79	.510	84	.594	89	.683	38
39		69	069	69	.138	71	.209	71	.280	74	.354	77	.431	78	509	84	.593	88	.681	39
+40		69	.069	69	.138	71	.209	71	.280	74	354	76	.430	79	.509	82	.591	[illegible]	.679	+40
δ / $s-\alpha$	0°	d.	5°	d.	10°	d.	15°	d.	20°	d.	25°	d.	30°	d.	35°	d.	40°	d.	45°	δ / $s-\alpha$

TABLE IV.

X

X has the same sign as $(s - \alpha_0)$.

δ \ $s-\alpha$	0°	d.	5°	d.	10°	d.	15°	d.	20°	d.	25°	d.	30°	d.	35°	d.	40°	d.	45°	$s-\alpha$ / δ
+40°	0.000	69	0.069	69	0.138	71	0.209	71	0.280	74	0.354	76	0.430	79	0.509	82	0.591	88	0.679	+40°
41		69	.069	70	0.139	70	.209	71	.280	74	.354	76	.430	78	0.508	83	.591	86	.677	41
42		69	.069	70	.139	70	.209	72	.281	73	.354	76	.430	78	0.508	82	.590	86	.676	42
43		69	.069	70	.139	70	.209	72	.281	73	.354	76	.430	78	0.508	82	.590	85	.675	43
44		69	.069	70	.139	70	.209	72	.281	73	.354	76	.430	78	.508	81	.589	85	.674	44
45		69	.069	70	.139	71	.210	72	.282	73	.355	75	.430	78	.508	81	.589	84	.673	45
46		70	.070	70	.140	70	.210	72	.282	73	.355	76	.431	77	.508	81	.589	84	.673	46
47		70	.070	70	.140	71	.211	72	.283	73	.356	75	.431	78	.509	80	.589	84	.673	47
48		70	.070	70	.140	71	.211	72	.283	74	.357	75	.432	77	.509	80	.589	84	.673	48
49		70	.070	71	.141	71	.212	72	.284	74	.358	75	.433	77	.510	80	.590	83	.673	49
+50		71	.071	70	.141	72	.213	72	.285	74	.359	75	.434	77	.511	80	.591	82	.673	+50
51		71	.071	71	.142	72	.214	72	.286	74	.360	75	.435	77	.512	80	.592	82	.674	51
52		71	.071	72	.143	71	.214	73	.287	74	.361	75	.436	77	.513	80	.593	82	.675	52
53		71	.071	72	.143	72	.215	73	.288	74	.362	75	.437	78	.515	79	.594	82	.676	53
54		72	.072	72	.144	72	.216	74	.290	74	.364	75	.439	77	.516	79	.595	82	.677	54
55		72	.072	72	.144	73	.217	74	.291	74	.365	76	.441	77	.518	79	.597	81	.678	55
56		72	.072	73	.145	73	.218	74	.292	75	.367	76	.443	77	.520	79	.599	81	.680	56
57		73	.073	73	.146	74	.220	74	.294	75	.369	76	.445	77	.522	79	.601	81	.682	57
58		73	.073	74	.147	74	.221	74	.295	76	.371	76	.447	77	.524	79	.603	81	.684	58
59		73	.073	75	.148	74	.222	75	.297	76	.373	76	.449	78	.527	79	.606	80	.686	59
+60		74	.074	75	.149	75	.224	75	.299	76	.375	77	.452	78	.530	78	.608	81	.689	+60
61		75	.075	75	.150	75	.225	76	.301	76	.377	77	.454	78	.532	79	.611	80	.691	61
62		75	.075	76	.151	76	.227	76	.303	77	.380	77	.457	78	.535	79	.614	80	.694	62
63		76	.076	76	.152	77	.229	76	.305	78	.383	77	.460	79	.539	79	.618	79	.697	63
64		77	.077	77	.154	77	.231	77	.308	77	.385	79	.464	78	.542	79	.621	80	.701	64
65		77	.077	78	.155	78	.233	77	.310	78	.388	79	.467	79	.546	79	.625	79	.704	65
66		78	.078	78	.156	79	.235	78	.313	78	.391	79	.470	80	.550	79	.629	79	.708	66
67		79	.079	79	.158	79	.237	79	.316	79	.395	79	.474	80	.554	79	.633	79	.712	67
68		80	.080	79	.159	80	.239	80	.319	79	.398	80	.478	80	.558	79	.637	80	.717	68
69		80	.080	81	.161	80	.241	81	.322	80	.402	80	.482	81	.563	79	.642	80	.722	69
+70		81	.081	82	.163	81	.244	81	.325	81	.406	81	.487	80	.567	80	.647	80	.727	+70
71		82	.082	82	.164	83	.247	81	.328	82	.400	82	.492	80	.572	81	.653	79	.732	71
72		83	.083	83	.166	83	.249	83	.332	82	.414	82	.496	81	.577	81	.658	80	.738	72
73		84	.084	84	.168	84	.252	84	.336	83	.419	82	.501	82	.583	81	.664	80	.744	73
74		85	.085	86	.171	84	.255	85	.340	84	.424	83	.507	82	.589	82	.671	79	.750	74
75		86	.086	87	.173	86	.259	85	.344	85	.429	84	.513	82	.595	82	.677	80	.757	75
76		88	.088	87	.175	87	.262	86	.348	86	.434	85	.519	83	.602	82	.684	80	.764	76
77		89	.089	89	.178	87	.265	88	.353	86	.439	86	.525	84	.609	82	.691	80	.771	77
78		90	.090	90	.180	89	.269	89	.358	87	.445	87	.532	84	.616	83	.699	80	.779	78
79		91	.091	92	.183	90	.273	90	.363	88	.451	87	.538	86	.624	83	.707	80	.787	79
+80		93	.093	92	.185	92	.277	91	.368	90	.458	88	.546	86	.632	83	.715	81	.796	+80
81		94	.094	94	.188	94	.282	92	.374	90	.464	89	.553	87	.640	84	.724	81	.805	81
82		96	.096	95	.191	95	.286	94	.380	92	.472	89	.561	88	.649	84	.733	81	.814	82
83		98	.098	97	.195	96	.291	95	.386	93	.479	91	.570	88	.658	85	.743	81	.824	83
84		99	.099	99	.198	98	.296	96	.392	95	.487	92	.579	89	.668	85	.753	81	.834	84
85		101	.101	101	.202	99	.301	98	.399	96	.495	93	.588	90	.678	86	.764	81	.845	85
86		103	.103	103	.206	101	.307	99	.406	98	.504	94	.598	91	.689	86	.775	82	.857	86
87		105	.105	105	.210	103	.313	101	.414	99	.513	95	.608	92	.700	87	.787	82	.869	87
88		107	.107	107	.214	105	.319	103	.422	101	.523	96	.619	93	.712	88	.800	82	.882	88
89		110	.110	108	.218	108	.326	105	.431	102	.533	98	.631	93	.724	89	.813	82	.895	89
+90		112	.112	111	.223	110	.333	107	.440	103	.543	100	.643	94	.737	90	.827	82	.909	+90
δ / $s-\alpha$	0°	d.	5°	d.	10°	d.	15°	d.	20°	d.	25°	d.	30°	d.	35°	d.	40°	d.	45°	$s-\alpha$ \ δ

TABLE V.

Y

δ \ $s-\alpha$	0°	d.	5°	d.	10°	d.	15°	d.	20°	d.	25°	d.	30°	d.	35°	d.	40°	d.	45°	$s-\alpha$ / δ
-40°																				-40°
39																				39
38																				38
37																				37
36																				36
35																				35
-34	+3.055	19	+3.074	59	+3.133															-34
33	2.883	18	2.901	53	2.954	95	+3.049													33
32	2.729	15	2.744	49	2.793	85	2.878	130	+3.008											32
31	2.588	14	2.602	44	2.646	78	2.724	118	2.842	170	+3.012									31
-30	+2.459	14	+2.473	40	+2.513	70	+2.583	108	+2.691	154	+2.845	219	+3.064							-30
29	2.341	13	2.354	37	2.391	64	2.455	99	2.554	140	2.694	198	2.892	282	+3.174					29
28	2.233	11	2.244	34	2.278	60	2.338	90	2.428	128	2.556	181	2.737	254	2.991					28
27	2.132	10	2.142	32	2.174	55	2.229	83	2.312	118	2.430	165	2.595	231	2.826	333	+3.159			27
26	2.038	10	2.048	30	2.078	51	2.129	77	2.206	109	2.315	151	2.466	211	2.677	300	2.977			26
25	1.952	9	1.961	27	1.988	48	2.036	71	2.107	101	2.208	139	2.347	194	2.541	273	2.814			25
24	1.871	8	1.879	26	1.905	44	1.949	67	2.016	93	2.109	129	2.238	178	2.416	249	2.665	363	+3.028	24
23	1.795	8	1.803	23	1.826	42	1.868	62	1.930	87	2.017	120	2.137	165	2.302	228	2.530	329	2.859	23
22	1.723	8	1.731	22	1.753	39	1.792	58	1.850	82	1.932	111	2.043	153	2.196	210	2.406	301	2.707	22
21	1.656	7	1.663	21	1.684	37	1.721	55	1.776	76	1.852	104	1.956	142	2.098	104	2.292	276	2.568	21
-20	+1.592	7	+1.599	20	+1.619	35	+1.654	52	+1.706	71	+1.777	98	+1.875	132	+2.007	180	+2.187	254	+2.441	-20
19	1.532	7	1.539	19	1.558	32	1.590	49	1.639	68	1.707	91	1.798	124	1.922	168	2.090	234	2.324	19
18	1.476	5	1.481	18	1.499	31	1.530	46	1.576	64	1.640	87	1.727	116	1.843	157	2.000	216	2.216	18
17	1.421	6	1.427	17	1.444	29	1.473	44	1.517	60	1.577	83	1.660	109	1.769	147	1.916	201	2.117	17
16	1.370	5	1.375	17	1.392	27	1.419	42	1.461	57	1.518	78	1.596	103	1.699	138	1.837	188	2.025	16
15	1.321	5	1.326	16	1.342	26	1.368	40	1.408	54	1.462	73	1.535	98	1.633	130	1.763	176	1.939	15
14	1.274	5	1.279	15	1.294	25	1.319	38	1.357	52	1.409	69	1.478	93	1.571	122	1.693	166	1.859	14
13	1.229	5	1.234	14	1.248	24	1.272	37	1.309	49	1.358	66	1.424	88	1.512	116	1.628	156	1.784	13
12	1.186	5	1.191	13	1.204	24	1.228	34	1.262	48	1.310	63	1.373	83	1.456	110	1.566	147	1.713	12
11	1.145	5	1.150	13	1.163	22	1.185	33	1.218	45	1.263	60	1.323	80	1.403	104	1.507	139	1.646	11
-10	+1.106	4	+1.110	12	+1.122	22	+1.144	31	+1.175	43	+1.218	58	+1.276	76	+1.352	99	+1.451	132	+1.583	-10
9	1.068	4	1.072	12	1.084	20	1.104	31	1.135	41	1.176	56	1.232	72	1.304	94	1.398	126	1.524	9
8	1.031	4	1.035	11	1.046	20	1.066	30	1.096	40	1.136	53	1.189	69	1.258	90	1.348	119	1.467	8
7	0.996	3	0.999	11	1.010	20	1.030	28	1.058	38	1.096	51	1.147	67	1.214	86	1.300	114	1.414	7
6	0.961	4	0.965	11	0.976	18	0.994	28	1.022	36	1.058	50	1.108	64	1.172	82	1.254	109	1.363	6
5	0.928	4	0.932	10	0.942	18	0.960	27	0.987	35	1.022	48	1.070	61	1.131	79	1.210	104	1.314	5
4	0.896	4	0.900	10	0.910	17	0.927	26	0.953	34	0.987	46	1.033	59	1.092	76	1.168	99	1.267	4
3	0.865	4	0.869	9	0.878	17	0.895	25	0.920	33	0.953	44	0.997	57	1.054	74	1.128	95	1.223	3
2	0.835	4	0.839	9	0.848	16	0.864	24	0.888	33	0.921	42	0.963	55	1.018	71	1.089	91	1.180	2
1	0.806	3	0.809	9	0.818	16	0.834	24	0.858	31	0.889	41	0.930	53	0.983	68	1.051	88	1.139	1
0	+0.778	3	+0.781	9	+0.790	15	+0.805	23	+0.828	30	+0.858	40	+0.898	51	+0.949	66	+1.015	85	+1.100	0
δ / $s-\alpha$	0°	d.	5°	d.	10°	d.	15°	d.	20°	d.	25°	d.	30°	d.	35°	d.	40°	d.	45°	δ \ $s-\alpha$

$Y = \tan(\Delta - \delta)$.

TABLE V.

Y

δ \ $s-\alpha$	0°	d.	5°	d.	10°	d.	15°	d.	20°	d.	25°	d.	30°	d.	35°	d.	40°	d.	45°	$s-\alpha$ / δ
0°	+0.778	3	+0.781	9	+0.790	15	+0.805	23	+0.828	30	+0.858	40	+0.898	51	+0.949	66	+1.015	85	+1.100	0°
1	0.750	3	0.753	9	0.762	15	0.777	22	0.799	29	0.828	39	0.867	50	0.917	63	0.980	82	1.062	1
2	0.723	3	0.726	9	0.735	14	0.749	22	0.771	28	0.799	38	0.837	48	0.885	62	0.947	78	1.025	2
3	0.697	3	0.700	8	0.708	14	0.722	21	0.743	28	0.771	37	0.808	47	0.855	59	0.914	76	0.990	3
4	0.671	3	0.674	8	0.682	14	0.696	20	0.716	28	0.744	35	0.779	46	0.825	58	0.883	73	0.956	4
5	0.646	3	0.649	8	0.657	13	0.670	20	0.690	27	0.717	35	0.752	44	0.796	56	0.852	72	0.924	5
6	0.622	2	0.624	8	0.632	13	0.645	20	0.665	26	0.691	34	0.725	43	0.768	54	0.822	70	0.892	6
7	0.598	2	0.600	8	0.608	13	0.621	19	0.640	25	0.665	33	0.698	42	0.740	54	0.794	67	0.861	7
8	0.574	3	0.577	7	0.584	13	0.597	18	0.615	25	0.640	33	0.673	41	0.714	52	0.766	65	0.831	8
9	0.551	3	0.554	7	0.561	13	0.574	18	0.592	24	0.616	32	0.648	40	0.688	50	0.738	64	0.802	9
+ 10	+0.529	2	+0.531	7	+0.538	13	+0.551	17	+0.568	24	+0.592	31	+0.623	39	+0.662	50	+0.712	62	+0.774	+ 10
11	0.507	2	0.509	7	0.516	12	0.528	17	0.545	24	0.569	30	0.599	38	0.637	49	0.686	60	0.746	11
12	0.485	2	0.487	7	0.494	12	0.506	17	0.523	23	0.546	30	0.576	37	0.613	47	0.660	59	0.719	12
13	0.464	2	0.466	7	0.473	11	0.484	17	0.501	23	0.524	29	0.553	36	0.589	46	0.635	58	0.693	13
14	0.442	3	0.445	7	0.452	11	0.463	16	0.479	23	0.502	28	0.530	36	0.566	45	0.611	56	0.667	14
15	0.422	2	0.424	7	0.431	11	0.442	16	0.458	22	0.480	28	0.508	35	0.543	45	0.588	55	0.643	15
16	0.401	3	0.404	6	0.410	11	0.421	16	0.437	21	0.458	28	0.486	35	0.521	43	0.564	54	0.618	16
17	0.381	2	0.383	7	0.390	11	0.401	16	0.417	21	0.438	27	0.465	34	0.499	43	0.542	52	0.594	17
18	0.361	3	0.364	6	0.370	11	0.381	15	0.396	21	0.417	27	0.444	33	0.477	42	0.519	52	0.571	18
19	0.342	2	0.344	6	0.350	11	0.361	15	0.376	21	0.397	26	0.423	33	0.456	41	0.497	51	0.548	19
+ 20	+0.322	3	+0.325	6	+0.331	10	+0.341	15	+0.356	20	+0.376	26	+0.402	33	+0.435	41	+0.476	50	+0.526	+ 20
21	0.303	2	0.305	6	0.311	11	0.322	15	0.337	20	0.357	25	0.382	33	0.415	39	0.454	49	0.503	21
22	0.284	2	0.286	6	0.292	11	0.303	14	0.317	20	0.337	25	0.362	32	0.394	39	0.433	49	0.482	22
23	0.266	2	0.268	6	0.274	10	0.284	14	0.298	20	0.318	25	0.343	31	0.374	39	0.413	47	0.460	23
24	0.247	2	0.249	6	0.255	10	0.265	14	0.279	20	0.299	24	0.323	31	0.354	39	0.393	46	0.439	24
25	0.229	2	0.231	5	0.236	10	0.246	15	0.261	19	0.280	24	0.304	31	0.335	38	0.373	46	0.419	25
26	0.210	2	0.212	6	0.218	10	0.228	14	0.242	19	0.261	24	0.285	31	0.316	37	0.353	45	0.398	26
27	0.192	2	0.194	6	0.200	10	0.210	14	0.224	18	0.242	25	0.267	29	0.296	37	0.333	45	0.378	27
28	0.174	2	0.176	6	0.182	9	0.191	15	0.206	18	0.224	24	0.248	30	0.278	36	0.314	44	0.358	28
29	0.156	2	0.158	6	0.164	9	0.173	14	0.187	20	0.207	22	0.229	30	0.259	36	0.295	44	0.339	29
+ 30	+0.138	2	+0.140	6	+0.146	9	+0.155	14	+0.169	20	+0.189	22	+0.211	29	+0.240	36	+0.276	44	+0.320	+ 30
31	0.120	2	0.122	6	0.128	10	0.138	13	0.151	20	0.171	22	0.193	29	0.222	35	0.257	43	0.300	31
32	0.103	2	0.105	5	0.110	10	0.120	14	0.134	19	0.153	22	0.175	29	0.204	35	0.239	42	0.281	32
33	0.085	2	0.087	6	0.093	9	0.102	14	0.116	18	0.134	23	0.157	29	0.186	34	0.220	43	0.263	33
34	0.068	2	0.070	5	0.075	10	0.085	13	0.098	18	0.116	23	0.139	29	0.168	34	0.202	42	0.244	34
35	0.050	2	0.052	6	0.058	9	0.067	14	0.081	18	0.099	22	0.121	29	0.150	34	0.184	42	0.226	35
36	0.033	2	0.035	5	0.040	10	0.050	13	0.063	18	0.081	23	0.104	28	0.132	34	0.166	41	0.207	36
37	0.015	2	0.017	6	0.023	9	0.032	14	0.046	18	0.064	22	0.086	28	0.114	34	0.148	41	0.189	37
38	−0.002	2	+0.000	5	+0.005	10	+0.015	13	0.028	18	0.046	23	0.069	28	0.097	33	0.130	41	0.171	38
39	−0.020	2	−0.018	6	−0.012	9	−0.003	14	+0.011	17	0.028	23	0.051	28	0.079	34	0.113	40	0.153	39
+ 40	−0.037	2	−0.035	5	−0.030	10	−0.020	13	−0.007	18	+0.011	23	+0.034	27	+0.061	34	+0.095	41	+0.136	+ 40
δ / $s-\alpha$	0°	d.	5°	d.	10°	d.	15°	d.	20°	d.	25°	d.	30°	d.	35°	d.	40°	d.	45°	δ \ $s-\alpha$

TABLE V.

Y

$s-\alpha$ / δ	0°	*d.*	5°	*d.*	10°	*d.*	15°	*d.*	20°	*d.*	25°	*d.*	30°	*d.*	35°	*d.*	40°	*d.*	45°	$s-\alpha$ / δ
+40°	-0.037	2	-0.035	5	-0.030	10	-0.020	13	-0.007	18	+0.011	23	+0.034	27	+0.061	34	+0.095	41	+0.136	+40°
41	0.055	2	0.053	6	0.047	9	-0.038	14	0.024	18	-0.006	22	+0.016	28	0.044	34	0.078	40	0.118	41
42	0.072	2	0.070	5	0.065	10	0.055	13	0.042	18	0.024	23	-0.001	27	0.026	34	0.060	40	0.100	42
43	0.090	2	0.088	6	0.082	9	0.073	14	0.059	18	0.041	22	0.019	28	+0.009	33	0.042	41	0.083	43
44	0.107	2	0.105	5	0.100	10	0.090	13	0.077	18	0.059	23	0.036	28	-0.008	33	0.025	40	0.065	44
45	0.125	2	0.123	5	0.118	10	0.108	14	0.094	18	0.076	22	0.054	28	0.026	34	+0.008	40	0.048	45
46	0.143	2	0.141	6	0.135	9	0.126	14	0.112	18	0.094	23	0.071	28	0.043	33	-0.010	40	0.030	46
47	0.161	2	0.159	6	0.153	10	0.143	13	0.130	18	0.112	23	0.089	28	0.061	34	0.027	40	+0.013	47
48	0.179	2	0.177	6	0.171	10	0.161	14	0.147	18	0.129	23	0.106	28	0.078	33	0.045	40	-0.005	48
49	0.197	2	0.195	6	0.189	10	0.179	14	0.165	18	0.147	23	0.124	28	0.096	34	0.062	40	0.022	49
+50	-0.215	2	-0.213	6	-0.207	10	-0.197	14	-0.183	18	-0.165	23	-0.142	28	-0.114	34	-0.080	40	-0.040	+50
51	0.233	2	0.231	6	0.225	9	0.216	15	0.201	18	0.183	23	0.160	29	0.131	33	0.098	41	0.057	51
52	0.252	2	0.250	6	0.244	10	0.234	14	0.220	19	0.201	23	0.178	29	0.149	34	0.115	40	0.075	52
53	0.270	2	0.268	6	0.262	10	0.252	14	0.238	19	0.219	23	0.196	29	0.167	34	0.133	41	0.092	53
54	0.289	2	0.287	6	0.281	10	0.271	14	0.257	19	0.238	24	0.214	29	0.185	34	0.151	41	0.110	54
55	-0.308	2	0.306	6	0.300	10	0.290	15	0.275	19	0.256	24	0.232	29	0.203	35	0.168	40	0.128	55
56	0.327	2	0.325	6	0.319	10	0.309	15	0.294	19	0.275	24	0.251	30	0.221	35	0.186	41	0.145	56
57	0.347	2	0.345	7	0.338	10	0.328	15	0.313	19	0.294	25	0.269	29	0.240	35	0.205	42	0.163	57
58	0.366	2	0.364	6	0.358	10	0.348	16	0.332	19	0.313	25	0.288	30	0.258	35	0.223	42	0.181	58
59	0.386	2	0.384	6	0.378	11	0.367	15	0.352	20	0.332	25	0.307	30	-0.277	36	0.241	42	0.199	59
+60	-0.406	2	-0.404	6	-0.398	11	-0.387	15	-0.372	20	-0.352	26	-0.326	30	-0.296	36	-0.260	42	-0.218	+60
61	0.427	2	0.425	7	0.418	11	0.407	15	0.392	21	0.371	25	0.346	31	0.315	36	0.279	43	0.236	61
62	0.448	2	0.446	7	0.439	11	0.428	16	0.412	21	0.391	26	0.365	31	0.334	36	0.298	44	0.254	62
63	0.469	2	0.467	7	0.460	11	0.449	17	0.432	20	0.412	27	0.385	31	0.354	37	0.317	44	0.273	63
64	0.490	2	0.488	7	0.481	11	0.470	17	0.453	21	0.432	26	0.406	32	0.374	38	0.336	44	0.292	64
65	0.512	2	0.510	7	0.503	12	0.491	16	0.475	22	0.453	27	0.426	32	0.394	39	0.355	44	0.311	65
66	0.535	3	0.532	7	0.525	12	0.513	17	0.496	22	0.474	27	0.447	33	0.414	39	0.375	45	0.330	66
67	0.557	2	0.555	7	0.548	13	0.535	17	0.518	22	0.496	28	0.468	33	0.435	40	0.395	45	0.350	67
68	0.580	2	0.578	7	0.571	13	0.558	18	0.540	22	0.518	29	0.489	34	0.455	39	0.416	47	0.369	68
69	0.604	3	0.601	7	0.594	13	0.581	18	0.563	23	0.540	29	0.511	34	0.477	41	0.436	47	0.389	69
+70	-0.628	3	-0.625	7	-0.618	13	-0.605	19	-0.586	23	-0.563	30	-0.533	35	-0.498	41	-0.457	47	-0.410	+70
71	0.652	2	0.650	8	0.642	13	0.629	19	0.610	24	0.586	30	0.556	36	0.520	42	0.478	48	0.430	71
72	0.678	3	0.675	8	0.667	13	0.654	20	0.634	24	0.610	31	0.579	36	0.543	43	0.500	49	0.451	72
73	0.704	3	0.701	9	0.692	13	0.679	20	0.659	25	0.634	31	0.603	38	0.565	43	0.522	50	0.472	73
74	0.730	3	0.727	8	0.719	14	0.705	21	0.684	26	0.658	31	0.627	38	0.589	45	0.544	50	0.494	74
75	0.757	3	0.754	9	0.745	14	0.731	21	0.710	26	0.684	33	0.651	39	0.612	45	0.567	51	0.516	75
76	0.785	3	0.782	9	0.773	15	0.758	21	0.737	27	0.710	34	0.676	39	0.637	46	0.591	53	0.538	76
77	0.813	3	0.810	9	0.801	15	0.786	22	0.764	28	0.736	34	0.702	40	0.662	48	0.614	53	0.561	77
78	0.843	3	0.840	10	0.830	16	0.814	22	0.792	28	0.764	35	0.729	42	0.687	48	0.639	55	0.584	78
79	0.873	3	0.870	10	0.860	16	0.844	23	0.821	29	0.792	36	0.756	43	0.713	49	0.664	57	0.607	79
+80	-0.904	3	-0.901	10	-0.891	17	-0.874	23	-0.851	31	-0.820	37	-0.783	43	-0.740	51	-0.689	57	-0.632	+80
81	0.937	4	0.933	10	0.923	17	0.906	25	0.881	31	0.850	38	0.812	45	0.767	52	0.715	59	0.656	81
82	0.970	4	0.966	10	0.956	18	0.938	25	0.913	32	0.881	40	0.841	46	0.795	53	0.742	61	0.681	82
83	1.004	3	1.001	11	0.990	19	0.971	26	0.945	33	0.912	40	0.872	48	0.824	55	0.769	62	0.707	83
84	1.040	4	1.036	11	1.025	19	1.006	27	0.979	34	0.945	42	0.903	49	0.854	57	0.797	63	0.734	84
85	1.077	4	1.073	12	1.061	20	1.041	27	1.014	36	0.978	43	0.935	51	0.884	58	0.826	65	0.761	85
86	1.116	5	1.111	12	1.099	21	1.078	28	1.050	37	1.013	45	0.968	52	0.916	60	0.856	67	0.789	86
87	1.156	5	1.151	13	1.138	21	1.117	30	1.087	38	1.049	46	1.003	55	0.948	61	0.887	69	0.818	87
88	1.197	4	1.193	14	1.179	22	1.157	31	1.126	40	1.086	48	1.038	56	0.982	64	0.918	71	0.847	88
89	1.241	5	1.236	14	1.222	23	1.199	33	1.166	41	1.125	50	1.075	58	1.017	66	0.951	73	0.878	89
+90	-1.286	5	-1.281	15	-1.266	24	-1.242	34	-1.208	43	-1.165	51	-1.114	61	-1.053	68	-0.985	76	-0.909	+90
δ / $s-\alpha$	0°	*d.*	5°	*d.*	10°	*d.*	15°	*d.*	20°	*d.*	25°	*d.*	30°	*d.*	35°	*d.*	40°	*d.*	45°	δ / $s-\alpha$

TABLE VI.

Δ

Δ is independent of the sign of $(s-\alpha)$.

$s-\alpha$	Δ °	′	diff.	$s-\alpha$	Δ °	′	diff.	$s-\alpha$	Δ °	′	diff.	$s-\alpha$	Δ °	′	diff.
0°	37	52		45°	47	43		90°	90	0		135°	132	17	
1	37	53	1	46	48	14	31	91	91	17	77	136	132	46	29
2	37	53	0	47	48	45	31	92	92	34	77	137	133	14	28
3	37	55	2	48	49	17	32	93	93	51	77	138	133	42	28
4	37	57	2	49	49	51	34	94	95	8	77	139	134	9	27
5	37	59	2	50	50	26	35	95	96	24	76	140	134	34	25
6	38	1	2	51	51	1	35	96	97	39	75	141	134	59	25
7	38	4	3	52	51	38	37	97	98	54	75	142	135	23	24
8	38	8	4	53	52	16	38	98	100	9	75	143	135	46	23
9	38	13	5	54	52	55	39	99	101	22	73	144	136	8	22
10	38	18	5	55	53	35	40	100	102	35	73	145	136	29	21
11	38	23	5	56	54	17	42	101	103	47	72	146	136	50	21
12	38	29	6	57	55	0	43	102	104	58	71	147	137	10	20
13	38	36	7	58	55	44	44	103	106	8	70	148	137	29	19
14	38	43	7	59	56	29	45	104	107	17	69	149	137	47	18
15	38	50	7	60	57	16	47	105	108	24	67	150	138	4	17
16	38	58	8	61	58	4	48	106	109	31	67	151	138	21	17
17	39	7	9	62	58	53	49	107	110	36	65	152	138	37	16
18	39	16	9	63	59	44	51	108	111	40	64	153	138	53	16
19	39	26	10	64	60	36	52	109	112	43	63	154	139	8	15
20	39	36	10	65	61	29	53	110	113	44	61	155	139	22	14
21	39	47	11	66	62	23	54	111	114	44	60	156	139	35	13
22	39	59	12	67	63	20	57	112	115	43	59	157	139	48	13
23	40	12	13	68	64	17	57	113	116	40	57	158	140	1	13
24	40	25	13	69	65	16	59	114	117	37	57	159	140	13	12
25	40	38	13	70	66	16	60	115	118	31	54	160	140	24	11
26	40	52	14	71	67	17	61	116	119	24	53	161	140	34	10
27	41	7	15	72	68	20	63	117	120	16	52	162	140	44	10
28	41	23	16	73	69	24	64	118	121	7	51	163	140	53	9
29	41	39	16	74	70	29	65	119	121	56	49	164	141	2	9
30	41	56	17	75	71	36	67	120	122	44	48	165	141	10	8
31	42	13	17	76	72	43	67	121	123	31	47	166	141	17	7
32	42	31	18	77	73	52	69	122	124	16	45	167	141	24	7
33	42	50	19	78	75	2	70	123	125	0	44	168	141	31	7
34	43	10	20	79	76	13	71	124	125	43	43	169	141	37	6
35	43	31	21	80	77	25	72	125	126	25	42	170	141	42	5
36	43	52	21	81	78	38	73	126	127	5	40	171	141	47	5
37	44	14	22	82	79	51	73	127	127	44	39	172	141	52	5
38	44	37	23	83	81	6	75	128	128	22	38	173	141	56	4
39	45	1	24	84	82	21	75	129	128	59	37	174	141	59	3
40	45	26	25	85	83	36	75	130	129	34	35	175	142	1	2
41	45	51	25	86	84	52	76	131	130	9	35	176	142	3	2
42	46	18	27	87	86	9	77	132	130	43	34	177	142	5	2
43	46	46	28	88	87	26	77	133	131	15	32	178	142	7	2
44	47	14	28	89	88	43	77	134	131	46	31	179	142	7	0
45	47	43	29	90	90	0	77	135	132	17	31	180	142	8	1
$s-\alpha$	Δ			$s-\alpha$	Δ			$s-\alpha$	Δ			$s-\alpha$	Δ		

$\tan \Delta = \tan \varphi \sec (s-\alpha)$.

TABLE VII.

$X \cos (\Delta - \delta)$

$X \cos (\Delta - \delta)$ has sign of $(s-\alpha)$. It is therefore positive when center of plate is west of meridian.

$s-\alpha$	$\log X \cos (\Delta-\delta)$	$X \cos (\Delta-\delta)$	$s-\alpha$	$s-\alpha$	$\log X \cos (\Delta-\delta)$	$X \cos (\Delta-\delta)$	$s-\alpha$
0°	$-\infty$	0.000	180°	45°	9.8278	0.673	135°
1	8.139	14	179	46	9.8387	690	134
2	8.440	28	178	47	9.8494	707	133
3	8.616	41	177	48	9.8599	725	132
4	8.741	55	176	49	9.8702	742	131
5	8.839	0.069	175	50	9.8804	0.759	130
6	8.918	83	174	51	9.8903	777	129
7	8.985	97	173	52	9.9001	795	128
8	9.0435	110	172	53	9.9096	812	127
9	9.0949	124	171	54	9.9190	830	126
10	9.1411	0.138	170	55	9.9282	0.848	125
11	9.1830	152	169	56	9.9372	866	124
12	9.2211	166	168	57	9.9461	884	123
13	9.2563	180	167	58	9.9547	901	122
14	9.2890	195	166	59	9.9633	919	121
15	9.3195	0.209	165	60	9.9716	0.937	120
16	9.3482	223	164	61	9.9796	.954	119
17	9.3751	237	163	62	9 9875	.972	118
18	9.4006	252	162	63	9.9952	.989	117
19	9.4248	266	161	64	0.0028	1.007	116
20	9.4478	0.280	160	65	0.0103	1.024	115
21	9.4698	294	159	66	0.0175	1.041	114
22	9.4908	309	158	67	0.0243	1.058	113
23	9.5109	324	157	68	0.0310	1 074	112
24	9.5302	339	156	69	0.0375	1.090	111
25	9.5488	0.354	155	70	0.0438	1.106	110
26	9.5668	369	154	71	0.0498	1.122	109
27	9.5842	384	153	72	0.0555	1.136	108
28	9.6009	399	152	73	0.0610	1.151	107
29	9.6172	414	151	74	0.0663	1.165	106
30	9.6330	0.430	150	75	0.0713	1.178	105
31	9.6484	446	149	76	0.0760	1.192	104
32	9.6633	461	148	77	0.0804	1.203	103
33	9.6778	477	147	78	0.0845	1.215	102
34	9.6919	492	146	79	0.0883	1.225	101
35	9.7057	0.508	145	80	0.0920	1.236	100
36	9.7192	524	144	81	0.0952	1.245	99
37	9.7323	540	143	82	0.0981	1.254	98
38	9.7452	556	142	83	0.1007	1.260	97
39	9.7578	572	141	84	0.1029	1.266	96
40	9.7700	0.589	140	85	0.1048	1.273	95
41	9.7821	606	139	86	0.1064	1.278	94
42	9.7938	623	138	87	0.1076	1.281	93
43	9.8054	640	137	88	0.1085	1.284	92
44	9.8167	656	136	89	0.1091	1.286	91
45	9.8278	0.673	135	90	0.1093	1.286	90
$s-\alpha$	$\log X \cos (\Delta-\delta)$	$X \cos (\Delta-\delta)$	$s-\alpha$	$s-\alpha$	$\log X \cos (\Delta-\delta)$	$X \cos (\Delta-\delta)$	$s-\alpha$

$X \cos (\Delta - \delta) = \tan (s-\alpha) \cos \Delta$.

TABLE VIII.

For Conversion of Degrees into Radians.

$k\left\{\begin{matrix}x\\y\end{matrix}\right\}$	$k\left\{\begin{matrix}x\\y\end{matrix}\right\} / 57.296$ (R)	Diff.
0.0	0.00000	174
.1	174	175
.2	349	175
.3	524	174
.4	698	175
.5	873	174
.6	1047	175
.7	1222	174
.8	1396	175
.9	1571	174
1.0	0.01745	175
1.1	1920	174
1.2	2094	175
1.3	2269	174
1.4	2443	175
1.5	2618	175
1.6	2793	174
1.7	2967	175
1.8	3142	174
1.9	3316	175
2.0	0.03491	174
2.1	3665	175
2.2	3840	174
2.3	4014	175
2.4	4189	174
2.5	4363	175
2.6	4538	174
2.7	4712	175
2.8	4887	174
2.9	5061	175
3.0	0.05236	175
3.1	5411	174
3.2	5585	175
3.3	5760	174
3.4	5934	175
3.5	6109	174
3.6	6283	175
3.7	6458	174
3.8	6632	175
3.9	6807	174
4.0	0.06981	175
4.1	7156	174
4.2	7330	175
4.3	7505	174
4.4	7679	175
4.5	7854	175
4.6	8029	174
4.7	8203	174
4.8	8377	175
4.9	8552	175
5.0	0.08727	
$k\left\{\begin{matrix}x\\y\end{matrix}\right\}$	$k\left\{\begin{matrix}x\\y\end{matrix}\right\} / 57.296$	

	174	175
1	17.4	17.5
2	34.8	35.0
3	52.2	52.5
4	69.6	70.0
5	87.0	87.5
6	104.4	105.0
7	121.8	122.5
8	139.2	140.0
9	156.6	157.5

TABLE IX.

Giving $x^2 + y^2$ with arguments x and y.

The quantities are expressed in radians. The tabulated values are given in units of the fourth decimal place.

y \ x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	.10
0.00	0	1	4	9	16	25	36	49	64	81	100
1	1	2	5	10	17	26	37	50	65	82	101
2	4	5	8	13	20	29	40	53	68	85	104
3	9	10	13	18	25	34	45	58	73	90	109
4	16	17	20	25	32	41	52	65	80	97	116
0.05	25	26	29	34	41	50	61	74	89	106	125
6	36	37	40	45	52	61	72	85	100	117	136
7	49	50	53	58	65	74	85	98	113	130	149
8	64	65	68	73	80	89	100	113	128	145	164
9	81	82	85	90	97	106	117	130	145	162	181
0.10	100	101	104	109	116	125	136	149	164	181	200

TABLE X.

D

$xX + yY$	D	Diff.	$xX + yY$	D	Diff.	$xX + yY$	D	Diff.	$xX + yY$	D	Diff.
−.00	+.0000	101	−25	+.3333	181	+.00	−.0000	99	+25	−.2000	64
1	.0101	103	26	.3514	185	1	.0099	97	26	.2064	62
2	.0204	105	27	.3699	190	2	.0196	95	27	.2126	62
3	.0309	107	28	.3889	195	3	.0291	94	28	.2188	60
4	.0416	110	29	.4084	202	4	.0385	91	29	.2248	60
5	.0526	112	−.30	+.4286	207	5	.0476	90	+.30	−.2308	58
6	.0638	115	31	.4493	213	6	.0566	88	31	.2366	58
7	.0753	117	32	.4706	219	7	.0654	87	32	.2424	57
8	.0870	119	33	.4925	227	8	.0741	85	33	.2481	56
9	.0989	122	34	.5152	233	9	.0826	83	34	.2537	56
−.10	+.1111	125	35	.5385	240	+.10	−.0909	82	35	.2593	54
11	.1236	128	36	.5625	248	11	.0991	80	36	.2647	54
12	.1364	130	37	.5873	256	12	.1071	79	37	.2701	53
13	.1494	134	38	.6129	264	13	.1150	78	38	.2754	52
14	.1628	137	39	.6393	274	14	.1228	76	39	.2806	51
15	.1765	140	−.40	+.6667	282	15	.1304	75	+.40	−.2857	51
16	.1905	143	41	.6949	292	16	.1379	74	41	.2908	50
17	.2048	147	42	.7241	303	17	.1453	72	42	.2958	49
18	.2195	151	43	.7544	313	18	.1525	72	43	.3007	49
19	.2346	154	44	.7857	325	19	.1597	70	44	.3056	47
−.20	+.2500	158	45	.8182	337	+.20	−.1667	69	45	.3103	48
21	.2658	162	46	.8519	349	21	.1736	67	46	.3151	46
22	.2820	167	47	.8868	363	22	.1803	67	47	.3197	46
23	.2987	171	48	.9231	377	23	.1870	66	48	.3243	46
24	.3158	175	49	.9608	392	24	.1936	64	49	.3289	44
25	.3333		−.50	+1.0000		25	.2000		+.50	−.3333	
$xX + yY$	D		$xX + yY$	D		$xX + yY$	D		$xX + yY$	D	

$$D = \frac{1}{1 + xX + yY} - 1$$

TABLE XI.

For Zenith Distance of Center of Plate.

$$Z = \tan^{-1} \sqrt{X^2 + Y^2}$$

X and Y are expressed in units of radius, Z in degrees.

Y \ X	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0	5.5	6.0
R 0.0	0°	27°	45°	56°	63°.4	68°.2	71°.6	74°.1	75°.97	77°.47	78°.69	79°.70	80°.54
0.5	27.	35.	48.	58.	64.1	68.6	71.8	74.2	76.07	77.55	78.75	79.74	80.57
1.0	45.	48.	55.	61.0	65.9	69.6	72.4	74.6	76.37	77.76	78.90	79.86	80.67
1.5	56.	58.	61.0	64.8	68.2	71.1	73.4	75.29	76.82	78.10	79.16	80.05	
2.0	63.4	64.1	65.9	68.2	70.5	72.7	74.5	76.07	77.40	78.52	79.48	80.30	
2.5	68.2	68.6	69.6	71.1	72.7	74.2	75.63	76.91	78.03	79.00	79.86		
3.0	71.6	71.8	72.4	73.4	74.5	75.63	76.74	77.76	78.69	79.53	80.27		
3.5	74.1	74.2	74.6	75.29	76.07	76.91	77.76	78.58	79.34	80.05			
4.0	75.97	76.07	76.37	76.82	77.40	78.03	78.69	79.34	79.97				
4.5	77.47	77.55	77.76	78.10	78.52	79.00	79.53	80.05	80.57				
5.0	78.69	78.75	78.90	79.16	79.48	79.86	80.27						
5.5	79.70	79.74	79.86	80.05	80.30	80.60							
6.0	80.54	80.57	80.67										

Table for Transformation of Decimals of a Degree to Minutes and Seconds of Arc or Time, and the Converse.

°	′	″	m	s
.1	6	0.	0	24.
.2	12	0.		48.
.3	18	0.	1	12.
.4	24	0.	1	36.
.5	30	0.	2	0.
.6	36	0.	2	24.
.7	42	0.	2	48.
.8	48	0.	3	12.
.9	54	0.	3	36.
°	′	″		s
.01	0	36.		2.4
.02	1	12.		4.8
.03	1	48.		7.2
.04	2	24.		9.6
.05	3	0.		12.0
.06	3	36.		14.4
.07	4	12.		16.8
.08	4	48.		19.2
.09	5	24.		21.6
°		″		
.001		3.6		
.002		7.2		
.003		10.8		
.004		14.4		
.005		18.0		
.006		21.6		
.007		25.2		
.008		28.8		
.009		32.4		

′	°
10	.1666667
20	.3333333
30	.5000000
40	.6666667
50	.8333333
′	
1	.0166667
2	.0333333
3	.0500000
4	.0666667
5	.0833333
6	.1000000
7	.1166667
8	.1333333
9	.1500000
″	
10	.0027778
20	.0055556
30	.0083333
40	.0111111
50	.0138889
″	
1	.0002778
2	.0005556
3	.0008333
4	.0011111
5	.0013889
6	.0016667
7	.0019444
8	.0022222
9	.0025000

m	°
1	.2500000
2	.5000000
3	.7500000
s	
10	.0416667
20	.0833333
30	.1250000
40	.1666667
50	.2083333
s	
1	.0041667
2	.0083333
3	.0125000
4	.0166667
5	.0208333
6	.0250000
7	.0291667
8	.0333333
9	.0375000

INVESTIGATION OF THE REPSOLD MEASURING APPARATUS.

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ROPORTIONAL PARTS.

7	8	9	11	12	13	14	15	16	17	18	19
.7	.8	.9	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.4	1.6	1.8	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8
2.1	2.4	2.7	3.3	3.6	3.9	4.2	4.5	4.8	5.1	5.4	5.7
2.8	3.2	3.6	4.4	4.8	5.2	5.6	6.0	6.4	6.8	7.2	7.6
3.5	4.0	4.5	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
4.2	4.8	5.4	6.6	7.2	7.8	8.4	9.0	9.6	10.2	10.8	11.4
4.9	5.6	6.3	7.7	8.4	9.1	9.8	10.5	11.2	11.9	12.6	13.3
5.6	6.4	7.2	8.8	9.6	10.4	11.2	12.0	12.8	13.6	14.4	15.2
6.3	7.2	8.1	9.9	10.8	11.7	12.6	13.5	14.4	15.3	16.2	17.1

27	28	29	31	32	33	34	35	36	37	38	39
2.7	2.8	2.9	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
5.4	5.6	5.8	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8
8.1	8.4	8.7	9.3	9.6	9.9	10.2	10.5	10.8	11.1	11.4	11.7
0.8	11.2	11.6	12.4	12.8	13.2	13.6	14.0	14.4	14.8	15.2	15.6
3.5	14.0	14.5	15.5	16.0	16.5	17.0	17.5	18.0	18.5	19.0	19.5
6.2	16.8	17.4	18.6	19.2	19.8	20.4	21.0	21.6	22.2	22.8	23.4
8.9	19.6	20.3	21.7	22.4	23.1	23.8	24.5	25.2	25.9	26.6	27.3
1.6	22.4	23.2	24.8	25.6	26.4	27.2	28.0	28.8	29.6	30.4	31.2
4.3	25.2	26.1	27.9	28.8	29.7	30.6	31.5	32.4	33.3	34.2	35.1

47	48	49	51	52	53	54	55	56	57	58	59
4.7	4.8	4.9	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
9.4	9.6	9.8	10.2	10.4	10.6	10.8	11.0	11.2	11.4	11.6	11.8
4.1	14.4	14.7	15.3	15.6	15.9	16.2	16.5	16.8	17.1	17.4	17.7
8.8	19.2	19.6	20.4	20.8	21.2	21.6	22.0	22.4	22.8	23.2	23.6
3.5	24.0	24.5	25.5	26.0	26.5	27.0	27.5	28.0	28.5	29.0	29.5
8.2	28.8	29.4	30.6	31.2	31.8	32.4	33.0	33.6	34.2	34.8	35.4
2.9	33.6	34.3	35.7	36.4	37.1	37.8	38.5	39.2	39.9	40.6	41.3
7.6	38.4	39.2	40.8	41.6	42.4	43.2	44.0	44.8	45.6	46.4	47.2
2.3	43.2	44.1	45.9	46.8	47.7	48.6	49.5	50.4	51.3	52.2	53.1

67	68	69	71	72	73	74	75	76	77	78	79
6.7	6.8	6.9	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
3.4	13.6	13.8	14.2	14.4	14.6	14.8	15.0	15.2	15.4	15.6	15.8
0.1	20.4	20.7	21.3	21.6	21.9	22.2	22.5	22.8	23.1	23.4	23.7
6.8	27.2	27.6	28.4	28.8	29.2	29.6	30.0	30.4	30.8	31.2	31.6
5	34.0	34.5	35.5	36.0	36.5	37.0	37.5	38.0	38.5	39.0	39.5
0.2	40.8	41.4	42.6	43.2	43.8	44.4	45.0	45.6	46.2	46.8	47.4
6.9	47.6	48.3	49.7	50.4	51.1	51.8	52.5	53.2	53.9	54.6	55.3
3.6	54.4	55.2	56.8	57.6	58.4	59.2	60.0	60.8	61.6	62.4	63.2
0.3	61.2	62.1	63.9	64.8	65.7	66.6	67.5	68.4	69.3	70.2	71.1

87	88	89	91	92	93	94	95	96	97	98	99
8.7	8.8	8.9	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
7.4	17.6	17.8	18.2	18.4	18.6	18.8	19.0	19.2	19.4	19.6	19.8
.1	26.4	26.7	27.3	27.6	27.9	28.2	28.5	28.8	29.1	29.4	29.7
.8	35.2	35.6	36.4	36.8	37.2	37.6	38.0	38.4	38.8	39.2	39.6
.5	44.0	44.5	45.5	46.0	46.5	47.0	47.5	48.0	48.5	49.0	49.5
.2	52.8	53.4	54.6	55.2	55.8	56.4	57.0	57.6	58.2	58.8	59.4
.9	61.6	62.3	63.7	64.4	65.1	65.8	66.5	67.2	67.9	68.6	69.3
9.6	70.4	71.2	72.8	73.6	74.4	75.2	76.0	76.8	77.6	78.4	79.2
8.3	79.2	80.1	81.9	82.8	83.7	84.6	85.5	86.4	87.3	88.2	89.1

PROPORTIONAL PARTS.

7	8	9	11	12	13	14	15	16	17	18	19
.7	.8	.9	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9
1.4	1.6	1.8	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8
2.1	2.4	2.7	3.3	3.6	3.9	4.2	4.5	4.8	5.1	5.4	5.7
2.8	3.2	3.6	4.4	4.8	5.2	5.6	6.0	6.4	6.8	7.2	7.6
3.5	4.0	4.5	5.5	6.0	6.5	7.0	7.5	8.0	8.5	9.0	9.5
4.2	4.8	5.4	6.6	7.2	7.8	8.4	9.0	9.6	10.2	10.8	11.4
4.9	5.6	6.3	7.7	8.4	9.1	9.8	10.5	11.2	11.9	12.6	13.3
5.6	6.4	7.2	8.8	9.6	10.4	11.2	12.0	12.8	13.6	14.4	15.2
6.3	7.2	8.1	9.9	10.8	11.7	12.6	13.5	14.4	15.3	16.2	17.1

27	28	29	31	32	33	34	35	36	37	38	39
2.7	2.8	2.9	3.1	3.2	3.3	3.4	3.5	3.6	3.7	3.8	3.9
5.4	5.6	5.8	6.2	6.4	6.6	6.8	7.0	7.2	7.4	7.6	7.8
8.1	8.4	8.7	9.3	9.6	9.9	10.2	10.5	10.8	11.1	11.4	11.7
.8	11.2	11.6	12.4	12.8	13.2	13.6	14.0	14.4	14.8	15.2	15.6
3.5	14.0	14.5	15.5	16.0	16.5	17.0	17.5	18.0	18.5	19.0	19.5
.2	16.8	17.4	18.6	19.2	19.8	20.4	21.0	21.6	22.2	22.8	23.4
.9	19.6	20.3	21.7	22.4	23.1	23.8	24.5	25.2	25.9	26.6	27.3
.6	22.4	23.2	24.8	25.6	26.4	27.2	28.0	28.8	29.6	30.4	31.2
4.3	25.2	26.1	27.9	28.8	29.7	30.6	31.5	32.4	33.3	34.2	35.1

47	48	49	51	52	53	54	55	56	57	58	59
4.7	4.8	4.9	5.1	5.2	5.3	5.4	5.5	5.6	5.7	5.8	5.9
9.4	9.6	9.8	10.2	10.4	10.6	10.8	11.0	11.2	11.4	11.6	11.8
4.1	14.4	14.7	15.3	15.6	15.9	16.2	16.5	16.8	17.1	17.4	17.7
.8	19.2	19.6	20.4	20.8	21.2	21.6	22.0	22.4	22.8	23.2	23.6
.5	24.0	24.5	25.5	26.0	26.5	27.0	27.5	28.0	28.5	29.0	29.5
.2	28.8	29.4	30.6	31.2	31.8	32.4	33.0	33.6	34.2	34.8	35.4
.9	33.6	34.3	35.7	36.4	37.1	37.8	38.5	39.2	39.9	40.6	41.3
.6	38.4	39.2	40.8	41.6	42.4	43.2	44.0	44.8	45.6	46.4	47.2
.3	43.2	44.1	45.9	46.8	47.7	48.6	49.5	50.4	51.3	52.2	53.1

67	68	69	71	72	73	74	75	76	77	78	79
.7	6.8	6.9	7.1	7.2	7.3	7.4	7.5	7.6	7.7	7.8	7.9
.4	13.6	13.8	14.2	14.4	14.6	14.8	15.0	15.2	15.4	15.6	15.8
.1	20.4	20.7	21.3	21.6	21.9	22.2	22.5	22.8	23.1	23.4	23.7
.8	27.2	27.6	28.4	28.8	29.2	29.6	30.0	30.4	30.8	31.2	31.6
.5	34.0	34.5	35.5	36.0	36.5	37.0	37.5	38.0	38.5	39.0	39.5
.2	40.8	41.4	42.6	43.2	43.8	44.4	45.0	45.6	46.2	46.8	47.4
.9	47.6	48.3	49.7	50.4	51.1	51.8	52.5	53.2	53.9	54.6	55.3
.6	54.4	55.2	56.8	57.6	58.4	59.2	60.0	60.8	61.6	62.4	63.2
.3	61.2	62.1	63.9	64.8	65.7	66.6	67.5	68.4	69.3	70.2	71.1

87	88	89	91	92	93	94	95	96	97	98	99
.7	8.8	8.9	9.1	9.2	9.3	9.4	9.5	9.6	9.7	9.8	9.9
.4	17.6	17.8	18.2	18.4	18.6	18.8	19.0	19.2	19.4	19.6	19.8
.1	26.4	26.7	27.3	27.6	27.9	28.2	28.5	28.8	29.1	29.4	29.7
.8	35.2	35.6	36.4	36.8	37.2	37.6	38.0	38.4	38.8	39.2	39.6
.5	44.0	44.5	45.5	46.0	46.5	47.0	47.5	48.0	48.5	49.0	49.5
.2	52.8	53.4	54.6	55.2	55.8	56.4	57.0	57.6	58.2	58.8	59.4
.9	61.6	62.3	63.7	64.4	65.1	65.8	66.5	67.2	67.9	68.6	69.3
.6	70.4	71.2	72.8	73.6	74.4	75.2	76.0	76.8	77.6	78.4	79.2
.3	79.2	80.1	81.9	82.8	83.7	84.6	85.5	86.4	87.3	88.2	89.1

UNIVERSITY OF CALIFORNIA PUBLICATIONS.

PUBLICATIONS OF THE LICK OBSERVATORY, VOL. VII, PART 5.

INVESTIGATION OF THE REPSOLD MEASURING APPARATUS.

BY BURT L. NEWKIRK.

In the winter of 1903–4 the Students' Observatory received from the firm of Repsold Sons of Hamburg a measuring apparatus of the type described page 148ff SCHEINER'S "*Photographie der Gestirne.*" The following investigations of this instrument have been made:

1. Division errors of the scales;
2. Irregularities of the micrometer screw used in measuring the X coördinate;
3. Straightness of the bar which guides the movable (X) microscope;
4. Straightness of the cylinder which guides the plate carriage in its motion, determining the Y axis;
5. Perpendicularity of these two directions of motion;
6. Parallelism of the plate with the two scales;
7. Distortion of field of objective of X microscope.

The following remarks will throw light upon the order of accuracy which it is desirable to attain in each of these investigations.

The apparatus is so constructed that it may be used to determine either position angle and distance or rectangular coördinates. Position angle and distance measures can be advantageously used in determining the parallax of any star, the line determined by the image of the star under investigation and the companion star being made as nearly as possible parallel to the direction of the X scale, and the distance directly measured, the position angle being read off roughly for the purpose of future identification, and the computation of refraction and aberration corrections. For this purpose it is desirable to know the corrections for the division lines of the X scale and the corrections due to irregularities of the micrometer screw to within a few ten-thousandths of a mm. It is also necessary to ascertain that the field of the object glass of the microscope is not subject to any distortion which would affect by so much as two or three ten-thousandths of a mm., measurements of distances up to 1 mm. made with the micrometer screw.

Measures of position angle might be carefully made so as (in combination with the distance measures) fully to determine the position of one star with reference to another, or with reference to the center of rotation of the plate, the right

ascension and declination of which would be determined in the reduction. The position angle measures would however not be as thoroughly independent of changes in relative position and length of various parts of the instrument, due to temperature variation or mechanical strains and inaccuracies of construction, as are the measures of distance, and would, therefore, be less trustworthy. Positions determined by measurements of angle and distance would be subject to the same degree of inaccuracy as the measures of position angle. The division errors of the circle, by means of which the position angles are measured, and the irregularities of the micrometer screws used in making the settings on the circle have not been investigated, the accuracy of their construction being presumed to be commensurate with the accuracy obtainable in the measures of position angle. Measures made in this way would be subject to errors of altogether inadmissible magnitude if the bar which carries the movable microscope were not made with particular care to insure its freedom from curvature. The straightness of this bar has therefore been tested and found to be free from curvature or irregularities large enough to introduce errors of a thousandth of a mm. in a position determined from measures of position angle and distance.

Rectangular coördinates can be measured either by setting on a star and reading off both coördinates on the X and Y scales respectively, or by measuring the X coördinate only, in the first position of the plate, and then rotating the plate through 90° so that the Y coördinate also may be determined by comparison with the X scale.

The latter method recommended by BAKHUYZEN (SCHEINER *Photographie der Gestirne*) is considered more reliable, the former being open to objections similar to those attached to the measurements of position angle. In the measurement of rectangular coördinates by the latter method, the limit of negligible error might be placed at about one thousandth of a mm. This necessitates in addition to the determination of scale corrections and corrections due to the irregularities of the micrometer, an investigation of the straightness of the cylinder which guides the plate carriage in its motion in the direction of the Y coördinate, and the effect of temperature variation upon the straightness of this cylinder.

The former of these two methods can be used when extreme accuracy is of secondary importance. For this purpose it is desirable to know the corrections for the division lines of both scales to within a few thousandths of a mm., to ascertain that the two scales are approximately parallel to the plate and perpendicular to each other and that the two directions of motion (of the movable microscope determining the X axis, and of the plate carriage determining the Y axis) be straight and approximately perpendicular. The investigation shows that errors introduced into the measures by maladjustment in the above-mentioned particulars will not exceed five thousandths of a mm.

DIVISION ERRORS OF THE TWO SCALES.

The two scales used in measuring X and Y coördinates are designated respectively as N. E. K. 10–1903 and N. E. K. 11–1903. I shall refer to them as "scale 10" and "scale 11." The division errors of these two scales were investigated and tabulated by the *Normal Aichungs Institute of Berlin.* Values were given to thousandths of a millimeter, accompanied by a note to the effect that they

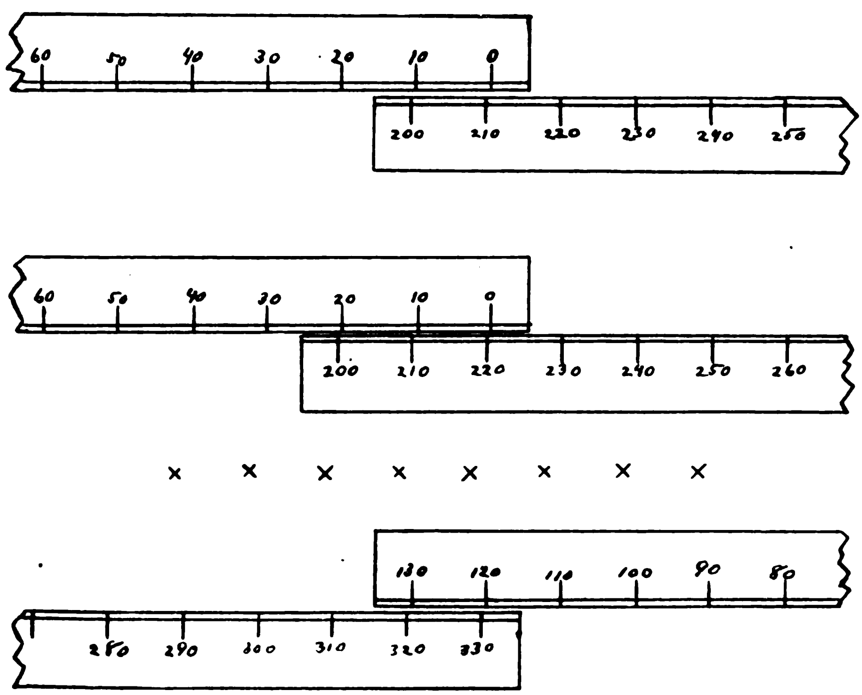

FIG. 1.

were probably reliable to within about .0015 mm. Inasmuch as *single settings* made with ordinary care upon a scale line differ from each other on the average by an amount considerably less than .0015 mm., it seemed to me that a more accurate determination of the division errors was essential if the instrument were to be used in work requiring the highest accuracy of which it is capable. I have therefore made an investigation of the division errors of some of the lines in the middle of each scale and tabulated the results to ten-thousandths of a mm. The tabulated values are probably reliable in the case of scale 10 to within three or four units of the last place. The tabulated corrections for lines near the center of scale 11 seem to be about five ten-thousandths of a mm. too small. It would not be difficult to remove this gradually accumulated error but it does not seem desirable to do so at the present time. See page 149.

The method employed was used by GILL in investigating the scales of the Cape heliometer and the scales of his Repsold measuring apparatus. The two scales are each approximately 130 mm. long and are made as nearly alike as possible except for the numbering of the division lines. The divisions of scale 10 are numbered from 1 to 130, and those of scale 11 from 200 to 330. The first step was to determine the correction for each tenth division line on both scales. By the correction corresponding to any division line, is meant that fraction of the unit of length which it is necessary to add to the number of the division line (less 200 in the case of scale 11) in order to obtain the distance of that line from the zero line of the scale, the unit of length being one one-hundred-and-thirtieth of the interval between the first and last lines of the scale. For example, the tabulated correction for line 263 of scale 11 is + .0196, and its distance from the zero line is therefore $\frac{63.0196}{130}$ of the distance between lines 200 and 330. These corrections for each tenth division line of both scales were deduced from a comparison of each 10 mm. interval of scale 10 with each 10 mm. interval of scale 11, the two scales being placed side by side, on the stage designed for the X scale, in the successive positions indicated in Figure 1.

Eight settings were made for the determination of the micrometer reading of a line. Let a_1 represent the micrometer reading of line 0 of scale 10 and b_1 the reading of line 210 of scale 11, the two readings having been taken without moving the micrometer carriage in the mean time. Then, $b_1 - a_1 = c_1$ represents the distance of the line 210 to the left, let us say, of the line 0. If now the microscope carriage be moved through an interval of approximately 10 mm., and a similar series of readings be taken on lines 10 and 200, designating the corresponding quantities by the subscript 2, the quantity $c_2 - c_1 = d$ represents the amount by which the interval 200–210 of scale 11 is larger than the interval 0–10 of scale 10. The comparison of each of the thirteen 10 mm. intervals of scale 10 with each 10 mm. interval of scale 11 yields 13^2 quantities such as d. Let these be designated by double subscripts, the first subscript referring to the interval of scale 10, the second to the interval of scale 11.

$$\begin{array}{cccccccc} d_{1\,1} & & d_{2\,1} & & d_{3\,1} & & d_{13\,1} \\ d_{1\,2} & & d_{2\,2} & & d_{3\,2} & & d_{13\,2} \\ d_{1\,3} & & d_{2\,3} & & d_{3\,3} & & d_{13\,3} \\ * & * & * & * & * & * & * \\ d_{1\,13} & & d_{2\,13} & & d_{3\,13} & & d_{13\,13} \end{array}$$

Now let D'_1, D'_2 . . . . D'_{13} represent the means of the quantities in the columns. D'_1 is the amount by which the interval 0–10 of scale 10 is shorter than one-thirteenth of the total length of scale 11. $\Sigma D'_i$ represents the total amount by which scale 10 is shorter than scale 11. The amounts by which the successive ten mm. intervals of scale 10 are shorter than one one-thirteenth of its own total length are obtained by applying with the opposite sign $\frac{1}{13}\Sigma D'_i$ to each D'.

Let: $D_1 = D'_1 - \frac{1}{13}\,\Sigma D'_i$, $D_2 = D'_2 - \frac{1}{13}\,\Sigma D'_i$, etc.

The correction for the nth line of scale 10 as defined above is:

Correction to nth line $= -(D_1 + D_2 + D_3 \ldots . + D_n)$. The correction to the thirteenth line is $-\Sigma D_i = 0$.

To obtain the corrections for the lines at 10 mm. intervals of scale 11, represent the means of the quantities in the rows by $\varDelta'_i$.

Let $\varDelta_1 = \varDelta'_1 - \frac{1}{13}\,\Sigma\varDelta'_i$, etc.

The $\varDelta$'s are the amounts by which the successive intervals of scale 11 are longer than one-thirteenth of the total length of the scale. We have, therefore,

Correction to nth line $= \varDelta_1 + \varDelta_2 + \ldots . \varDelta_n$.

The observations were made with the high power microscope, 10 revolutions of the micrometer being equivalent to one scale division. The readings were made to thousandths of a revolution so that the quantities c were obtained directly in units approximately equal to .0001 of a scale division. Settings were made upon the scale line on that side of the line running the length of the scale which is nearest the other scale in the positions indicated (see figure) and the attention was fixed upon that portion of the scale line about one-twentieth mm. distant from the long line. The intervals c measured by the micrometer were small and nearly uniform, so that the quantities d were also small and the error of run was negligible throughout most of the investigation. The same portion of the micrometer was used in measuring all the intervals c in any one position of the scales so that the irregularities of pitch or eccentricity of the head would not affect the results.

The following is an array of the quantities d with the values of the D's and $\varDelta$'s. When d is positive the interval of scale 11 is larger.

TABLE I.

$d_{i\,j}$

Interval of Sc. 11. \ Interval of Sc. 10.	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120	120–130	Σ	$\Sigma/_{13} = \varDelta'_i$.
200–210	+.04001	−.00250	+.00057	−.00817	+.00052	+.00034	−.00041	−.00117	−.00108	+.00077	−.00110	−.00202	+.00046	+.02622	+.00202
210–220	4065	− 168	+ 199	− 671	+ 246	+ 189	+ 48	+ 97	+ 11	+ 148	+ 30	− 5	+ 268	+.04457	+.00343
220–230	4208	− 29	+ 244	− 642	+ 290	+ 361	+ 138	+ 132	+ 55	+ 162	+ 17	− 62	+ 166	+.05040	+.00387
230–240	3975	− 246	+ 121	− 774	+ 28	+ 32	− 179	− 96	− 120	+ 8	− 142	− 181	+ 87	+.02513	+.00193
240–250	4174	− 31	+ 243	− 634	+ 296	+ 311	+ 164	+ 67	+ 14	+ 205	+ 31	+ 12	+ 197	+.05049	+.00388
250–260	4078	− 18	+ 201	− 602	+ 205	+ 325	+ 60	+ 77	− 13	+ 119	− 64	− 42	+ 135	+.04461	+.00343
260–270	4088	+ 9	+ 226	− 627	+ 185	+ 277	+ 72	+ 66	+ 66	+ 174	+ 34	− 7	+ 254	+.04817	+.00371
270–280	3760	− 382	− 109	− 963	− 70	− 37	− 233	− 306	− 273	− 141	− 266	− 434	− 78	+.00468	+.00036
280–290	4033	− 81	+ 139	− 736	+ 33	+ 178	+ 23	− 41	− 104	+ 86	− 63	− 203	+ 84	+.03348	+.00257
290–300	4711	+ 560	+ 870	− 1	+ 859	+ 977	+ 707	+ 718	+ 685	+ 870	+ 673	+ 652	+ 902	+.13183	+ 01014
300–310	3926	− 185	+ 51	− 847	− 11	+ 68	− 19	− 151	− 242	+ 28	− 174	− 236	+ 47	+.02255	+.00173
310–320	4094	− 96	+ 291	− 738	+ 240	+ 274	+ 5	+ 52	00	+ 138	− 34	− 102	+ 227	+.04351	+.00335
320–330	0056	− 4151	− 3802	− 4728	− 3811	− 3697	− 3990	− 4025	− 4012	− 3891	− 4029	− 4137	− 3849	−.48066	−.03698
Σ	+.49169	−.05068	−.01269	−.12780	−.01458	−.00708	−.03245	−.03527	−.04041	−.02017	−.04097	−.04947	−.01514		
$\Sigma/_{13} = D'_i$.	+.03782	−.00390	−.00098	−.00983	−.00112	−.00054	−.00250	−.00271	−.00311	−.00155	−.00315	−.00381	−.00117		

$$\Sigma D'_i = \Sigma \varDelta'_i = +.00345$$

$$\frac{\Sigma D'_i}{13} = +.000265$$

The unit is 1 mm.

TABLE II.

Table for check see p. 146.

Interval of Sc. 11. \ Interval of Sc. 10.	0–10	10–20	20–30	30–40	40–50	50–60	60–70	70–80	80–90	90–100	100–110	110–120	120–130
200–210	+.03958	−.00214	+.00078	−.00807	+.00064	+.00122	−.00074	−.00095	−.00135	+.00021	−.00139	−.00205	+.00059
210–220	4099	− 73	+ 219	− 666	+ 205	+ 263	+ 67	+ 46	+ 6	+ 62	+ 2	− 64	+ 200
220–230	4143	− 29	+ 263	− 622	+ 249	+ 307	+ 111	+ 90	+ 50	+ 206	+ 46	− 20	+ 244
230–240	3949	− 223	+ 69	− 816	+ 55	+ 113	− 83	− 104	− 144	+ 12	− 148	− 214	+ 50
240–250	4144	− 28	+ 264	− 621	+ 250	+ 308	+ 112	+ 91	+ 51	+ 207	+ 47	− 19	+ 245
250–260	4099	− 73	+ 219	− 666	+ 205	+ 263	+ 67	+ 46	+ 6	+ 162	+ 2	− 64	+ 200
260–270	4127	− 45	+ 247	− 638	+ 233	+ 291	+ 95	+ 74	+ 34	+ 190	+ 30	− 36	+ 228
270–280	3792	− 380	− 88	− 973	− 102	− 44	− 240	− 261	− 301	− 145	− 305	− 371	− 107
280–290	4013	− 159	+ 133	− 752	+ 119	+ 177	− 19	− 40	− 80	+ 76	− 84	− 150	+ 114
290–300	4770	+ 598	+ 890	+ 5	+ 876	+ 934	+ 738	+ 717	+ 677	+ 833	+ 673	+ 607	+ 871
300–310	3929	− 243	+ 49	− 836	+ 35	+ 93	− 103	− 124	− 164	− 8	− 168	− 234	+ 30
310–320	4091	− 81	+ 211	− 674	+ 197	+ 255	+ 59	+ 38	− 2	+ 54	− 6	− 72	+ 192
320–330	+ 58	− 4114	− 3822	− 4707	− 3836	− 3778	− 3974	− 3995	− 4035	− 3879	− 4039	− 4105	− 3841

The unit is 1 mm.

The D's and the $\varDelta$'s with the determination of the corrections to the scale lines follow.

D_1	$+.03755_5$	D_6	$-.00080_5$	D_{11}	$-.00341_5$
D_2	$-.00416_5$	D_7	$-.00276_5$	D_{12}	$-.00407_5$
D_3	$-.00124_5$	D_8	$-.00297_5$	D_{13}	$-.00143_5$
D_4	$-.01009_5$	D_9	$-.00337_5$		
D_5	$-.00138_5$	D_{10}	$-.00181_5$		

$\varDelta_1$	$+.00175_5$	$\varDelta_6$	$+.00316_5$	$\varDelta_{11}$	$+.00146_5$
$\varDelta_2$	$+.00316_5$	$\varDelta_7$	$+.00344_5$	$\varDelta_{12}$	$+.00308_5$
$\varDelta_3$	$+.00360_5$	$\varDelta_8$	$+.00009_5$	$\varDelta_{13}$	$-.03724_5$
$\varDelta_4$	$+.00166_5$	$\varDelta_9$	$+.00230_5$		
$\varDelta_5$	$+.00361_5$	$\varDelta_{10}$	$+.00987_5$		

CORRECTIONS FOR LINES OF SCALE 10.

Line.	Cor.	Line	Cor.	Line.	Cor.
10	− 375.6	60	− 198.7	110	− 55.2
20	− 334.0	70	− 171.0	120	− 14.4
30	− 321.5	80	− 141.3	130	.0
40	− 220.6	90	− 107.5		
50	− 206.7	100	− 89.4		

CORRECTIONS FOR LINES OF SCALE 11.

Line.	Cor.	Line.	Cor.	Line.	Cor.
210	+ 17.6	260	+ 169.7	310	+ 341.6
220	+ 49.2	270	+ 204.2	320	+ 372.4
230	+ 85.3	280	+ 205.1	330	.0
240	+ 101.9	290	+ 228.2		
250	+ 138.1	300	+ 326.9		

The unit is one ten-thousandth of a scale interval.

These tables of corrections form the basis of the following determination of corrections for intermediate lines.

When corrections for the lines at the beginning and end of any 10 mm. interval are known, corrections for the intervening lines are as follows.

Correction for nth line = Correction for first line

$$-\left\{\left[D_1+\frac{(D)}{10}\right]+\left[D_2+\frac{(D)}{10}\right]+\ .\ .\ .\ +\left[D_n+\frac{(D)}{10}\right]\right\},$$

where (D) = amount by which the 10 mm. interval is smaller than one-thirteenth of the whole scale, D_1, D_2, etc. = amounts by which successive mm. intervals are smaller than one-tenth of the whole interval.

Since $D_n = D'_n - \frac{1}{10}\Sigma D'$.

Correction for nth line = Correction for first line

$$-\left\{\left[D'_1+\frac{(D)-\Sigma D'_i}{10}\right]+\left[D'_2+\frac{(D)-\Sigma D'_i}{10}\right]+\ .\ .\ .\ .\ +\left[D'_n+\frac{(D)-\Sigma D'_i}{10}\right]\right.$$

The corresponding formula for the correction of a line of scale 11 is:
Correction for nth line = Correction for first line

$$+\left\{\left[\Delta'_1+\frac{(\Delta)-\Sigma\Delta'_i}{10}\right]+\left[\Delta'_2+\frac{(\Delta)-\Sigma\Delta'_i}{10}\right]+\ .\ .\ \text{etc.}\right\}$$

where (Δ) represents the amount by which the 10 mm. interval is larger than one-thirteenth of the whole scale.

Corrections were next determined for lines 35 and 95 of scale 10 and for lines 235 and 295 of scale 11, and these were made the basis for a determination of corrections for the intervening lines at intervals of 10 mm. The corrections for lines 35 and 295 were obtained by comparing the intervals 30–35–40 with the intervals 290–295–300 of the other scale and also by a comparison of the intervals 20–35–50 with the intervals 280–295–300. Similar comparisons were used to determine the corrections for lines 95 and 235. The adopted values are means of three investigations.

First adopted values of correction for line: 35 = −310.4
95 = −105.2
235 = +105.0
295 = +315.7

A second series of comparisons, made some months later, yielded the following results:

Correction for line: 35 = −314.0
95 = −106.2
235 = +105.5
295 = +313.8

The largest discrepancy is 3.6 ten-thousandths of a mm.

There is a very serious objection to the plan of using the corrections for the lines at 10 mm. intervals between lines 35 and 95 in conjunction with the corrections previously obtained for the lines 30, 40, 50, etc., as the basis for the determination of corrections for the lines within the successive 5 mm. intervals. This objection arises from the fact that the error of the determination of scale correction by this method is cumulative toward the center of the portion of the scale under investigation. Thus we should expect the error of the first determination to accumulate in the neighborhood of lines 60 and 70 and the error of the second determination to accumulate toward the line 65. If the two cumulative errors have opposite sign, the table of corrections would show the interval 60–65 to be considerably shorter than it really is, and the interval 65–70 correspondingly too long or vice versa. This is a serious matter in the measurement of short distances in which cumulative error of scale correction ought to be eliminated.

To guard against such error in the assumed lengths of the 5 mm. intervals two other determinations were made of corrections for lines at 5 mm. intervals

from 35 to 95 of scale 10, and from 235 to 295 of scale 11. In the first of these, corrections were determined for lines at 5 mm. intervals from 40–60 and from 60–90 of scale 10, and from 240–260 and from 260–290 of scale 11. In the second, corrections were determined for lines at 5 mm. intervals from 35 to 60 and from 70 to 95 of scale 10, and from 235 to 260 and 270 to 295 of scale 11. The method ordinarily used in determining corrections for any lines within an interval of the scale bases these corrections upon the corrections already determined for the lines at the beginning and end of that interval. In this case the method was so modified as to base the work upon the corrections obtained in the first investigation for the lines 40, 50, 60, 70, 80, 90 of scale 10 and lines 240, 250, 260, 270, 280, 290 of scale 11 and upon these lines only. The way in which this was accomplished may best be explained by reference to a particular example. In determining the corrections for the lines from 35 to 60 inclusive the D''s were:

Interval:	35–40;	40–45;	45–50;	50–55;	55–60
D'_i	-62.4	$+11.1$	$+22.3$	$+18.5$	$+18.6$

$$
\begin{array}{ll}
\text{Assumed correction for line 35} & -310.8 \quad +5x \\
\text{Assumed correction for line 60} & -198.7 \quad +5y \\
(D) & -112.1-5\,(y-x) \\
\Sigma D'_i & +\quad 8.1 \\
(D)-\Sigma D'_i & -120.2-5\,(y-x) \\
\dfrac{(D)-\Sigma D'_i}{5} & -\ 24.0-\ (y-x)
\end{array}
$$

$$
\begin{array}{rcccccc}
\text{Interval:} & 35\text{–}40; & 40\text{–}45; & 45\text{–}50; & 50\text{–}55; & 55\text{–}60 \\
D'_i & -62.4 & +11.1 & +22.3 & +18.5 & +18.6 \\
D'_i+\dfrac{(D)-\Sigma D'_i}{5} & \begin{bmatrix}-86.4\\-y+x\end{bmatrix} & \begin{bmatrix}-12.9\\-y+x\end{bmatrix} & \begin{bmatrix}-1.7\\-y+x\end{bmatrix} & \begin{bmatrix}-5.5\\-y+x\end{bmatrix} & \begin{bmatrix}-5.4\\-y+x\end{bmatrix}
\end{array}
$$

$$
\begin{array}{rcccccc}
\text{Line:} & 35 & 40 & 45 & 50 & 55 & 60 \\
\text{Correction to Line:} & \begin{bmatrix}-310.8\\+5x\end{bmatrix} & \begin{bmatrix}-224.4\\+y+4x\end{bmatrix} & \begin{bmatrix}-211.5\\+2y+3x\end{bmatrix} & \begin{bmatrix}-209.8\\+3y+2x\end{bmatrix} & \begin{bmatrix}-204.3\\+4y+x\end{bmatrix} & \begin{bmatrix}-198.9\\+5y\end{bmatrix}
\end{array}
$$

The corrections given in the table for lines 40, 50, 60 were:

Line 40	— 220.5
Line 50	— 206.7
Line 60	— 198.7

This leads to the equations:

$$y+4x=+224.4-220.5=+3.9$$
$$3y+2x=+209.8-206.7=+3.1$$
$$5y=+198.9-198.7=+\ .2$$

Taking the sum of the second and third minus the first, and the sum of the first and second minus the third, and solving, we have

$$7y - 2x = - .6$$
$$y - 6x = - 6.8$$
$$x = + 1.18$$
$$y = + .25$$

Giving as the result of the determination:

Line	35	40	45	50	55	60
Correction . .	— 304.9	— 219.4	— 207.5	— 206.7	— 202.1	— 197.7

The final table of corrections for lines at 5 mm. intervals, between lines 30 and 100 of scale 10 and lines 230 and 300 of scale 11, is obtained by taking the average of the results of the three determinations just described. These results with their averages are inserted here in tabular form. These corrections, with those for the lines 30, 100, 230, 300, given page 140, form the basis of the determination of corrections for the intervening lines at intervals of 1 mm.

SCALE 10.

Line. / Cor.	35	40	45	50	55	60	65	70	75	80	85	90	95
	- 310.7	- 220.5	- 213.5	- 206.7	- 206.3	- 198.7	- 188.7	- 171.0	- 159.4	- 141.3	- 135.6	- 107.5	- 105.2
		- 219.7	- 209.0	- 208.4	- 200.6	- 197.9							
						- 198.5	- 187.6	- 171.3	- 161.0	- 143.6	- 131.5	- 105.1	
	- 304.9	- 219.4	- 207.5	- 206.7	- 202.1	- 197.7		- 169.3	- 158.4	- 141.4	- 134.0	- 105.7	- 104.0
Aver.	- 307.8	- 219.8	- 210.0	- 207.2	- 203.0	- 198.2	- 188.2	- 170.5	- 159.6	- 142.1	- 133.7	- 106.1	- 104.6

SCALE 11.

Line. / Cor.	235	240	245	250	255	260	265	270	275	280	285	290	295
	+ 105.0	+ 101.9	+ 130.0	+ 138.1	+ 162.3	+ 169.7	+ 195.5	+ 204.2	+ 202.6	+ 205.1	+ 215.8	+ 228.2	+ 315.7
		+ 103.9	+ 133.2	+ 136.5	+ 159.1	+ 170.3	+ 195.4	+ 203.6					
								+ 204.4	+ 207.2	+ 205.8	+ 221.3	+ 228.5	
	+ 101.0	+ 104.0	+ 133.6	+ 138.3	+ 159.3	+ 168.3		+ 201.2	+ 202.2	+ 205.9	+ 217.2	+ 225.9	+ 315.5
Aver.	+ 103.0	+ 103.3	+ 132.3	+ 137.7	+ 160.2	+ 169.5	+ 195.4	+ 203.4	+ 204.0	+ 205.7	+ 217.8	+ 227.6	+ 315.6

The unit is one ten-thousandth of a scale interval.

I insert here, as an example, the array of d's for the interval 45–50 of scale 10 and 281–5 of scale 11, with the check array as described on page 146, and the determination of the corrections to the lines of the two scales.

LINES 46–50, 281–5.

Interval.	45–6	46–7	47–8	48–9	49–50
280–1	− 52	+ 32	+ 97	− 73	+ 39
281–2	+ 22	+ 15	+ 48	− 79	+ 41
282–3	− 23	+ 21	+ 64	− 79	+ 87
283–4	− 138	− 77	+ 20	− 215	− 22
284–5	+ 133	+ 157	+ 167	+ 22	+ 212
Σ	− 58	+ 148	+ 396	− 424	+ 357
$D'_i = \Sigma/5$ . .	− 12	+ 30	+ 79	− 85	+ 71
$D'_i + \frac{(D) - \Sigma D'_i}{5}$	− 34	+ 8	+ 57	− 107	+ 49
Cor.	− 206.6	− 207.4	− 213.1	− 202.4	− 207.3

Sum = + 83

Cor. for line 45	− 2100
Cor. for line 50	− 2072
(D)	− 28
$\Sigma D'_i$	+ 83
	5) − 111
$\frac{(D) - \Sigma D'_i}{5}$	− 22

Int.	45–6	46–7	47–8	48–9	49–50
280–1	− 20	+ 22	+ 71	− 93	+ 63
281–2	− 20	− 22	+ 71	− 93	+ 63
282–3	− 15	+ 27	+ 76	− 88	+ 68
283–4	− 115	− 73	− 24	− 188	− 32
284–5	+ 109	+ 151	+ 200	+ 36	+ 192

Cor. for line 280	+ 2057
Cor. for line 285	+ 2178
(Δ)	+ 121
$\Sigma \Delta'_i$	+ 83
	5) + 38
$\frac{(\Delta) - \Sigma \Delta'_i}{5}$	+ 8

Interval.	280–1	281–2	282–3	283–4	284–5
45– 6	− 52	+ 22	− 23	− 138	+ 133
46– 7	+ 32	+ 15	+ 21	− 77	+ 157
47– 8	+ 97	+ 48	+ 64	+ 20	+ 167
48– 9	− 73	− 79	− 79	− 215	+ 22
49–50	+ 39	+ 41	+ 87	− 22	+ 212
Σ	+ 43	+ 47	+ 70	− 432	+ 691
$\Delta'_i = \Sigma/5$ . .	+ 9	+ 9	+ 14	− 86	+ 138
$\Delta'_i + \frac{(\Delta) - \Sigma \Delta'_i}{5}$	+ 17	+ 17	+ 22	− 78	+ 146
Cor.	+ 207.4	+ 209.1	+ 211.3	+ 203.5	+ 218.1

Sum = + 84

Following are the tables of corrections so far as they have been determined:

CORRECTIONS FOR SCALE 10.

Subtract the correction from observed reading.
The unit is one ten-thousandth of a scale division.

Units. / Tens.	0	1	2	3	4	5	6	7	8	9	10	Units. / Tens.
0	0										376	0
10	376										334	10
20	334										322	20
30	322	316	305	318	311	308	310	306	290	288	220	30
40	220	213	220	219	214	210	207	207	213	202	207	40
50	207	199	208	208	204	203	201	194	210	205	198	50
60	198	195	197	193	188	188	192	202	169	182	170	60
70	170	167	172	181	168	160	162	146	151	140	142	70
80	142	144	144	128	134	134	122	113	112	110	106	80
90	106	105	101	107	101	105	100	101	101	90	89	90
100	89										55	100
110	55										14	110
120	14										0	120

CORRECTIONS FOR SCALE 11.

Add the corrections to the observed reading.
The unit is one ten-thousandth of a scale division.

Units. / Tens.	0	1	2	3	4	5	6	7	8	9	10	Units. / Tens.
200	0										18	200
210	18										49	210
220	49										85	220
230	85	89	94	107	99	103	103	107	106	109	103	230
240	103	111	120	118	121	132	131	137	158	155	160	240
250	160	143	142	146	155	160	173	177	174	166	170	250
260	170	178	172	196	194	195	192	200	195	201	204	260
270	204	203	214	202	206	204	208	210	205	199	206	270
280	206	207	209	211	204	218	212	227	233	224	228	280
290	228	288	299	320	317	316	307	324	314	326	327	290
300	327										342	300
310	342										372	310
320	372										0	320

METHODS OF CHECKING WORK AND ACCURACY OF DETERMINATION.

The quantities D_1, D_2, D_3, etc., are the amounts by which the successive 10 mm. intervals of scale 10 are found to be smaller than one-thirteenth of that scale. The quantities Δ'_1, Δ'_2, Δ'_3, etc., are the amounts by which the successive 10 mm. intervals of scale 11 are found to be larger than one-thirteenth of scale 10. $D_i + \Delta'_j (= d_{ij})$ is therefore the amount by which the *ith* interval of scale 10 is smaller than the *jth* interval of scale 11, according to our determination This affords a convenient method of checking the observation and reduction as far as the formation of the table of values of the *d*'s, given page 139. The table for check (p. 140) is formed by adding each Δ' to each D. Disagreement between numbers in this table and corresponding numbers in the table of *d*'s is due to error in observation or reduction. The unit employed being approximately one mm., it is seen that no single *d* is in error by as much as one thousandth of a mm. When this check was first applied, several larger discrepancies were found. After investigation, one value of *d* was discarded, being affected with a large error, the cause which could not be determined. This and five other intervals were recompared, the mean of the original and newly determined values of *d* being taken in the other five cases.

Another check is given by the relation:

$$\Sigma D' = \Sigma \Delta'.$$

Similar checks were applied in the determination of corrections for the lines at intervals of 10 mm. from line 35 to line 95 and from line 265 to line 295, and in the investigation of the intervening lines in each of the 5 mm. intervals.

Several different tests have been applied with a view to determining the extent to which the final results can be trusted, and all point to the conclusion previously stated that they are reliable to within three or four ten-thousandths of a mm. in the case of scale 10 and only slightly less reliable in the case of scale 11. The values of D at the foot of Table I, page 139, are the arithmetic means of thirteen quantities each. Inasmuch as the thirteen values of each column result from a comparison of one of the intervals of scale 10 with thirteen different intervals of scale 11 it would not do to regard the differences between one of the D's and the thirteen quantities of which it is the average as the residuals from which to obtain a probable error of the value of D. However, the differences between the quantities in the body of Table I and the corresponding quantities of Table II may be

regarded as residuals due to errors of observation. This would be rigorously true if the values of the D's and the Δ's used in building up Table II were absolutely correct. They are undoubtedly correct to within a few ten-thousandths of a mm., so that the errors with which they are affected are negligible as compared with the error of a single comparison. An estimate of the probable error of one of the D's may therefore be obtained by applying the formula:

$$p.\ e. = \tfrac{2}{3} \sqrt{\frac{\Sigma\, v\, v}{n\,(n-1)}}$$

v representing the difference between corresponding values in Tables I and II. This yields for the following intervals taken at random:

Interval	$p.\ e.$
230–240	0.84 ten-thousandths of a mm.
260–270	0.55
290–300	1.05

The probable error of one of the D's is therefore about eight-tenths of a ten-thousandth of a mm.

The probable error of $\Sigma\, D'_i$ would be $\sqrt{13 \times (0.00008)^2}$ mm. = .0003 mm. $\Sigma\, D'_i$ is the difference in length of the two scales, which can be directly observed. A comparison with the directly observed value yields the following:

Computed value $= \Sigma\, D'_i$	$=$	+0.0034 mm. (scale 11 longer).
Directly observed	$=$	0.0036 mm. (scale 11 longer).

Thus the actual difference between the computed and observed values is somewhat less than the probable error of the computed value.

The values of the corrections are obtained in such a way that the accumulated error is zero at the extremities of the interval. The probable errors are therefore largest for the lines near the center of the interval and are symmetrical on both sides of the center. The largest actual errors are therefore to be expected in the neighborhood of lines 65 and 265. I have made an independent determination of the corrections for lines 65 and 265 by comparing one half of each scale with each half of the other scale. The mean of two separate determinations yields the following results:

	Correction to line.	
	65	265
Independent determination	—.0190	+.0193 mm.
Tabulated value	—.0188	+.0195

In the investigation of the lines at intervals of 1 mm. in groups of five, only four settings were made on each line instead of eight, as in the case of the lines at intervals of 10 mm. the corrections for which form the basis of the subsequent work. I have found values of the probable errors of the D's used in determining

corrections for lines in the group 45–50 by the method employed in finding the probable errors of the *D*'s in the investigation of the lines at 10 mm. intervals. They are as follows:

	D_1	D_2	D_3	D_4	D_5	
p. e.	0.7	0.3	0.8	0.6	0.7	unit = 0.0001 mm.,

the average being 0.6. The probable accumulated error in any 5 mm. interval is therefore less than one ten-thousandth of a mm.

As a final and most searching test of the order of accuracy of the tabulated corrections, I have made an essentially independent determination of the correction for certain lines in the center of each scale. Corrections were first determined for lines at intervals of 26 mm., *i. e.*, one-fifth of the scale length, using the method of comparing each 26 mm. interval of one scale with each 26 mm. interval of the other scale as described above. In order to guard against accumulated error four independent determinations were made and the averages taken. The results are as follows:

SCALE 10.

Line.	26	52	78	104
	336.3	206.9	153.1	80.9
	333.6	206 7	151.9	79.0
	332.9	212.7	151.8	83.9
	330.8	211.1	150.0	83.8
Ave.	333.4	209.4	151.7	81.9

SCALE 11.

Line.	226	252	278	104
	81.3	145.7	210.5	343.6
	79.3	145.8	214 5	342.8
	82.2	148.4	211.3	340.6
	78.8	141.2	209.2	341.7
Ave.	80.4	145.3	211.4	342.2

Corrections for the lines 77 and 277 were derived from the corrections for lines 78 and 278 by comparing the interval 77–78 with each one mm. interval from line 230 to line 235 of scale 11, and the interval 277–8 with each 1 mm. interval between lines 30 and 35 of scale 10. In thus determining the amounts by which the intervals 77–78 and 277–8 are larger or smaller than one one-hundred-and-thirtieth of their respective scales, use is made of the previously determined value of the amount by which the 5 mm. intervals used in the comparison differ from one twenty-sixth of the length of their respective scales. Inasmuch, however, as any error of the previous determination (probably not exceeding three ten-thousandths of a mm.) enters into the present determination divided by five, this circumstance does not vitiate the essential independence of this test, of the previous determinations of scale corrections. The resulting values of the corrections for lines 77 and 277 are:

Correction for line 77 = — 145.
Correction for line 277 = + 217.

Corrections were next determined by the ordinary method, described on page 140, for lines at 5 mm. intervals from 52 to 77 and from 252 to 277, basing the determi-

nation on the values given by this independent investigation of the corrections for lines 52, 77, 252, 277. The comparison of the tabulated values of scale corrections with the values obtained by this independent determination follows.

SCALE 10.

Comparison of tabulated with independently determined corrections.

Line.	52	57	62	67	72	77	78
Tabulated . .	-208	-194	-197	-202	-172	-146	-151
Independent .	-209	-196	-196	-205	-168	-145	-152
T - I	+1	+2	-1	+3	-4	-1	+1

SCALE 11.

Line.	252	257	262	267	272	277	278
Tabulated . .	+142	+177	+172	+200	+214	+210	+205
Independent .	+145	+181	+180	+207	+222	+217	+211
T - I	-3	-4	-8	-7	-8	-7	-6

The unit is one ten-thousandth of a scale division.

In the case of scale 10, the discrepancies do not exceed four ten-thousandths of a mm., of which a part may reasonably be supposed to be due to error in each of the two values. A constant correction of six ten-thousandths of a mm. to the tabulated values of the corrections for scale 11 would make the agreement better than it is in the case of scale 10. This accumulated error in the tabulated values of the corrections for scale 11, amounting in the maximum to about six ten-thousandths of a mm., being constant throughout an interval of 26 mm., would not affect the measurement of distances shorter than 26 mm. if the center of the scale were used. Further investigation might be made to determine the amount of this error affecting the tabulated corrections for lines outside the interval 252–278 and corrections applied to the tabulated quantities. This does not seem worth while, however, in view of the fact that the errors of the present tabulated values are entirely negligible in most investigations in which the measuring apparatus is to be employed, and measurements requiring the highest degree of accuracy are made dependent upon scale 10 alone, which is to be used as the X scale. In fact, the investigation of the Y scale would have been omitted altogether were it not for the fact that to include it in the investigation required no more observations and only a little more numerical work in reduction than would have been required for the investigation of the X scale alone.

This final test therefore confirms the indications of probable error given by various tests mentioned at the beginning of this section: Namely, that the tabulated corrections for scale 10 are probably reliable to within three or four ten-thousandths of a mm.

At various times during this investigation, the scales were carefully cleaned, notably before the final test just described.

The accumulation of particles of dust about any scale line might cause an error in the determination of the correction for that line but would not produce such a cumulative error as seems to affect the tabulated corrections for scale 11.

Comparisons of the total lengths of the two scales at various times while the scales were under investigation aroused a suspicion that changes had occurred in the length of one or both, of the order of magnitude of the changes in length of the pendula used in determinations of the force of gravity. If such a change were due to a uniform expansion or contraction of the metal throughout its entire length, the values of the tabulated corrections would be unaffected, since each interval would be changed in the same proportion as the whole scale which is the unit. (See definition of "Correction for Scale Line," p. 138.) If, however, the change in the length of the scale were due to a local expansion or contraction of the metal, the effect would be to produce an apparent accumulated error such as appears in the case of scale 11. It should be remarked in this connection that the independent determination of the correction for line 265, mentioned on page 147, was made about July 1, 1904, soon after the determination of corrections for lines at intervals of 10 mm. throughout the scales. This measure indicates that the tabulated correction was at that time certainly not too small. The final check which seems to indicate that the tabulated values of the corrections for lines near the middle of scale 11 are all too small was made about a year later, and it is during this time that the relative length of the two scales may have changed.

Temperature changes, if not too rapid, do not affect this determination, the two scales being made of the same material and having the same coefficient of expansion.

IRREGULARITIES OF THE MICROMETER USED IN MEASURING THE *X* COÖRDINATE. The method employed in this investigation is that described in BRÜNNOW'S *Sphärsche Astronomie* (p. 426 vierte Auflage, Part 7, Par. 5). The two scales were placed side by side so that division line 50 of scale 10 was about one-tenth mm. ($=\frac{1}{5}$ rev.) distant from line 276 of scale 11. This interval was then measured with each succeeding portion of the screw corresponding to one-tenth mm. from 10.0 to 13.0, *i. e.*, through three revolutions. The measures were then repeated in the reverse order. The means of the two measures of the interval made with the same portion of the screw were taken, there being thirty

such values. Brünnow designates these means by the letters a', a'', . . . a^n and sets.

$$a = \frac{a' + a'' + \ . \ . \ . \ a^n}{n}$$

The corrections which must be applied to the readings of the micrometer head are then

Reading	Correction
10.0	0
10.1	$a - a'$
10.2	$2a - a' - a''$
10.3	$3a - a' - a'' - a'''$
.	.
.	.
.	.
13.0	$30a - a' - a'' \ . \ . \ . - a^n = 0$

Eight series of observations were made and combined by averaging the resulting corrections into two sets of four each. These values of the corrections were plotted on coördinate paper and two curves constructed through the plotted points. This was done to determine whether the smaller waves that characterize the curves are real or accidental and therefore to be disregarded in making up a

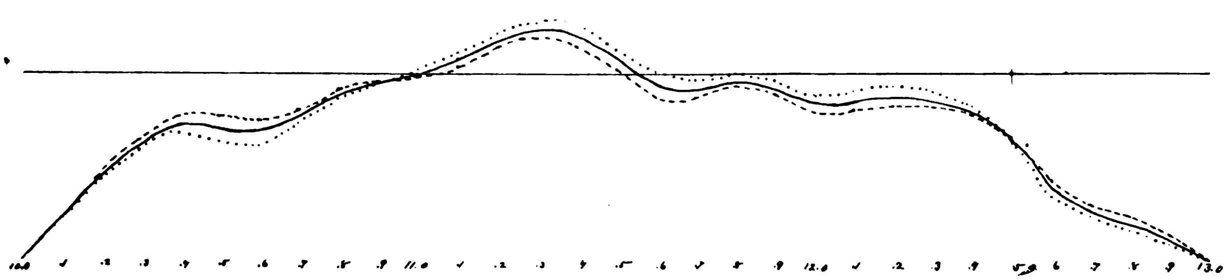

FIG. 2.

table of corrections. The two curves were found to agree surprisingly well, the waves corresponding without exception. See Figure 2. A third curve was drawn representing the mean of the eight determinations and a table of corrections read off from this curve. It is proposed to make measurements of star plates in such a way that the reading on the scale line shall lie about 11.0. The table of corrections has been so constructed, therefore, that the correction is zero for the micrometer reading 11.0. Below are given the observed corrections and the table constructed from them with the help of the plotted curve. The unit is one ten-thousandth mm.

Unit = 1 ten-thousandth mm.

Mic Reading.	Observed Cor. Ave 1st 4	Observed Cor. Ave. 2d 4	Table of Cor
10.0	0	0	− 23
.1			− 18
.2	+ 11	+ 14	− 14
.3			− 10
.4	+ 16	+ 18	− 6
.5			− 7
.6	+ 14	+ 18	− 8
.7			− 6
.8	+ 21	+ 22	− 2
.9			− 1
11.0	+ 29	+ 23	0
.1			+ 2
.2			+ 4
.3	+ 29	+ 28	+ 6
.4	+ 30	+ 26	+ 5
.5			+ 2

Mic. Reading.	Observed Cor. Ave. 1st 4	Observed Cor. Ave. 2d 4	Table of Cor.
11.5			+ 2
.6	+ 23	+ 20	− 2
.7			− 2
.8	+ 23	+ 22	− 1
.9			− 2
12.0	+ 21	+ 18	− 4
.1			− 4
.2	+ 22	+ 20	− 3
.3			− 3
.4	+ 19	+ 18	− 4
.5			− 8
.6	+ 7	+ 9	− 14
.7	+ 4	+ 5	− 17
.8			− 19
.9			21
18.0	0	0	− 24

STRAIGHTNESS OF THE GUIDING BAR OF THE MOVABLE MICROSCOPE. The high power microscope (used ordinarily in measuring the Y coördinate) which gives an enlargement of about 38 diameters was placed in the movable microscope carriage, and the X scale, designated as N. E. K. 10 was placed on its stage below. Microscope and scale were so adjusted that the double hairs of the micrometer were parallel to the direction of the line which runs the entire length of the scale, cutting the millimeter division lines at right angles, and this line was nearly parallel with the direction of the bar (carrying the microscope) whose straightness was to be tested. Settings were made upon the long scale line at its points of intersection with the millimeter division lines at intervals of 13 mm. throughout its entire length, the scale being in the position "numbers down" as seen through the microscope. The scale was then turned end for end so as to be in the position "numbers up" and the settings were repeated. The table below gives in columns two and three the means of the readings. The number of settings combined in each mean varies from eight to twelve. From each of these readings I have subtracted the reading taken at the division 130 and then applied a further correction proportional to the distance of the division where the setting was made from division 130, to remove the effect of the inclination of the line to the direction of the bar. The readings so corrected are given in columns four and five. Column six is the sum of these two columns. The values of column six are thus freed from the effect of symmetrical curvature of the line upon which the settings were made. The unit throughout is the ten-thousandth of a millimeter. The values of column six show no regular variation

such as would be due to a curvature of the bar under investigation and they may be regarded as due to accidental errors of the settings.

Results of investigation of straightness of bar carrying X microscope.

Div.	Settings. Nos. Down.	Settings. Nos. Up.	Cor. for Incl. Nos. Down.	Cor. for Incl. Nos. Up.	Sum.
0	530	582	+ 6	− 10	− 4
13	510	596	− 11	+ 12	+ 1
26	497	591	− 18	+ 15	− 3
39	506	569	− 9	0	− 9
52	517	575	+ 6	+ 14	+ 20
65	522	537	+ 15	− 16	− 1
78	500	540	− 4	− 6	− 10
91	506	530	+ 5	− 8	− 3
104	505	532	+ 7	+ 2	+ 9
117	498	520	+ 4	− 3	+ 1
130	491	515	0	0	0

The unit is .0001 mm.

IRREGULARITIES OR CURVATURE OF CYLINDER WHICH GUIDES THE PLATE CARRIAGE IN ITS MOTION IN THE DIRECTION OF THE Y COÖRDINATE. The cylinder is of steel and has evidently been ground into its present shape. It is 34 cm. long and 3.3 cm. in diameter, and therefore very rigid. Slight local irregularities such as dents or kinks of any considerable size are not to be expected. The plate carriage bears upon the cylinder by means of two Y supports 21 cm. apart and each in contact with the cylinder along two lines for a distance of 1 cm. The effect of scratches or very slight local irregularities upon the motion of the plate carriage would be reduced by the contact of the supports along these lines.

A defect much more to be feared is curvature of the bar, by which its middle portion is displaced laterally with respect to the two ends. This might arise if the bar were in a state of pressure, due either to faulty construction or unequal temperature expansion. Let us suppose that the axis of the cylinder is an arc, the projection of which on the plane of the plate is the circular ark ab (See Figure 3) with center at C. Let c and d represent the two points of the cylinder at which the two supports of the plate carriage are in contact with it. As the plate carriage moves along the guiding cylinder in the general direction of the Y axis, the line cd remaining fixed as regards the plate, every point of the plate instead of suffering a rectilinear displacement, as it should if the cylinder were straight, moves in an arc of a circle whose center is at C. The points that fall successively under the cross lines of the microscope as the plate carriage is moved in the Y direction lie on such a circular arc, and all measured X coördinates are measured along radii of this arc. (See definition of coördinate system, next paragraph.)

Let the origin of a rectangular coördinate system on the plate be some arbitrarily chosen point which remains approximately fixed when the plate is rotated, and let the Y axis be a line tangent at this point to an arc of the circle with radius R and center at C. It is assumed that the axis determined by the direction of motion of the X microscope is perpendicular to this line. Inasmuch as only small values of X are to be measured in this investigation, this condition need be only approxi-

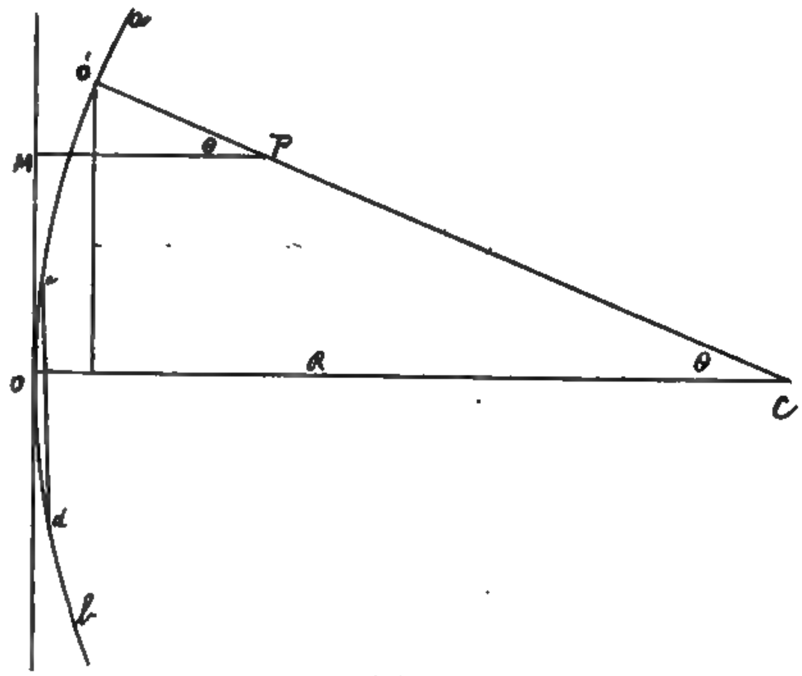

FIG. 3.

mately fulfilled. Let P represent any point on the plate, and aOb the locus of points of the plate which fall successively under the cross-hairs of the microscope when the plate is moved in the Y direction. Then if:

$MP = \xi$, $OM = \eta$

$O'P = X =$ measured coördinate

$\theta =$ angle OCP

$OC = R$ (regarded as negative if C lies to the left of O)

we have

$$\xi = X \cos\theta + R \operatorname{vers}\theta$$

(ξ and X being positive or negative respectively according as P lies to the right or left of the lines OM and OO').

If the plate be rotated through $180° + \varphi$ and the new coördinates of P be denoted by the subscript π

$$\begin{aligned}\xi_\pi &= -\xi + a\,\xi + b\,\eta \\ &= X_\pi \cos\theta_\pi + R_\pi \operatorname{vers}\theta_\pi\end{aligned}$$

we have, approximately:

$$\mathrm{R} = \mathrm{R}_\pi,\ \theta = -\theta_\pi = \frac{\eta}{R} \text{ (since } OM = OM_\pi = OO' \text{ approximately)}$$

$$\xi + \xi_\pi = X \cos\theta + X_\pi \cos\theta + 2R \operatorname{vers}\theta$$

$$= X + X_\pi - \tfrac{1}{2}(X + X_\pi - 2\mathrm{R})\left(\frac{\eta}{R}\right)^2$$

$$\xi - \xi_\pi = X - X_\pi - \tfrac{1}{2}(X - X_\pi)\left(\frac{\eta}{R}\right)^2$$

$$X + X_\pi = a\,\xi + b\,\eta + \tfrac{1}{2}(X + X_\pi - 2R)\left(\frac{\eta}{R}\right)^2$$

$$X - X_\pi - 2\,\xi = -a\,\xi - b\,\eta + \tfrac{1}{2}(X - X_\pi)\left(\frac{\eta}{R}\right)^2$$

a and b are small if φ is made small, and if we choose points for which ξ is small, the term $a\,\xi$ may be dropped. R will amount to some hundreds of meters. For points for which ξ is small, we write, therefore:

$$X + X_\pi = b\,\eta - \frac{1}{R}\,\eta^2$$

and determine R with the help of this equation by observing the X coördinates of points near the η axis at the upper and lower edges of the plate. The quantity

$$X - X_\pi - 2\,\xi = -a\,\xi - b\,\eta + \tfrac{1}{2}(X - X_\pi)\left(\frac{\eta}{R}\right)^2$$

represents the error in the assumption ordinarily made in reduction that

$$\frac{X - X_\pi}{2} = \xi$$

and that part of the error of the ξ coördinate which is due to curvature of the cylinder is:

$$\tfrac{1}{4}(X - X_\pi)\left(\frac{\eta}{R}\right)^2$$

The investigation for the determination of curvature was made as follows: With a needle, well defined marks were made in the gelatine of a photographic plate near the η axis and at the top, middle, and bottom of the plate, and at two intermediate stations. The X coördinates of nine or fifteen such points in groups of three were accurately measured in two positions of the plate, differing in orientation by 180°. The Y coördinates were read off to the nearest tenth millimeter. The measures of the group of three points at the middle of the plate were used to determine the scale reading of the zero of the X coördinates, and the groups at the top and bottom of the plate yielded the absolute terms of the two equations of conditions necessary to determine b and $1/R$ in the above

expression for $X + X_\pi$. Four determinations of curvature were made at widely different temperatures. The results are:

Thermometer—F.	R	
52	937 m	Center to right
59	555 m	Center to right
62	$>$ 1,000 m	Center to left
80	$>$ 1,000 m	Center to left.

The last two determinations are probably much more trustworthy than the other two, greater care having been taken to secure sharply defined marks for the pointings.

Substituting $R = 500$ m. in the above expression the error in the ξ coördinate due to such curvature of the cylinder is found to be altogether negligible.

PERPENDICULARITY OF THE TWO DIRECTIONS OF MOTION. If observations are to be made according to the second method as described on page 136, the four constant method of reduction being used, it is essential that the directions of motions which define the X and Y axes be perpendicular to each other. The following plan was adopted for the accomplishment of this adjustment. Two crosses were scratched in the gelatine of a plate about 10 mm. apart. The plate was then placed in the machine and oriented in such a way that the two crosses fell successively under the cross-hairs of the microscope when the plate was moved in the Y direction. The plate was then turned through 90° (which angle was measured with the circle) and the arm which carries the movable microscope adjusted so that the two crosses fell successively under the cross-hairs when the microscope was moved along its track, *i. e.* in the direction of the X axis. After this adjustment had been made, its accuracy was tested by taking circle readings in the two positions of the plate. The two directions were found to be perpendicular to each other within about 20″. The maximum error introduced into any measured coördinate by the lack of perpendicularity of the directions of motion defining the two axes is, for the X coördinate: $Y \sin 20''$, or for the Y coördinate: $X \sin 20''$. The scale being 130 mm. long, the maximum value of X or Y is 65 mm., so that the greatest error due to this source would be $.0001 \times 65$ mm. $= 0.006$ mm. The use of the six constant method of reduction would eliminate errors due to this cause.

PARALLELISM OF SCALES TO PLATE. It is only necessary that the two scales make equal angles with the plate so that the scale values may be the same in both coördinates. This has been tested by measuring the coördinates of well defined points on a plate in two positions, differing by 90° in orientation, so that the same distances were compared with both scales. The result indicates that a distance of a hundred mm. when measured as an X coördinate yields a value

larger by about .0058 mm. than the value obtained when the same distance is measured as a Y coördinate. Two determinations were made, nine distances being used in each. Following are the results in the form X measure — Y measure.

	X Measure – Y Measure.	
Dist. No.	1st Deter.	2d Deter.
1	+.0080	+.0099
2	+ 90	+ 87
3	+ 96	+ 73
4	+ 80	+ 121
5	+ 90	+ 109
6	+ 96	+ 95
7	+ 51	+ 118
8	+ 59	+ 116
9	+ 65	+ 92
Mean	+ 79	+ 101

Average of two determinations .0090 mm

Applying a correction of —.0023 for difference in length of scales, and reducing to a value corresponding to 100 mm. of distance (the actual measured distance was about 116 mm. in every case) we have the value .0058 mm. given above.

DISTORTION OF THE FIELD OF THE OBJECT GLASS OF THE X MICROSCOPE. The micrometer is to be used only in the measurement of distances less than 1 mm. Consequently, little is to be feared from distortion of the field of the objective. A short investigation of the matter, however, has been made. The two scales were placed side by side on the stage and so that the division lines of the one scale about bisected the intervals of the other scale. The interval between line 278 of scale 11 and line 14 of scale 10 was then measured with different portions of the micrometer screw, thus utilizing different portions of the field of the objective along the line of a diameter of the field. The results are given below and indicate that no serious distortion exists but that the pitch of the screw varies, which indication is borne out by the results of the more extended investigation, pages 150–152.

r = distance in mm. from part of field corresponding to 11 revolutions;
D = measured distance;
v is expressed in ten-thousandths of a mm.

r	D	v
— 2.0	0.5001 mm.	— 7
— 1.5	6	— 2
— 1.0	6	— 2
— .5	0	— 8
.0	12	+ 4
+ .5	11	+ 3
+ 1.0	8	0
+ 1.5	18	+ 10
+ 2.0	14	+ 6
Mean . . .	0.5008	

UNIVERSITY OF CALIFORNIA PUBLICATIONS.

PUBLICATIONS OF THE LICK OBSERVATORY, VOL. VII, PART 6.

ON ASTRONOMICAL REFRACTION.

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PREFACE.

That the tables for computing refractions need modification has long been a recognized fact. It has been evident from observations made at the Lick Observatory that, not only are Bessel's refractions too large, but that the Pulkowa refractions are also. Consequently, in March, 1899, Dr. KEELER, late Director of the Lick Observatory, suggested that I should undertake to reduce a series of observations that had been made for refraction by Professor SCHAEBERLE, to discuss them completely, and to construct therefrom refraction tables for the Lick Observatory. This was to be done for a dissertation in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of California.

After consultation with him, and with his advice and consent, it was decided to forego the reduction of Professor SCHAEBERLE'S observations and to investigate the Constant of Refraction from observations to be made by myself, and to be reduced according to the method hereinafter set forth.

RUSSELL TRACY CRAWFORD.

Berkeley Astronomical Department,
March, 1907.

CONTENTS.

INTRODUCTION.

1. The observations for this investigation were made with the 6.4-inch Meridian Circle of the Lick Observatory. A detailed account of this instrument and the room in which it is housed has been given by Astronomer TUCKER in Volume IV of these *Publications*, 1900.

Two independent sets were made under different meteorological conditions; one during the summer months June and July, the other during the autumn months October and November. The summer series comprises observations of 31 stars on seventeen nights, the autumn 43 stars on ten nights. Other stars were also observed for the purpose of determining bisection error. The detailed reduction of the summer series has already been published as No. 8, Vol. I, Third Series "*Proceedings of the California Academy of Sciences.*" This will not be duplicated, but the reduction of the autumn series is here given in full. The final solution is based upon both series.

2. *Meteorological Observations and Conditions.* To make quite sure of the condition of the atmosphere at any time during the observations, the thermometers were read, on the average, three times an hour (at nearly equal intervals); and the barometer was observed every hour. The reading of the wet-bulb thermometer was also taken when the dry was read. The relative humidity has not been introduced into the reductions, but it was thought desirable to have it for possible future reductions.

The barometer, Green 2839, hangs on the north wall of the observing room. The least reading of its vernier is one two-hundredths of an inch. The dry and the wet-bulb thermometers (F.) hang in the air-space between the north walls. The dry-bulb thermometer, used to indicate the external temperature, is Green 494. This thermometer has been calibrated at the Yale Observatory. The corrections which have been applied to all the readings have been taken from the following table sent from Yale Observatory:

t (F.)	Cor.
°	°
0	+ 0.1
32	− 0.2
52	− 0.1
72	− 0.2
112	− 0.1

The table which follows contains the *uncorrected* temperatures (t), the readings of the attached thermometer (T), of the barometer (B), and the times at which they were taken. The readings of the wet-bulb thermometer are not given. The unit of B is one two-hundredths of an inch.

TABLE OF BAROMETER AND THERMOMETER READINGS.

Sid. T	October 17.			October 24.			October 26.			October 31.			November 1.		
h	t	T	B	t	T	B	t	T	B	t	T	B	t	T	B
	°	°		°	°		°	°		°	°		°	°	
23.9	53.6	53	5155	45.6	45	5187	50.8	51½	5194	44.8	45	5185	51.5	53	5181
0.3	53.0			46.0			50.9			45.8			51.6		
0.7	53.5			46.6			50.9			46.3			51.9		
1.0	53.0	53	5153	46.3	46	5186	50.2	51	5192	46.0	45	5184	51.7	53	5181
1.4	52.8			46.5			50.2			45.8			51.6		
1.9	52.4	53	5153	46.1	46	5187	50.7	51	5190	46.0	45	5183	52.0	53	5181
2.3	51.9			46.1			50.8			46.5			52.0		
2.7	52.0			45.9			50.2			50.8			51.6		
3.0	52.9	52	5151	46.0	46	5187	50.0	51	5189	51.7	47	5177	51.5	53	5180
3.5	53.2			46.1			49.8			52.8			51.6		
3.8	53.0			45.9			49.0			50.9	48½	5185	51.0		
4.3	53.0			45.4			49.5						50.5		
4.6	53.0	53	5148	45.0	46	5186	49.9	50½	5185				50.0	52	5180

Sid. T	November 6.			November 22.			November 23.			December 1.			December 6.		
h	t	T	B	t	T	B	t	T	B	t	T	B	t	T	B
	°	°		°	°		°	°		°	°		°	°	
23.9	47.0	47	5183	40.5	41	5158	44.0	44	5172	56.2	54	5186	49.0	46½	5167
0.3	47.1			41.1			44.2			55.4			49.9		
0.7	47.1			41.5			44.9			56.0			49.9		
1.0	47.4	47½	5182	41.5	41	5160	45.0	45	5175	55.6	54½	5185	49.8	47½	5167
1.4	48.2			41.2			45.3			55.5			50.0		
1.9	48.0	48½	5183	42.0	40½	5160	45.5	45½	5178	56.3	54½	5185	50.8	48	5166
2.3	47.8			42.0			45.8			55.8			50.8		
2.7	48.0			41.8			46.5			56.0			50.4		
3.0	48.5	48½	5183	41.2	40½	5164	45.4	46	5178	55.9	54½	5185	50.2	48½	5161
3.5	48.9			41.0			47.0			55.6			50.0		
3.8	48.9			41.6			47.5			55.3			49.5		
4.3	48.8			42.3			47.0			55.0			49.3		
4.6	48.5	49	5185	42.8	41	5163	47.0	47½	5178	54.7	54½	5184	49.1	50	5159

The maximum temperature of the two series was 74°.0, July 3; the minimum was 40°.5, November 22, giving a range of 33°.5. The total range of the barometer for the two series was one-quarter of an inch.

Besides the regular thermometers in the air-space between the north walls, three other thermometers were suspended from the ceiling of the observing room. All three were swung under the observing slit, near the plane of the meridian. One was directly over the instrument, and three or four feet from the ceiling. The other two were hung, one north and one south, about halfway between the instrument and the north and south walls respectively, and at such a distance above the floor that the plane of the axis of the instrument and the line of sight

of the telescope, pointed at about 83° zenith distance (north and south respectively), would intersect the thermometers near their bulbs.

Before being thus placed, these thermometers were compared with Green 494, so that their readings could be reduced for comparison with those of the external thermometer (Green 494).

During the course of an evening's observations these three thermometers were read just after reading the regular thermometer. The average difference between the inside and the outside thermometers was found to be the same for all three, and is 0°.3 (F.) for the summer series, and even less for the autumn. It is nearly always the case (in this hemisphere) that the southern part of a room is a trifle warmer than the northern. But this is not the case on Mount Hamilton. The temperature of the air inside is, on the average, very uniform and but very little (0°.3) warmer than the air outside. In his "*Untersuchung über die Astronomische Refraction u. s. w.*," Dr. BAUSCHINGER notes that the southern part of his observing room in Munich was warmer than the northern, and that at night the average difference between the inside and the outside temperatures is 1°.3 (C.). From his investigations, he concludes that the temperature of the air *within* the observing room should be taken into account.

Because of this difficulty, many observers have seriously considered the idea of mounting their instruments under a movable house, so that when at work the instrument may be entirely out of doors. But this would needlessly endanger the instrument. To accomplish the same purpose, the Meridian Circle house built at Kiel is constructed in the shape of a cylinder whose axis coincides with the axis of the instrument. This is undoubtedly the best form of construction.

For the efficiency of the Meridian Circle house on Mount Hamilton, the difference between the inside and the outside thermometers can speak. As has been noted, the average difference (in the sense Inside – Outside) is + 0°.3 (F.) for the summer series, and even less for the autumn. For the first series the *maximum* difference observed is + 1°.1 (F.). It may be noted further that this observation was made a few minutes before the sun had set. *This maximum is less than half the average at Munich.* For the other series the maximum difference is + 2°.1 (F.), which is less than the average at Munich. There was one still larger difference, viz., — 3°.7 (F.), which may be discarded for it occurred on a poor night, immediately after observing had been stopped because of clouds and poor "seeing." The hot wave, which caused the outside temperature to rise suddenly, undoubtedly destroyed the "seeing." It is safe to assume, therefore, that after the observing room has been completely opened for an hour and a half, the temperature inside is practically the same as it is outside.

3. *Plan for Observing.* The method of determining the refractions may be stated as being a quasi converse to TALCOTT'S method of determining latitude.

Instead of eliminating the refractions to get the latitude, the method is to determine the refractions by eliminating the latitude, as follows:

Let

z_s = the zenith distance of a southern star,
z_n = the zenith distance of a northern star,
z'_s = the apparent zenith distance of the southern star,
z'_n = the apparent zenith distance of the northern star,
δ_s = the declination of the southern star,
δ_n = the declination of the northern star,
r_s = the refraction of the southern star,
r_n = the refraction of the northern star,
φ = the latitude of the Meridian Circle.

Then

$$\delta_n = \varphi + z_n = \varphi + (z'_n + r_n) \tag{1}$$

$$\delta_s = \varphi - z_s = \varphi - (z'_s + r_s) \tag{2}$$

$$\delta_n - \delta_s = z'_s + z'_n + r_s + r_n \tag{3}$$

Let

$$A = \delta_n - \delta_s \tag{4}$$

$$B = z'_s + z'_n \tag{5}$$

Then

$$A = B + r_s + r_n \tag{6}$$

or

$$r_s + r_n = A - B \tag{7}$$

If now, the southern and northern zenith distances were the same, and if, at the times of observing them, the conditions of the atmosphere were the same, the two refractions would be the same, *i. e.*

$$r_s = r_n.$$

In this case we have

$$2r = A - B \tag{I}$$

In practice these ideal conditions are only approximately satisfied. We therefore proceed as follows:

From (7) we have

$$2r_s - r_s + r_n = A - B \tag{8}$$

whence

$$2r_s = (A - B) + (r_s - r_n)$$

and

also

$$\left.\begin{aligned} r_s &= \tfrac{1}{2}(A - B) + \tfrac{1}{2}(r_s - r_n) \\ r_n &= \tfrac{1}{2}(A - B) + \tfrac{1}{2}(r_n - r_s) \end{aligned}\right\} \tag{II}$$

In case the northern star is at lower culmination we shall have:

$$\delta_n = 180^\circ - z_n - \varphi \tag{9}$$

$$\delta_s = \varphi - z_s \tag{10}$$

$$\delta_n + \delta_s = 180^\circ - z_n - z_s \tag{11}$$

$$= 180^\circ - [z'_n + r_n + z'_s + r_s]. \tag{12}$$

Hence

$$r_n + r_s = 180^\circ - [z'_n + z'_s] - [\delta_n + \delta_s] \tag{13}$$

and

$$2r_s = 180^\circ - [z'_n + z'_s] - [\delta_n + \delta_s] + [r_s - r_n]. \tag{14}$$

Calling

$$A' = \delta_n + \delta_s \tag{15}$$

and since

$$B = z'_s + z'_n \tag{5}$$

we have

and

$$\left.\begin{aligned} r_s &= 90^\circ - \tfrac{1}{2}[A' + B] + \tfrac{1}{2}[r_s - r_n] \\ r_n &= 90^\circ - \tfrac{1}{2}[A' + B] + \tfrac{1}{2}[r_n - r_s] \end{aligned}\right\} \tag{III}$$

In order to obtain the refractions from (II) and (III) it is necessary to know the declinations of the stars, their apparent zenith distances (or rather the sums of the zenith distances of the pairs of north and south stars), and the differences between the refractions of the pairs. The stars chosen for this work are all fundamental, and in a first approximation their declinations are to be considered absolute. The list of stars, given later, has been taken from Professor NEWCOMB'S "*Catalogue of Fundamental Stars for 1875 and 1900, reduced to an absolute System.*" The apparent zenith distances, or the sums of the zenith distances of the several pairs, are obtained from the Meridian Circle observations; and the differences in the refractions are found by computing the refractions from some standard table. In this work the Pulkowa tables have been used. The term $\frac{1}{2}(r_s - r_n)$ being of the nature of a differential refraction, any error in the constant of refraction of the table used will have practically no effect upon this difference. The more nearly ideal conditions (*i. e.*, when $r_s = r_n$) are approached, of course, the better the determination of the refractions will be.

This method has both its advantages and its disadvantages. Among the former, the most important are: first, the complete elimination of the latitude and hence also of its variation; second, the elimination of the nadir, since $(z'_s + z'_n)$ is nothing more nor less than the difference between the circle readings, and is therefore independent of the zenith point; third, there is no wait of twelve hours or of six months in order to observe the same star at both culminations, as is usually done; fourth, only one half of any error in an observed zenith distance enters into the observed refraction; and fifth, the simplicity of the reductions.

The greatest disadvantage in the method lies in the fact that the declinations of the stars have to be considered known. But by taking fundamental stars, such as those whose places are given by Professor NEWCOMB'S new Fundamental Catalogue, and by taking a large number of these stars, this difficulty will be nearly completely eliminated. It should be noted that the stars observed at great northern zenith distances are observed at lower culmination, and that their positions have been determined from observations made at upper culmination when the effect of refraction is small; and that the stars observed at great southern zenith distances are those whose positions have been determined at observatories in the southern hemisphere where they culminated near the zenith and where again the effects of refraction are small.

Having now the new refractions, the correction to the constant of the table used (Pulkowa) is found from the following equation [eq. (701) pg. 672, Vol. 1, CHAUVENET, "*Spherical and Practical Astronomy*"]:

$$dr = A d\alpha + B d\beta,$$

where

$$A = \frac{r}{\alpha}$$

and

$$B = \sin^2 z \sqrt{\frac{2}{\beta}} \left(\frac{dQ}{d\beta} - \frac{Q}{2\beta}\right)$$

For the Lick Observatory, whose altitude is 4,209 feet and where the mean annual pressure is less than 26 inches, an investigation into the effect of the higher powers of $\Delta\beta$ involved in the factor $\beta = \frac{b}{B} = 1 + \frac{b-B}{B} = 1 + \frac{\Delta b}{B}$ (in BESSEL'S notation for r) was necessary. In his memoir, "*Untersuchungen über die Constitution der Atmosphäre und die Strahlenbrechung in Derselben*," St. Petersburg, 1866, GYLDEN has neglected the squares and higher powers of $\frac{\Delta b}{B}$, since for places at low altitudes $\frac{\Delta b}{B}$ is a very small quantity. This investigation was made by Professor COMSTOCK (Vol. I, "*Publications of the Lick Observatory*"). From his investigation the conclusion is drawn that "the Pulkowa Refraction Tables may be used for atmospheric pressures as low as 25 inches without taking into account the squares and higher powers of Δb, and the quantities so neglected will not be sensible at zenith distances less than 80°." The minimum reading of the barometer during these observations was 25.72 inches, so that in these reductions no modification of the factor of the refraction depending upon the barometer need be made.

This question having been disposed of, the assumption is here made that all of the error in the refractions is due to an error in the constant of refraction. This amounts to assuming the constant β to be correct or that $d\beta = 0$. The equation above then reduces to the very simple expression

$$dr = A d\alpha = \frac{r}{a} d\alpha;$$

hence

$$\frac{d\alpha}{\alpha} = \frac{dr}{r},$$

or

$$d\log\alpha = d\log r.$$

Having dlogr from the reductions, we thus have dlogα, and hence dα.

The assumption would perhaps seem somewhat risky for stars whose zenith distances are greater than 80°. But at the conclusion of the reductions, the value of dlogα deducted from such stars was found to fit in very well with those deducted from the other stars. Futhermore, down to 85° zenith distance the "seeing" was very good. In consequence of these facts it was decided to take into account all the stars observed. The zenith distances of the stars in the list for the summer series range from 21° 21′ to 89° 12′, and for the other series from 15° 46′ to 87° 49′ (apparent).

Below 85° zenith distance the quality of the "seeing" decreases quite rapidly. This can be seen from the following table of average weights. These were derived from the probable errors of the individual determinations of dlogα.

Z. D.	Av. Wt.	Z. D.	Av. Wt.
20° to 30°	2.0	70 to 80°	11.8
50 to 60	7.5	80 to 85	14.8
60 to 70	7.5	below 85	3.6

The small weight for the small zenith distance is due to the fact that in the expression for da the refraction occurs in the denominator. The small weight for the stars at zenith distances greater than 85° is, of course, due to uncertainties in observing at such low altitudes.

OBSERVATIONS.

1. List.—The following list of 43 stars was observed on ten nights, from October 17 to December 6, 1899, inclusive, and have been reduced according to the plan outlined in the preceding section.

The numbers of the stars are those of NEWCOMB'S "*Catalogue of Fundamental Stars for 1875 and 1900, reduced to an Absolute System.*"

No.	α (1900).			δ (1900).		
	h	m	s	°	′	″
5	0	4	20	−46	17	57.15
9	0	6	39	−35	41	34.42
788	12	25	17	+58	57	21.47
793	12	29	13	+70	20	22.03
38	0	36	36	−46	38	2.99
49		43	30	+7	2	27.19
58	0	53	47	−29	53	52.78
66	1	1	37	−47	15	16.03
76		8	30	+7	2	47.77
79		12	38	+3	5	16.48
83		18	52	+67	36	29.32
89		22	33	+88	46	26.61
96		30	31	+72	31	49.56
103		34	56	+67	32	14.25
108	1	40	7	+8	39	16.19
885	14	1	41	+64	51	13.52
136	2	6	38	+66	3	20.75
139		8	30	−31	11	34.97
149		20	49	+66	57	10.47
153	2	22	50	+8	0	42.99
911	14	27	44	+76	8	26.20
161	2	33	8	+21	31	44.60
174		39	22	−14	16	55.59
179	2	44	54	−32	49	33.16
944	14	51	0	+74	33	51.05
187	2	54	28	−40	42	19.13
190	2	57	33	+53	6	53.92
202	3	7	49	−29	22	52.49
210	3	15	56	−43	27	7.78
976	15	20	53	+72	11	23.33
216	3	22	37	−41	59	15.22
223		29	36	−50	43	4.76
235	3	38	27	−10	6	6.83
1002	15	45	8	+62	54	30.82
1014	15	55	25	+55	1	56.23
1019	16	0	1	+58	49	56.19
264	4	5	6	+85	17	29.06
270	4	10	41	−42	32	27.86
1034	16	13	40	+76	7	45.78
1045		20	25	+75	59	9.05
1050	16	22	38	+61	44	25.78
290	4	31	40	−30	46	1.53
297	4	37	20	−42	3	17.36

2. *Details of Observations.*—A night's program consisted in observing the above list, together with three nadirs, one before, one during, and one after the observing of the stars. As has been pointed out, the nadirs are not necessary for the refraction determinations, but were taken for the reduction of the latitude, which is a problem practically inseparable from the main one undertaken here.

No transits were observed during these observations, the whole attention being devoted to the observations for zenith distance. The telescope was set to the nearest 2′ and not disturbed until the observation had been completed. The bisection was made (with but a very few exceptions) at the central transit wire, by means of the declination micrometer. For the sake of uniformity every star was bisected but once during its transit. Because of unavoidable circumstances a few of the stars had passed the meridian before the bisection could have been made. In these cases the readings have been reduced to the meridian.

For the position of the circle four microscopes were read. Settings were made upon two scratches under every microscope. The circle microscopes were usually read after the star had been bisected. In a few cases, because of a following star culminating very soon, the microscopes were read before the bisection. In such cases the position of the circle was quickly checked after the bisection.

The correction for runs for a night was obtained from all of the microscope readings of the night. This correction has been applied to all of the observations. Its values for the several nights of observing are given in the following table:

Date.	R.	Date.	R.
October 17 . . .	+0″.10	November 6 . . .	+0″.08
24 . . .	+0 .10	22 . . .	+0 .08
26 . . .	+0 .09	23 . . .	+0 .10
31 . . .	+0 .07	December 1 . . .	+0 .11
November 1 . . .	+0 .08	6 . . .	+0 .11

These corrections were applied to the circle readings to reduce them to the mean position of the two scratches: so that for a reading of 0″ the correction is + R, for 60″ it is 0, and for 120″ it is — R.

In the few cases where the bisections were made a little late the reductions to the meridian were computed from the formula,

$$\delta = \delta' - \frac{\sin^2 \frac{1}{2} (\tau - m)}{\sin 1''} \sin 2\delta'$$

The horizontal flexure in this instrument is very small. In his work published in Volume IV of these *Publications*, Astronomer TUCKER adopts the correction 0″.1 sin Z. D., which was determined from a series of observations extending over two and a half years. In this work the mean of three determinations of

flexure being less than the probable error of a single determination, the flexure is considered zero. The mean of the values of one revolution of the declination micrometer, determined at the same time, is 48″.11. The value adopted is 48″.10.

For the computation of the preliminary refractions (called r′ in the reductions) the Pulkowa tables have been used. The reductions for the barometer, for the attached, and for the external thermometers were taken from Volume I of these *Publications*.

At the time these reductions were made the graduation errors of the 1° divisions of the fixed circle had been determined by Astronomer TUCKER. His results are given in Volume IV of these *Publications*. He says there, in part: "The probable error of a reading upon four divisions of the fixed circle due to graduation may be adopted as ±0″.15. * * * * There is some evidence of periodic character in the errors, and it may be assumed, in absence of further data, that the probable error due to errors of graduation is not diminished by reading upon two adjoining divisions under each microscope. * * * * The largest error measured is 0″.7 for the mean of four divisions." In view of this no corrections have been applied for division error.

Three nadirs were observed every night. The changes during a night were usually very small. The following table gives the means of the three determinations on the several nights:

Date.	Nadir 134° 57′	t	Date.	Nadir 134° 57′	t
October 17 . .	19″.87	53°	November 6 . .	22″.95	48°
24 . .	21 .71	46	22 . .	24 .02	42
26 . .	21 .20	50	23 . .	23 .40	46
31 . .	22 .52	47	December 1 . .	22 .46	56
November 1 . .	22 .12	51	6 . .	23 .07	50

All of the observations were taken with the fixed circle west. Had more time been available the instrument would have been reversed.

Weights, ranging from 5, the highest, to 1 (occasionally ½), the lowest, were arbitrarily assigned to all the observations. Judgment on a weight was formed from the steadiness of the image during the observation. These weights have been applied all through the reductions.

REDUCTION OF OBSERVATIONS.

1. Explanation of Reduction.—The first thing done on the reductions was to take the means of the microscope readings and to apply the micrometer corrections, giving the circle readings (called C′ in the tables following). The means of the microscopes were checked by taking the difference of every microscope reading from the mean of the four. If the sums of these differences for the

two opposite pairs of microscopes were the same, the mean was correct. The corrections for the micrometers were checked by duplicating this part of the work.

From the readings C′ the quantity B [equations (II) and (III)] is obtained. The terms A and A′ of these equations are obtained from the declinations.

The declinations have been reduced to 1899.0 by means of the data furnished in NEWCOMB'S Catalogue. The reductions to apparent places were computed by using the Besselian Star Numbers from the *American Ephemeris*. The factors a′, b′, c′ and d′ were computed from the *American Ephemeris* data. The reductions to apparent places for the first night (October 17) were computed by means of the Independent Star Numbers also. The places for the remaining nights were checked by differences. The apparent declinations are placed in the columns δ of the tables given later.

The following table exhibits the stars' approximate zenith distances and the stars with which they are grouped in the reductions for the refractions:

Star No.	Z. D. South.	Z. D. North.	Grouped with Star No.	Star No.	Z. D. South.	Z. D. North.	Grouped with Star No.
5 . . .	83° 31′	. . .	788, 1019	944 . . .	. . .	68° 4′	58, 139, 179, 202, 290
9 . . .	72 59	. . .	793	187 . . .	77 59	. . .	885
788 . . .	. . .	83 36	5, 38, 66	190 . . .	. . .	15 46	161
793 . . .	. . .	72 17	9, 179	202 . . .	66 41	. . .	911, 944, 1034, 1045
38 . . .	83 51	. . .	788, 1019	210 . . .	80 43	. . .	1002, 1050
49 . . .	30 17	. . .	83, 103	976 . . .	. . .	70 26	139, 179
58 . . .	67 12	. . .	911, 944	216 . . .	79 15	. . .	1002, 1050
66 . . .	84 28	. . .	788, 1019	223 . . .	87 49	. . .	1014
76 . . .	30 17	. . .	83, 103	235 . . .	47 25	. . .	264
79 . . .	34 14	. . .	96	1002 . . .	. . .	79 40	210, 216, 270, 297
83 . . .	. . .	30 16	49, 76	1014 . . .	. . .	87 24	223
89 . . .	. . .	51 25	174	1019 . . .	. . .	83 43	5, 38, 66
96 . . .	. . .	35 11	79	264 . . .	. . .	47 56	235
103 . . .	. . .	30 11	49, 76	270 . . .	79 48	. . .	1002, 1050
108 . . .	28 41	. . .	136, 149	1034 . . .	. . .	66 30	202, 290
885 . . .	. . .	77 45	187	1045 . . .	. . .	66 38	202, 290
136 . . .	. . .	28 43	108, 153	1050 . . .	. . .	80 50	210, 216, 270, 297
139 . . .	68 30	. . .	911, 944, 976	290 . . .	68 4	. . .	944, 1034, 1045
149 . . .	. . .	29 36	108, 153	297 . . .	79 19	. . .	1002, 1050
153 . . .	29 20	. . .	136, 149				
911 . . .	. . .	66 29	58, 139, 202				
161 . . .	15 48	. . .	190				
174 . . .	51 36	. . .	89				
179 . . .	70 8	. . .	793, 944, 976				

The following tables show the reductions for the new refractions. The column p contains the means of the weights assigned to the stars. The other columns have already been explained. In the grouping of the pairs on the several dates the southern star is written first and the northern star below it. The numbers of the stars given at the top of each table are arranged in the same order. The pairs which have their northern stars at upper culmination are placed in the first few tables. It will be noticed that the headings for these are slightly different from those containing the lower culmination stars.

Because of very bad "seeing," or of other circumstances, some of the stars were not observed on some nights. In such cases blanks appear after the dates. No observations have been rejected.

Stars No. { 49, 83 }

2. *Computation of Refractions.*

Date.	δ	A B	C′	r′	$\frac{r'_s - r'_n}{2}$	$\frac{A - B}{2}$	r	log r	p
1899.									
Oct. 17	+7° 2′ 37″.32	60° 33′ 58″.91	345° 14′ 38″.89	0′ 28″.95	0″.00	0′ 29″.27	0′ 29″.27	1.46642	4
	+67 36 36.23	60 33 0.37	284 41 38.52	0 28.95	. .	. .	0 29.27	1.46642	4
24	37.45	34 1.10	39.64	29.55	+0.01	0 30.09	0 30.10	1.47857	1½
	38.55	33 0.91	38.73	29.53	. .	. .	0 30.08	1.47828	1½
26	37.50	1.76	39.68	29.33	0.00	0 29.37	0 29.37	1.46790	4
	39.26	3.02	36.66	29.32	. .	. .	0 29.37	1.46790	4
31	37.37	3.35	41.00	29.56	0.00	0 29.45	0 29.45	1.46909	2½
	40.72	4.45	36.55	29.55	. .	. .	0 29.45	1.46909	2½
Nov. 1	37.34	3.65	40.98	29.19	0.00	0 29.07	0 29.07	1.46345	3½
	40.99	5.51	35.47	29.18	. .	. .	0 29.07	1.46345	3½
6	37.34	5.20	41.26	29.49	+0.03	0 29.52	0 29.56	1.47070	4½
	42.54	6.15	35.11	29.42	. .	. .	0 29.49	1.46967	4½
22	37.01	10.02	42.90	29.72	+0.01	0 29.86	0 29.87	1.47524	2½
	47.03	10.29	32.61	29.70	. .	. .	0 29.85	1.47494	2½
23	36.97	10.31	42.04	29.58	+0.01	0 29.76	0 29.78	1.47392	2
	47.28	10.79	31.25	29.55	. .	. .	0 29.75	1.47349	2
Dec. 1	36.45	11.46	41.93	28.98	0.00	0 28.91	0 28.91	1.46105	3
	48.91	13.64	28.29	28.98	. .	. .	0 28.91	1.46105	3
Dec. 6	+7 2 36.35	60 34 13.72	345 14 42.11	0 29.25	+0.01	0 29.53	0 29.55	1.47056	3
	+67 36 50.07	60 33 14.65	284 41 27.46	0 29.22	. .	. .	0 29.52	1.47012	3

Stars No. { 49, 103 }

Date.	δ	A B	C′	r′	½ (r′ₛ − r′ₙ)	½ (A − B)	r	log r	p
1899.									
Oct. 17	+7° 2′ 37″.32	60° 29′ 42″.53	345° 14′ 38″.89	0′ 28″.95	+0″.03	0′ 29″.39	0′ 29″.42	1.46864	4
	+67 32 19.85	60 28 43.75	284 45 55.14	0 28.88	. .	. .	0 29.36	1.46776	4
24	37.45	44.70	39.64	29.55	+0.05	0 30.07	0 30.12	1.47885	1½
	22.15	44.56	55.08	29.45	. .	. .	0 30.02	1.47741	1½
26	37.50	45.35	39.68	29.33	+0.04	0 29.49	0 29.53	1.47026	4
	22.85	46.37	53.31	29.24	. .	. .	0 29.45	1.46909	4
31	37.37	46.95	41.00	29.56	+0.04	0 29.73	0 29.77	1.47378	2½
	24.32	47.49	53.51	29.47	. .	. .	0 29.69	1.47261	2½
Nov. 1	37.34	47.25	40.98	29.19	+0.04	0 29.19	0 29.23	1.46583	3½
	24.59	48.87	52.11	29.10	. .	. .	0 29.15	1.46464	3½
6	37.34	48.82	41.26	29.49	+0.08	0 29.71	0 29.79	1.47407	4½
	26.16	49.39	51.87	29.32	. .	. .	0 29.63	1.47173	4½
22	37.01	53.74	42.90	29.72	+0.05	0 29.79	0 29.84	1.47480	3
	30.75	54.15	48.75	29.61	. .	. .	0 29.74	1.47334	3
23	36.97	54.05	42.04	29.58	+0.05	0 29.91	0 29.96	1.47654	2
	31.02	54.22	47.82	29.47	. .	. .	0 29.86	1.47509	2
Dec. 1	36.45	56.30	41.93	28.98	+0.05	0 29.52	0 29.57	1.47085	3
	32.75	57.26	44.67	28.88	. .	. .	0 29.47	1.46938	3
Dec. 6	+7 2 36.35	60 29 57.62	345 14 42.11	0 29.25	+0.07	0 29.67	0 29.74	1.47334	3
	+67 32 33.97	60 28 58.28	284 45 43.83	0 29.11	. .	. .	0 29.60	1.47129	3

Stars No. $\begin{cases} 76 \\ 83 \end{cases}$

Date.	δ	A B	C′	r′	$\frac{r'_s - r'_n}{2}$	$\frac{A-B}{2}$	r	log r	p
1899.									
Oct. 17	+ 7° 2′ 57″.52	60° 33′ 38″.71	345° 14′ 18″.78	0′ 28″.97	+0″.01	0′ 29″.22	0′ 29″.23	1.46583	4
	+67 36 36.23	60 32 40.26	284 41 38.52	0 28.95	. .	. .	0 29.21	1.46553	4
24	57.63	40.92	20.31	29.55	+0.01	0 29.67	0 29.68	1.47246	1
	38.55	41.58	38.73	29.53	. .	. .	0 29.66	1.47217	1
26	57.68	41.58	19.69	29.35	+0.01	0 29.27	0 29.28	1.46657	4
	39.26	43.03	36.66	29.32	. .	. .	0 29.26	1.46627	4
31	57.54	43.18	20.04	29.57	+0.01	0 29.84	0 29.85	1.47494	2
	40.72	43.49	36.55	29.55	. .	. .	0 29.83	1.47465	2
Nov. 1	57.50	43.49	20.88	29.20	+0.01	0 29.04	0 29.05	1.46315	3½
	40.99	45.41	35.47	29.18	. .	. .	0 29.03	1.46285	3½
6	57.48	45.06	21.44	29.46	+0.02	0 29.36	0 29.38	1.46805	4
	42.54	46.33	35.11	29.42	. .	. .	0 29.34	1.46746	4
22	57.13	49.90	22.72	29.72	+0.01	0 29.89	0 29.90	1.47567	2
	47.03	50.11	32.61	29.70	. .	. .	0 29.88	1.47538	2
23	57.10	50.18	21.85	29.58	+0.01	0 29.79	0 29.80	1.47422	2
	47.28	50.60	31.25	29.55	. .	. .	0 29.78	1.47392	2
Dec. 1	56.56	52.35	21.90	28.99	+0.01	0 29.37	0 29.38	1.46805	3
	48.91	53.61	28.29	28.98	. .	. .	0 29.36	1.46776	3
Dec. 6	+ 7 2 56.45	60 33 53.62	345 14 22.84	0 29.24	+0.01	0 29.12	0 29.13	1.46434	3
	+67 36 50.07	60 32 55.38	284 41 27.46	0 29.22	. .	. .	0 29.11	1.46404	3

Stars No. $\begin{cases} 76 \\ 103 \end{cases}$

Date.	δ	A B	C′	r′	$\frac{r'_s - r'_n}{2}$	$\frac{A-B}{2}$	r	log r	p
1899.									
Oct. 17	+ 7° 2′ 57″.52	60° 29′ 22″.33	345° 14′ 18″.78	0′ 28″.97	+0″.04	0′ 29″.33	0′ 29″.37	1.46790	4
	+67 32 19.85	60 28 23.64	284 45 55.14	0 28.88	. .	. .	0 29.29	1.46672	4
24	57.63	24.52	20.31	29.55	+0.05	0 29.64	0 29.69	1.47261	1
	22.15	25.23	55.08	29.45	. .	. .	0 29.59	1.47114	1
26	57.68	25.17	19.69	29.35	+0.05	0 29.39	0 29.44	1.46894	4
	22.85	26.38	53.31	29.24	. .	. .	0 29.34	1.46746	4
31	57.54	26.78	20.04	29.57	+0.05	0 30.12	0 30.17	1.47958	2
	24.32	26.53	53.51	29.47	. .	. .	0 30.07	1.47813	2
Nov. 1	57.50	27.09	20.88	29.20	+0.05	0 29.16	0 29.21	1.46553	3½
	24.59	28.77	52.11	29.10	. .	. .	0 29.11	1.46404	3½
6	57.48	28.68	21.44	29.46	+0.07	0 29.55	0 29.62	1.47159	4
	26.16	29.57	51.87	29.32	. .	. .	0 29.48	1.46953	4
22	57.13	33.62	22.72	29.72	+0.05	0 29.82	0 29.87	1.47524	2½
	30.75	33.97	48.75	29.61	. .	. .	0 29.77	1.47378	2½
23	57.10	33.92	21.85	29.58	+0.05	0 29.94	0 29.99	1.47698	2
	31.02	34.03	47.82	29.47	. .	. .	0 29.89	1.47553	2
Dec. 1	56.56	36.19	21.90	28.99	+0.05	0 29.48	0 29.53	1.47026	3
	32.75	37.23	44.67	28.88	. .	. .	0 29.43	1.46879	3
Dec. 6	+ 7 2 56.45	60 29 37.52	345 14 22.84	0 29.24	+0.06	0 29.25	0 29.31	1.46702	3
	+67 32 33.97	60 28 39.01	284 45 43.83	0 29.11	. .	. .	0 29.19	1.46523	3

Stars No. { 79 / 96 }

Date.	δ	A / B	C′	r′	½ (r′ₛ − r′ₙ)	½ (A − B)	r	log r	p
1899.									
Oct. 17	+ 3° 5′ 26″.02	69° 26′ 28″.94	349° 11′ 45″.74	0′ 33″.77	−0″.60	0′ 34″.19	0′ 33″.59	1.52621	4
	+72 31 54.96	69 25 20.56	279 46 25.18	0 34.97	. .	. .	0 34.79	1.54145	4
24	25.95	31.42	46.93	34.45	−0.61	35.25	0 34.64	1.53958	1
	57.37	20.91	26.02	35.67	. .	. .	0 35.86	1.55461	1
26	25.96	32.14	46.01	34.21	−0.60	35.04	0 34.44	1.53706	4
	58.10	22.06	23.95	35.41	. .	. .	0 35.64	1.55194	4
31	25.70	33.95	48.10	34.47	−0.61	35.06	0 34.45	1.53719	2
	59.65	23.82	24.28	35.70	. .	. .	0 35.67	1.55230	2
Nov. 1	5 25.64	34.30	47.57	34.03	−0.60	34.74	0 34.14	1.53326	3½
	31 59.94	24.81	22.76	35.24	. .	. .	0 35.34	1.54827	3½
6	5 25.50	36.10	47.84	34.33	−0.60	35.22	0 34.62	1.53933	4
	32 1.60	25.65	22.19	35.52	. .	. .	0 35.82	1.55413	4
22	24.81	41.65	49.30	34.64	−0.62	35.92	0 35.30	1.54777	2½
	6.46	29.81	19.49	35.88	. .	. .	0 36.54	1.56277	2½
23	24.76	41.98	49.33	34.47	−0.61	35.26	0 34.65	1.53970	2
	6.74	31.46	17.87	35.69	. .	. .	0 35.87	1.55473	2
Dec. 1	24.08	44.51	49.94	33.80	−0.60	34.34	0 33.74	1.52815	3
	8.59	35.83	14.11	35.00	. .	. .	0 34.94	1.54332	3
Dec. 6	+ 3 5 23.93	69 26 45.95	349 11 49.70	0 34.08	−0.60	34.92	0 34.32	1.53555	3
	+72 32 9.88	69 25 36.11	279 46 13.59	0 35.27	. .	. .	0 35.52	1.55047	3

Stars No. { 174 / 89 }

Date.	δ	A / B	C′	r′	½ (r′ₛ − r′ₙ)	½ (A − B)	r	log r	p
1899.									
Oct. 17	−14° 16′ 45″.47	103° 3′ 16″.23	366° 33′ 28″.62	1′ 2″.62	+0″.25	1′ 2″.60	1′ 2″.85	1.79831	4
	+88 46 30.76	103 1 11.02	263 32 17.60	1 2.11	. .	. .	1 2.35	1.79484	4
24	46.92	20.34	29.63	3.85	+0.25	3.88	1 4.13	1.80706	1½
	33.42	12.58	17.05	3.34	. .	. .	1 3.63	1.80366	1½
26	47.14	21.35	30.52	3.29	+0.19	3.03	1 3.22	1.80085	3
	34.21	15.28	15.24	2.91	. .	. .	1 2.84	1.79824	3
31	47.92	23.88	31.90	3.21	−0.10	3.56	1 3.46	1.80250	1½
	35.96	16.75	15.15	3.41	. .	. .	1 3.66	1.80387	1½
Nov. 1	48.11	24.40	32.56	3.01	+0.20	2.38	1 2.58	1.79644	4
	36.29	19.63	12.93	2.60	. .	. .	1 2.18	1.79365	4
6	48.90	27.05	33.77	3.53	+0.21	3.24	1 3.45	1.80243	4½
	38.15	20.57	13.20	3.10	. .	. .	1 3.03	1.79955	4½
22	51.39	35.04	36.71	4.13	+0.20	3.54	1 3.74	1.80441	3½
	43.65	27.95	8.76	3.74	. .	. .	1 3.34	1.80168	3½
23	51.55	35.54	36.49	3.68	+0.15	3.86	1 4.01	1.80625	2
	43.99	27.82	8.67	3.38	. .	. .	1 3.71	1.80421	2
Dec. 1	53.06	39.22	38.13	2.52	+0.17	2.30	1 2.47	1.79567	3½
	46.16	34.61	3.52	2.17	. .	. .	1 2.13	1.79330	3½
Dec. 6	−14 16 53.71	103 3 41.37	366 33 38.14	1 2.96	+0.15	1 2.88	1 3.03	1.79955	3
	+88 46 47.66	103 1 35.60	263 32 2.54	1 2.67	. .	. .	1 2.73	1.79748	3

Stars No. {108, 136}

Date.	δ	A / B	C′	r′	½ (r′ₛ − r′ₙ)	½ (A − B)	r	log r	p
1899.	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″		
Oct. 17	+ 8 39 25.41	57 23 58.48	343 37 52.54	0 27.15	−0.03	0 27.34	0 27.31	1.43632	4
	+66 3 23.89	57 23 3.79	286 14 48.75	27.22	. .	. .	0 27.37	1.43727	4
24	25.52	24 0.59	54.01	27.69	−0.02	27.90	27.88	1.44529	1
	26.11	23 4.79	49.22	27.74	. .	. .	27.92	1.44592	1
26	25.60	1.17	52.87	27.47	−0.01	27.59	27.58	1.44059	3
	26.77	5.99	46 88	27.49	. .	. .	27.60	1.44091	3
31	25.51	2.72	54.73	27.70	0.00	27.50	27 50	1.43933	2½
	28.23	7.72	47.01	27.71	. .	. .	27.51	1.43949	2½
Nov. 1	25.46	3.03	53.87	27.34	−0.01	27.45	27.44	1.43838	3½
	28.49	8.13	45 74	27.37	. .	. .	27 46	1.43870	3½
6	25.45	4.57	54.69	27.57	−0.02	27.81	27.79	1.44389	4
	30.02	8.94	45.75	27.62	. .	. .	27.83	1.44451	4
22	25.18	9.49	56.01	27.83	−0.01	28.28	28.27	1.45133	3
	34.67	12.92	43.09	27.85	. .	. .	28.29	1.45163	3
23	25.16	9.80	55.13	27.69	−0.01	28.15	28.14	1.44932	2½
	34.96	13.49	41.64	27.72	. .	. .	28.16	1.44963	2½
Dec. 1	24.66	12.12	55.44	27.14	−0.02	27.57	27.55	1.44012	3
	36.78	16.97	38.47	27.18	. .	. .	27.59	1.44075	3
Dec. 6	+ 8 39 24.60	57 24 13.49	343 37 55.38	0 27.35	−0.01	0 27.79	0 27.78	1.44373	3
	+66 3 38.09	57 23 17.91	286 14 37.47	0 27.37	. .	. .	0 27.80	1.44404	3

Stars No. {108, 149}

Date.	δ	A / B	C′	r′	½ (r′ₛ − r′ₙ)	½ (A − B)	r	log r	p
1899.	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″		
Oct. 17	+ 8 39 25.41	58 17 46.90	343 37 52.54	0 27.15	−0.55	0 28.20	0 27.65	1.44170	4
	+66 57 12.31	58 16 50.49	285 21 2.05	0 28.25	. .	. .	0 28.75	1.45864	4
24	25.52	48.97	54.01	27.69	−0.55	28.53	27.98	1.44685	1½
	14.49	51.90	2.11	28.79	. .	. .	29.08	1.46359	1½
26	25.60	49.58	37 52.87	27.47	−0.53	28.51	27.98	1.44685	3
	15.18	52.56	21 0.31	28.53	. .	. .	29.04	1.46300	3
31	25.51	51.14	37 54.73	27.70	−0 50	28.01	27.51	1.43949	1½
	16.65	55.12	20 59.61	28.70	. .	. .	28.51	1.45500	1½
Nov. 1	25.46	51.45	53.87	27.34	−0.53	27.88	27.35	1.43696	3½
	16.91	55.68	58.19	28.41	. .	. .	28.41	1.45347	3½
6	25.45	17 52.99	54.69	27.57	−0.54	28.36	27.82	1.44436	4
	18.44	16 56.26	58.43	28.66	. .	. .	28.90	1.46090	4
22	25.18	17 58.00	56.01	27.83	−0.53	28.73	28.20	1.45025	3
	23.18	17 0.53	55.48	28.90	. .	. .	29.26	1.46627	3
23	25.16	17 58.32	55.13	27.69	−0.53	28.53	28.00	1.44716	1½
	23.48	17 1.26	53.87	28.75	. .	. .	29.06	1.46330	1½
Dec. 1	24.66	18 0.72	55.44	27.14	−0.53	27.67	27.14	1.43361	3
	25 38	17 5.38	50.06	28.21	. .	. .	28.21	1.45040	3
Dec. 6	+ 8 39 24.60	58 18 2.14	343 37 55.38	0 27.35	−0.52	0 28.31	0 27.79	1.44389	3
	+66 57 26.74	58 17 5.51	285 20 49.87	0 28.39	. .	. .	0 28.83	1.45984	3

Stars No. {153, 136}

Date.	δ	A B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	$\frac{1}{2}(A - B)$	r	log r	p
1899. Oct. 17									
24	+8° 0′ 51.43″	58° 2′ 34.68″	344° 16′ 27.19″	0′ 28.45″	+0.35″	0′ 28.35″	0′ 28.70″	1.45788	1½
	+66 3 26.11	58 1 37.97	286 14 49.22	0 27.74	. .	. .	0 28.00	1.44716	1½
26	51.46	35.31	26.80	28.20	+0.35	27.69	28.04	1.44778	2
	26.77	39.92	46.88	27.49	. .	. .	27.34	1.43680	2
31	51.34	36 89	28.16	28.33	+0.31	27.87	28.18	1.44994	2
	28.23	41.15	47.01	27.71	. .	. .	27.56	1.44028	2
Nov. 1	51.29	37.20	27.93	28.07	+0.35	27.50	27.85	1.44483	3
	28.49	42.19	45.74	27.37	. .	. .	27.15	1.43377	3
6	51.16	38.86	28.58	28.33	+0.35	28.01	28.36	1.45271	4½
	30.02	42.83	45.75	27.62	. .	. .	27.66	1.44185	4½
Nov. 22	50.75	43.92	29.97	28.57	+0.36	28.52	28.88	1.46060	3
	34.67	46.88	43.09	27.85	. .	. .	28.16	1.44963	3
23	50.73	44.23	29.45	28.41	+0.34	28.21	28.55	1.45561	2
	34.96	47.81	41.64	27.72	. .	. .	27.87	1.44514	2
Dec. 1	50.17	46.61	29.46	27.88	+0.35	27.81	28.16	1.44963	3
	36.78	50.99	38.47	27.18	. .	. .	27.46	1.43870	3
Dec. 6	+8 0 50.05	58 2 48.04	344 16 30.03	0 28.06	+0.34	0 27.74	0 28.08	1.44840	3
	+66 3 38.09	58 1 52.56	286 14 37.47	0 27.37	. .	. .	0 27.40	1.43775	3

Stars No. {153, 149}

Date.	δ	A B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	$\frac{1}{2}(A - B)$	r	log r	p
1899. Oct. 17									
24	+8° 0′ 51.43″	58° 56′ 23.06″	344° 16′ 27.19″	0′ 28.45″	−0.17″	0′ 28.99″	0′ 28.82″	1.45969	2
	+66 57 14.49	58 55 25.08	285 21 2.11	0 28.79	. .	. .	29.16	1.46479	2
26	51.46	23.72	16 26.80	28.20	−0.16	28.61	28.45	1.45408	2
	15.18	26.49	21 0.31	28.53	. .	. .	28.77	1.45894	2
31	51.34	25.31	16 28.16	28.33	−0.18	28.38	28.20	1.45025	1
	16.65	28.55	20 59.61	28.70	. .	. .	28.56	1.45576	1
Nov. 1	51.29	25.62	27.93	28.07	−0.17	27.94	27.77	1.44358	3
	16.91	29.74	58.19	28.41	. .	. .	28.11	1.44886	3
6	51.16	27.28	28.58	28.33	−0.16	28.56	28.40	1.45332	4½
	18.44	30.15	58.43	28.66	. .	. .	28.72	1.45818	4½
22	50.75	32.43	29.97	28.57	−0.16	28.97	28.81	1.45954	3
	23.18	34.49	55.48	28.90	. .	. .	29.13	1.46434	3
23	50.73	32.75	29.45	28.41	−0.17	28.58	28.41	1.45347	1
	23.48	35.58	53.87	28.75	. .	. .	28.75	1.45864	1
Dec. 1	50.17	35.21	29.46	27.88	−0.16	27.90	27.74	1.44311	3
	25.38	39.40	50.06	28.21	. .	. .	28.06	1.44809	3
Dec. 6	+8 0 50.05	58 56 36.69	344 16 30.03	0 28.06	−0.16	28.26	0 28.10	1.44871	3
	+66 57 26.74	58 55 40.16	285 20 49.87	0 28.39	. .	. .	0 28.42	1.45362	3

Stars No. { 161, 190 }

Date.	δ	A / B	C′	r′	½ (r′ₛ − r′ₙ)	½ (A − B)	r	log r	p
1899.	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″		
Oct. 17	+21 31 51.51	31 35 3.46	330 45 39.84	0 14.07	+0.02	0 14.55	0 14.57	1.16346	4
	+53 6 54.97	31 34 34.35	299 11 5.49	0 14.02	. .	. .	0 14.53	1.16227	4
24	52.05	4.55	40.12	14 35	+0.02	14.82	14.84	1.17143	1½
	56.60	34.91	5 21	14.31	. .	. .	14.80	1.17026	1½
26	52.27	4.87	39.94	14.22	+0.01	14.71	14.72	1.16791	3
	57.14	35.45	4.49	14.20	. .	. .	14.70	1.16732	3
31	52.55	5.73	41.32	14.24	+0.05	15.44	15.49	1.19005	1½
	58.28	34.84	6.48	14.13	. .	. .	15.39	1.18724	1½
Nov. 1	52.58	5.89	39.73	14 15	+0.01	14.16	14.17	1.15137	3
	58.47	36.57	3.16	14.13	. .	. .	14.15	1.15076	3
6	31 52.88	6.75	40.54	14.28	+0 02	14.78	14.80	1.17026	4½
	6 59.63	37.19	3.35	14.23	. .	. .	14.76	1.16909	4½
22	31 53.72	9.61	45 41.30	14.40	0.00	15.03	15.03	1.17696	3½
	7 3.33	39.55	11 1.75	14 40	. .	. .	15.03	1.17696	3½
23	53.78	9.80	45 40.50	14.31	0.00	14.46	14.46	1.16017	2
	3.58	40.87	10 59.63	14.30	. .	. .	14.46	1.16017	2
Dec. 1	53.83	11.23	39.09	14.05	+0.01	14.63	14.64	1.16554	3
	5.06	41.97	57.12	14.02	. .	. .	14.62	1.16495	3
Dec. 6	+21 31 54.07	31 35 12.08	330 45 39.15	0 14.15	+0.01	0 14.30	0 14.31	1.15564	2½
	+53 7 6.15	31 34 43.47	299 10 55.68	0 14.12	. .	. .	0 14.29	1.15503	2½

Stars No. { 235, 264 }

Date.	δ	A / B	C′	r′	½ (r′ₛ − r′ₙ)	½ (A − B)	r	log r	p
1899.	° ′ ″	° ′ ″	° ′ ″	′ ″	″	′ ″	′ ″		
Oct. 17	−10 5 57.29	95 23 16.80	362 22 48.68	0 53.90	−0.48	0 54.51	0 54 03	1.73263	3
	+85 17 19.51	95 21 27.77	267 1 20.91	0 54.86	. .	. .	0 54.99	1.74028	3
24	58.16	19.53	49 86	55.09	−0.51	55.92	55.41	1.74359	2
	21.37	27.69	22.17	56.11	. .	. .	56.43	1.75151	2
26	58.31	20.34	50.00	54.69	−0.49	55.72	55.23	1.74128	3
	22.03	28.90	21.10	55.67	. .	. .	56.21	1.74981	3
31									
Nov. 1	59.19	22.98	52.01	54.41	−0 52	55.14	54.62	1.73735	3½
	23.79	32.70	19.31	55.45	. .	. .	55.66	1.74544	3½
6	5 59.94	25.22	53.24	54.73	−0.49	55.42	54.93	1.73981	4
	17 25.28	34.38	18.86	55.72	. .	. .	55.91	1.74749	4
22	6 2.34	32.96	55.74	55.39	−0.46	56.39	55.93	1.74764	4½
	17 30.62	40.17	15.57	56.31	. .	. .	56.85	1.75473	4½
23	2.49	33.48	56.00	54.85	−0.49	55.42	54.93	1.73981	2
	30.99	42.64	13.36	55.84	. .	. .	55.91	1.74749	2
Dec. 1	3.96	37 49	57.08	54.01	−0.50	54.99	54.49	1.73632	3
	33.53	47.51	9.57	55.01	. .	. .	55.49	1.74421	3
Dec. 6	−10 6 4.65	95 23 39.96	362 22 58.54	0 54.39	−0.50	0 55.09	0 54.59	1.73711	3½
	+85 17 35.31	95 21 49.77	267 1 8.77	0 55.40	. .	. .	0 55.59	1.74500	3½

Stars No. { 5 / 788 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′ + B)	r	log r	p
1899.									
Oct. 17	−46° 17′ 55″.65	12° 39′ 21″.73	398° 29′ 1″.41	6′ 44″.57	−1″.89	6′ 42″.77	6′ 40″.88	2.60301	4
	+58 57 17.38	167 7 12.73	231 21 48.68	6 48.36	. .	. .	6 44.66	2.60709	4
24	57.20	39 17.63	28 56.09	53.94	−1.71	52.07	50.36	2.61316	2
	14.83	6 58.23	21 57.86	57.36	. .	. .	53.78	2.61677	2
26	57.63	39 16.44	57.70	49.86	−1.76	48.96	47.20	2.60981	4
	14.07	7 5.64	52.06	53.38	. .	. .	50.72	2.61355	4
31	58.86	39 13 60	28 55.12	54.30	−1.55	54.18	52.63	2.61556	2
	12.46	6 58.03	21 57.09	57.41	. .	. .	55.73	2.61881	2
Nov. 1	17 59.11	39 13.04	29 1.52	48.24	−1.75	46.77	45.02	2.60748	3
	57 12.15	7 13.41	21 48.11	51.75	. .	. .	48.52	2.61121	3
6	18 0.06	10.30	28 55.65	52.49	−1.81	53.37	51.56	2.61443	3
	57 10.36	2.95	21 52.70	56.11	. .	. .	55.18	2.61824	3
22	2.86	2.22	58.52	56.52	−1.68	56.71	55.03	2.61808	4
	5.08	4.36	54.16	59.88	. .	. .	58.39	2.62158	4
23	3.01	39 1.77	28 57.07	54.46	−1.88	52.74	50.86	2.61369	1½
	4.78	7 12.75	21 44.32	58.22	. .	. .	54.62	2.61765	1½
Dec. 1	4.28	38 58.39	29 11.22	45.15	−1.95	43.72	41.77	2.60398	3
	2.67	7 34.17	21 37.05	49.05	. .	. .	45.67	2.60817	3
Dec. 6	−46 18 4 67	12 38 56.62	398 29 6.81	6 49.41	−1.61	6 47.50	6 45 89	2.60841	2
	+58 57 1.29	167 7 28.37	231 21 38.44	6 52.63	. .	. .	6 49.11	2.61184	2

Stars No. { 5 / 1019 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′ + B)	r	log r	p
1899.									
Oct. 17	−46° 17′ 55″.65	12° 32′ 12″.58	398° 29′ 1″.41	6′ 44″.57	−4″.97	6′ 45″.97	6′ 41″.00	2.60314	3½
	+58 50 8.23	167 14 15.47	231 14 45.94	6 54.51	. .	. .	6 50.94	2.61378	3½
24	57.20	9.14	28 56.09	6 53.94	−5.16	55.54	6 50.38	2.61319	2½
	6.34	59.77	14 56.32	7 4.27	. .	. .	7 0.70	2 62397	2½
26	57.63	8.05	28 57.70	6 49.86	−5.59	52.79	6 47.20	2.60981	2½
	5.68	6.36	14 51.34	7 1.04	. .	. .	6 58.38	2.62157	2½
31									
Nov. 1	17 59.11	4.76	29 1.52	6 48.24	−5.42	49.77	6 44.35	2.60676	3½
	50 3.87	15.70	14 45.82	6 59.09	. .	. .	6 55.19	2.61825	3½
6	18 0.06	32 2.23	28 55.65	6 52.49	−4.39	54.52	6 50.13	2.61292	2½
	50 2.29	14 8.72	14 46.93	7 1.28	. .	. .	6 58.91	2.62212	2½
22	18 2.86	31 53.71	28 58.52	6 56.52	−4.96	57.91	6 52.95	2.61590	4
	49 56.57	14 10.47	14 48.05	7 6.45	. .	. .	7 2.87	2.62621	4
23									
Dec. 1	4.28	49.12	29 11.22	6 45.15	−4.74	47 80	6 43.06	2.60537	3
	53.40	35.28	14 35.94	6 55.64	. .	. .	6 52.54	2.61547	3
Dec. 6	−46 18 4.67	12 31 46.78	398 29 6.81	6 49 41	−4.85	6 50.19	6 45.34	2.60782	2
	+58 49 51.45	167 14 32.84	231 14 33.97	6 59.11	. .	. .	6 55.04	2.61809	2

Stars No. { 9 / 793 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−35° 41′ 30″.86	34° 38′ 45″.50	387° 56′ 37″.38	2′ 40″.20	+3″.32	2′ 35″.83	2′ 39″.15	2.20181	4
	+70 20 16.36	145 16 2.84	242 40 34.54	2 33.55	. .	. .	2 32.51	2.18330	4
24	32.17	38 41.53	36.14	43.76	+3.47	39.56	43.03	2.21227	2
	13.70	15 59.34	36.80	36.82	. .	. .	36.09	2.19337	2
26	32 51	38 40.41	37.30	42.25	+3.40	38.10	41.50	2.20817	4
	12.92	16 3 38	33.92	35.44	. .	. .	34.70	2.18949	4
31	33.60	37.63	38.11	43.89	+3.53	39.46	42.99	2.21116	3
	11.23	3.45	34.66	36.83	. .	. .	35.93	2.19297	3
Nov. 1	33.82	37.09	41.34	41.57	+3.37	37.10	40.47	2.20539	3
	10.91	8.70	32.64	34.83	. .	. .	33.73	2.18676	3
6	34.66	34.39	40.50	43.18	+3.39	38.90	42.29	2.21029	3
	9.05	7.80	32.70	36.40	. .	. .	35.51	2.19176	3
22	37.17	26.56	44.26	44.61	+3.48	39.65	43.13	2.21253	4
	3.73	14.13	30.13	37.64	. .	. .	36.17	2.19360	4
23	37.32	26.11	42.33	43.87	+3.45	39.89	43.34	2.21209	1½
	3.43	14.11	28.22	36.97	. .	. .	36.44	2.19435	1½
Dec. 1	41 38 55	22.80	46.45	40.37	+3.30	36.35	39.65	2.20317	3
	20 1.35	24.49	21.96	33.76	. .	. .	33.05	2.18483	3
Dec. 6	−35 41 38.95	34 38 20.98	387 56 46.59	2 41.91	+3.42	2 37.96	2 41.38	2.20785	3
	+70 19 59.93	145 16 24.09	242 40 22.50	2 35.07	. .	. .	2 34.54	2.18804	3

Stars No. { 38 / 788 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−46° 37′ 59″.90	12° 19′ 17″.48	398° 48′ 46″.48	7′ 3″.22	+7″.43	6′ 52″.36	6′ 59″.79	2.62303	4
	+58 57 17.38	167 26 57.80	231 21 48.68	6 48.36	. .	. .	6 44.93	2.60738	4
24	38 1.58	13.25	36.46	12.38	+7.51	7 4.07	7 11.58	2.63506	2
	57 14.83	38.60	57.86	57.36	. .	. .	6 56.56	2.61968	2
26	2.03	12.04	41.95	8.69	+7.65	6 59.03	7 6.68	2.63010	4
	14.07	49.89	52.06	53.38	. .	. .	6 51.38	2.61424	4
31	3.37	9.09	40.86	12.62	+7.60	7 3.57	7 11.17	2.63465	3
	12.46	43.77	57.09	57.41	. .	. .	6 55.97	2.61806	3
Nov. 1	3.65	19 8.50	48.95	6.89	+7.57	6 55.33	7 2.90	2.62624	3
	12.15	27 0.84	48.11	51.75	. .	. .	6 47.76	2.61040	3
6	4.74	19 5.62	41.89	11.45	+7.67	7 2.59	7 10.26	2.63373	3
	10.36	26 49.19	52.70	56.11	. .	. .	6 54.92	2.61796	3
22	7.99	18 57.09	45.38	15.34	+7.73	7 5.84	7 13.57	2.63606	3½
	5.08	26 51.22	54.16	59 88	. .	. .	6 58.11	2.62129	3½
23	8.17	18 56.61	44.83	13.25	+7.51	7 1.44	7 8.95	2.63241	2
	4.78	27 0.51	44.32	58.22	. .	. .	6 53.93	2.61693	2
Dec. 1	9.71	52.96	56.53	3.63	+7.29	6 53.78	7 1.07	2.62435	3
	2.67	19.48	37.05	49.05	. .	. .	6 46.49	2.60905	3
Dec. 6	−46 38 10.27	12 18 51.02	398 48 52.50	7 7.84	+7.60	6 57.46	7 5.06	2.62845	2
	+58 57 1.29	167 27 14.06	231 21 38.44	6 52.63	. .	. .	6 49.86	2.61264	2

Stars No. { 66 / 788 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−47° 15′ 11″.67	11° 42′ 5″.71	399° 25′ 19″.71	7′ 42″.68	+27″.16	7′ 11″.63	7′ 38″.79	2.66161	4
	+58 57 17.38	168 3 31.03	231 21 48.68	6 48.36	. .	. .	6 44.47	2.60689	4
24	13.45	1.38	11.20	52.71	+27.67	22.64	50.31	2.67238	1½
	14.83	13.34	57.86	57 36	. .	. .	54.97	2.61802	1½
26	13.94	42 0.13	15.38	49.11	+27.86	18.27	46.13	2.66851	4
	14.07	3 23.32	52.06	53.38	. .	. .	50.41	2.61321	4
31	15.34	41 57.12	13.95	52.94	+27.76	23.01	50.77	2.67281	2
	12.46	3 16.86	57.09	57.41	. .	. .	55.25	2.61831	2
Nov. 1	15.63	56.52	20.57	46.63	+27.44	15.51	42.95	2.66553	3
	12.15	32.46	48.11	51.75	. .	. .	48.07	2.61073	3
6	16.84	53.52	19.26	51.34	+27.61	19.96	47.57	2.66985	3½
	10.36	26.56	52.70	56.11	. .	. .	52.35	2.61527	3½
22									
23	20.63	44.15	17.77	53.34	+27 56	21.15	48.71	2.67090	2
	4.78	33.45	44.32	58.22	. .	. .	53.59	2.61657	2
Dec. 1	22.38	40.29	28.09	43.22	+27.08	14.33	41.41	2.66409	3
	2.67	51.04	37.05	49.05	. .	. .	47.25	2.60986	3
Dec. 6	−47 15 23.07	11 41 38.22	399 25 26.95	7 47.73	+27.55	7 16.63	7 44.18	2.66669	2½
	+58 57 1.29	168 3 48.51	231 21 38.44	6 52.63	. .	. .	6 49.08	2.61181	2½

Stars No. { 179 / 793 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−32° 49′ 23″.05	37° 30′ 53″.31	385° 4′ 52″.53	2′ 16″.28	−8″.63	2′ 24″.35	2′ 15″.72	2.13264	4
	+70 20 16.36	142 24 17.99	242 40 34.54	2 33.55	. .	. .	2 32.98	2.18464	4
24	24.64	49.06	53.88	19.03	−8.84	26.93	18.09	2.14016	2
	13.70	17.08	36.80	36.82	. .	. .	35.77	2.19248	2
26	25.03	47.89	53.92	17.83	−8.80	26.05	17.25	2.13725	4
	12.92	20.00	33.92	35 44	. .	. .	34 85	2.18991	4
31	26.29	44.94	58.65	17.40	−9.71	25.53	15.82	2.13296	2
	11.23	23.99	34.66	36.83	. .	. .	35.24	2.19100	2
Nov. 1	26.57	44.34	57.55	17.19	−8.82	25.37	16.55	2.13529	3½
	10.91	24.91	32.64	34.83	. .	. .	34.19	2.18806	3½
6	27.80	41.25	4 57.65	18.29	−9.05	26.90	17.85	2.13941	3½
	9.05	24.95	40 32.70	36.40	. .	. .	35.95	2.19298	3½
22	31.67	32.16	5 1.99	19.68	−8.98	27.99	19.01	2.14305	4
	3.73	31.86	40 30.13	37.64	. .	. .	36.97	2.19582	4
23	31.90	32.53	1.50	18.69	−9.14	27.09	17.95	2.13972	2
	3.43	33.28	28.22	36.97	. .	. .	36.23	2.19376	2
Dec. 1	49 34.01	27.34	4.90	16.12	−8.82	24.86	16.04	2.13367	2½
	20 1.35	42.94	21.96	33.76	. .	. .	33.68	2.18662	2½
Dec. 6	−32 49 35.02	37 30 24.91	385 5 6.24	2 17.11	−8.98	2 25.67	2 16.69	2.13574	3
	+70 19 59.93	142 24 43.74	242 40 22.50	2 35.07	. .	. .	2 34.65	2.18934	3

Stars No. { 38 / 1019 l. c.

Date.	δ	A′ / B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A'+B)$	r	log r	p
1899.									
Oct. 17	−46° 37′ 59″.90	12° 12′ 8″.33	398° 48′ 46″.48	7′ 3″.22	+4″.35	6′ 55″.56	6′ 59″.91	2.62316	3½
	+58 50 8.23	167 34 0.54	231 14 45.94	6 54.51	..	..	6 51.21	2.61406	3½
24	38 1.58	12 4.76	36.46	7 12.38	+4.05	7 7.55	7 11.60	2.63508	2½
	50 6.34	33 40.14	56.32	7 4.27	..	..	7 3.50	2.62685	2½
26	2.03	3.65	41.95	7 8.69	+3.82	7 2.87	7 6.69	2.63011	2½
	5.68	50.61	51.34	7 1.04	..	..	6 59.05	2.62227	2½
31									
Nov. 1	3.65	12 0.22	48.95	7 6.89	+3.90	6 58.32	7 2.22	2.62554	3½
	3.87	34 3.13	45.82	6 59.09	..	..	6 54.42	2.61744	3½
6	38 4.74	11 58.55	41.89	7 11.45	+4.58	7 3.24	7 7.82	2.63125	2½
	50 2.29	33 54.96	46.93	7 1.28	..	..	6 58.66	2.62186	2½
22	38 7.99	11 48.58	45.38	7 15.34	+4.49	7 7.04	7 11.53	2.63501	3½
	49 56.57	33 57.33	48.05	7 6.45	..	..	7 2.55	2.62585	3½
23									
Dec. 1	9.71	11 43.69	56.53	7 3.63	+3.99	6 57.86	7 1.85	2.62516	3
	53.40	34 20.59	35.94	6 55.64	..	..	6 53.87	2.61686	3
Dec. 6	−46 38 10.27	12 11 41.18	398 48 52.50	7 7.84	+4.36	7 0.14	7 4.50	2.62788	2½
	+58 49 51.45	167 34 18.53	231 14 33.97	6 59.11	..	..	6 55.78	2.61897	2½

Stars No. { 58 / 911 l.c.

Date.	δ	A′ / B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	$90^\circ - \frac{1}{2}(A'+B)$	r	log r	p
1899.									
Oct. 17	−29° 53′ 46″.43	46° 14′ 43″.33	382° 9′ 34″.42	1′ 57″.26	+1″.83	1′ 55″.58	1′ 57″.41	2.06971	4
	+76 8 29.76	133 41 25.50	248 28 8.92	1 53.59	..	..	1 53.75	2.05595	4
24	47.71	39.49	35.48	59.70	+1.95	57.34	59.29	2.07660	2
	27.20	25.82	9.66	55.79	..	..	55.39	2.06217	2
26	48.06	38.33	34.92	58.83	+2.05	57.31	59.36	2.07686	3
	26.39	27.05	7.87	54.73	..	..	55.26	2.06168	3
31	49.14	35.45	37.43	59.74	+2.25	57.35	59.60	2.07773	2½
	24.59	29.84	7.59	55.23	..	..	55.10	2.06108	2½
Nov. 1	49.36	34.88	38.40	58.23	+1.99	55.75	57.74	2.07093	3
	24.24	33.61	4.79	54.24	..	..	53.76	2.05599	3
6	50.28	32.06	39.01	1 59.38	+2.06	57.34	1 59.40	2.07700	5
	22.34	33.26	5.75	1 55.25	..	..	1 55.28	2.06175	5
22	53.08	23.09	9 42.66	2 0.37	+2.04	58.22	2 0.26	2.08012	2½
	16.17	40.47	28 2.19	1 56.29	..	..	1 56.18	2.06513	2½
23	53.25	22.53	9 42.10	1 59.80	+2.10	57.48	1 59.58	2.07766	2½
	15.78	42.50	27 59.60	1 55.59	..	..	1 55.38	2.06213	2½
Dec. 1	54.72	18.35	44.23	57.41	+1.99	55.82	57.81	2.07119	3½
	13.07	50.00	54.23	53.43	..	..	53.83	2.05625	3½
Dec. 6	−29 53 55.28	46 14 15.92	382 9 44.06	1 58.44	+2.11	1 56.81	1 58.92	2.07525	3
	+76 8 11.20	133 41 50.46	248 27 53.60	1 54.21	..	..	1 54.70	2.05956	3

Stars No. { 58
944 l. c.

Date.	δ	A′ B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899. Oct. 17	−29° 53′ 46″.43 +74 33 56.81	44° 40′ 10″.38 135 15 48.51	382° 9′ 34″.42 246 53 45.91	1′ 57″.26 2 2.43	−2″.58 . .	2′ 0″.55 . .	1′ 57″.97 2 3.13	2.07177 2.09036	4 4
24	47.71 54.36	6.65 48.89	35.48 46.59	59.70 4.97	−2.63 . .	2.23 . .	59.60 4.86	2.07773 2.09642	2 2
26	48.06 53.56	5.50 51.04	34.92 43.88	58.83 3.92	−2.54 . .	1.73 . .	59.19 4.27	2.07624 2.09437	3 3
31	49.14 51.79	2.65 53.35	37.43 44.08	59.74 3.43	−2.34 . .	2.00 . .	59.66 4.34	2.07795 2.09461	2 2
Nov. 1	49.36 51.45	40 2.09 15 57.46	38.40 40.94	58.23 3.32	−2.54 . .	0.22 . .	57.68 2.76	2.07070 2.08906	3½ 3½
6	50.28 49.58	39 59.30 15 56.76	39.01 42.25	59.38 4.29	−2.45 . .	1.97 . .	59.52 4.42	2.07744 2.09489	4½ 4½
22	53.08 43.40	39 50.32 16 3.54	42.66 39.12	60.37 5.63	−2.63 . .	3.07 . .	60.44 5.70	2.08077 2.09933	3 3
23	53.25 43.00	49.75 5.76	42.10 36.34	59.80 4.75	−2.47 . .	2.24 . .	59.77 4.71	2.07835 2.09590	2½ 2½
Dec. 1	54.72 40.22	45.50 12.97	44.23 31.26	57.41 2.37	−2.48 . .	0.76 . .	58.28 3.24	2.07291 2.09075	3½ 3½
Dec. 6	−29 53 55.28 +74 33 38.28	44 39 43.00 135 16 14.87	382 9 44.06 246 53 29.19	1 58.44 2 3.26	−2.41 . .	2 1.06 . .	1 58.65 2 3.47	2.07427 2.09156	3 3

Stars No. { 66
1019 l. c.

Date.	δ	A′ B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899. Oct. 17	−47° 15′ 11″.67 +58 50 8.23	11° 34′ 56″.56 168 10 33.77	399° 25′ 19″.71 231 14 45.94	7′ 42″.68 6 54.51	+24″.08 . .	7′ 14″.83 . .	7′ 38″.91 6 50.75	2.66173 2.61358	3½ 3½
24	13.45 6.34	52.89 14.88	11.20 56.32	7 52.71 7 4.27	+24.22 . .	26.11 . .	7 50.33 7 1.89	2.67240 2.62520	2 2
26	13.94 5.68	51.74 24.04	15.38 51.34	7 49.11 7 1.04	+24.03 . .	22.11 . .	7 46.14 6 58.08	2.66852 2.62126	2½ 2½
31									
Nov. 1	15.63 3.87	48.24 34.75	20.57 45.82	7 46.63 6 59.09	+23.77 . .	18.50 . .	7 42.27 6 54.73	2.66490 2.61777	3¼ 3½
6	15 16.84 50 2.29	45.45 32.33	19.26 46.93	7 51.34 7 1.28	+25.03 . .	21.11 . .	7 46.14 6 56.08	2.66852 2.61918	3 3
22									
23									
Dec. 1	15 22.38 49 53.40	31.02 52.15	28.09 35.94	7 43.22 6 55.64	+23.79 . .	18.41 . .	7 42.20 6 54.62	2.66483 2.61765	3 3
Dec. 6	−47 15 23.07 +58 49 51.45	11 34 28.38 168 10 52.98	399 25 26.95 231 14 33.97	7 47.73 6 59.11	+24.31 . .	7 19.32 . .	7 43.63 6 55.01	2.66617 2.61806	3 3

Stars No. { 187 / 885 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−40° 42′ 8″.67 +64 51 16.10	24° 9′ 7″.43 155 43 24.47	392° 56′ 7″.84 237 12 43.37	3′ 47″.46 3 43.35	+2″.05 . .	3′ 44″.05 . .	3′ 46″.10 3 42.00	2.35430 2.34635	4 4
24	10.47 13.53	3.06 18.65	6.69 48.04	52.26 47.82	+2.22 . .	49.14 . .	51.36 46.92	2.36429 2.35587	1 1
26	10.93 12.74	9 1.81 43 23.32	7.68 44.36	50.33 45.74	+2.29 . .	47.43 . .	49.72 45.14	2.36120 2.35245	3 3
31									
Nov. 1	12.64 10.60	8 57.96 43 31.76	12.11 40.35	49.25 44.73	+2.26 . .	45.14 . .	47.40 42.88	2.35679 2.34807	3 3
6	14.05 8.69	54.64 30.40	12.37 41.97	50.96 46.75	+2.10 . .	47.48 . .	49.58 45.38	2.36093 2.35291	4 4
22	42 18.43 51 2.59	44.16 34.90	14.38 39.48	53.63 48.77	+2.43 . .	50.47 . .	52.90 48.04	2.36717 2.35801	3½ 3½
23									
Dec. 1	42 21.04 50 59.54	38.50 51.77	21.71 29.94	47.44 43.03	+2.20 . .	44.86 . .	47.06 42.66	2.35614 2.34764	2½ 2½
Dec. 6	−40 42 22.21 +64 50 57.70	24 8 35.49 155 43 52.10	392 56 21.49 237 12 29.39	3 49.09 3 44.79	+2.15 . .	3 46.20 . .	3 48.35 3 44.05	2.35860 2.35034	2½ 2½

Stars No. { 189 / 911 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−31° 11′ 26″.07 +76 8 29.76	44° 57′ 3″.69 134 58 57.72	383° 27′ 6″.64 248 28 8.92	2′ 5″.25 1 53.59	+5″.83 . .	1′ 59″.29 . .	2′ 5″.12 1 53.46	2.09733 2.05484	4 4
24	27.57 27.20	56 59.83 58 57.03	6.69 9.66	7.67 55.79	+5.94 . .	2 1.57 . .	7.51 55.63	2.10554 2.06307	2 2
26	27.96 26.39	56 58.43 58 59.64	7.51 7.87	6.50 54.73	+5.88 . .	0.96 . .	5.84 54.08	2.09982 2.05721	2 2
31	29.16 24.59	56 55.43 59 2.61	10.20 7.59	7.52 55.23	+6.14 . .	2 0.98 . .	7.12 54.84	2.10421 2.06009	2 2
Nov. 1	29.43 24.24	54.81 6.06	10.83 4.79	5.96 54.24	+5.86 . .	1 59.56 . .	5.42 53.70	2.09837 2.05576	3 3
6	30.55 22.34	51.79 5.69	11.44 5.75	7.10 55.25	+5.92 . .	2 1.26 . .	7.18 55.34	2.10442 2.06198	4½ 4½
22	34.08 16.17	42.09 13.58	27 15.77 28 2.19	8.19 56.29	+5.95 . .	2.16 . .	8.11 56.21	2.10758 2.06524	3 3
23	34.29 15.78	41.49 15.20	27 14.80 27 59.60	7.57 55.59	+5.99 . .	2 1.65 . .	7.64 55.66	2.10599 2.06318	1½ 1½
Dec. 1	36.19 13.07	36.88 23.60	17.83 54.23	5.06 53.43	+5.81 . .	1 59.76 . .	5.57 53.95	2.09889 2.05671	3½ 3½
Dec. 6	−31 11 37.05 +76 8 11.20	44 56 34.15 134 59 25.03	383 27 18.63 248 27 53.60	2 5.94 1 54.21	+5.86 . .	2 0.41 . .	2 6.27 1 54.55	2.10130 2.05899	3 3

Stars No. { 139 / 944 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−31° 11′ 26.07″	43° 22′ 30.74″	383° 27′ 6.64″	2′ 5.25″	+1.41″	2′ 4.26″	2′ 5.67″	2.09923	4
	+74 33 56.81	136 33 20.73	246 53 45.91	2 2.43	. .	. .	2 2.85	2.08937	4
24	27.57	26.85	6.69	7.67	+1.35	6.52	7.87	2.10677	2
	54.36	20.10	46.59	4.97	. .	. .	5.17	2.09750	2
26	27.96	25.60	7.51	6.50	+1.29	5.38	6.67	2.10267	2
	53.56	23.63	43.88	3.92	. .	. .	4.09	2.09374	2
31	29.16	22.63	10.20	7.52	+2.04	5.62	7.66	2.10605	1½
	51.79	26.12	44.08	3 43	. .	. .	3.58	2.09195	1½
Nov. 1	29.43	22.02	10.83	5.96	+1.32	4.04	5.36	2.09816	3½
	51.45	29.89	40.94	3.32	. .	. .	2.72	2.08891	3½
6	30.55	19.03	11.44	7.10	+1.40	6.39	7.79	2.10650	4
	49.58	28.19	42.25	4.29	. .	. .	4.99	2.09687	4
22	34.08	9.32	15.77	8.19	+1.28	7.01	8.29	2.10819	3½
	43.40	36.65	39.12	5.63	. .	. .	5.73	2.09944	3½
23	34.29	8.71	14.80	7.57	+1.41	6.41	7.82	2.10660	1½
	43.00	38.46	36.34	4.75	. .	. .	5.00	2.09691	1½
Dec. 1	36.19	4.03	17.83	5.06	+1.34	4.70	6.04	2.10051	3½
	40.22	46.57	31.26	2.37	. .	. .	3.36	2.09117	3½
Dec. 6	−31 11 37.05	43 22 1.23	383 27 18.63	2 5.94	+1.34	2 4.66	2 6.00	2.10037	3
	+74 33 38.28	136 33 49.44	246 53 29.19	2 3.26	. .	. .	2 3.32	2.09103	3

Stars No. { 139 / 976 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−31° 11′ 26.07″	41° 0′ 5.71″	383° 27′ 6.64″	2′ 5.25″	−6.49″	2′ 11.74″	2′ 5.25″	2.09778	3½
	+72 11 31.78	138 55 30.81	244 31 35.83	2 18.24	. .	. .	2 18.23	2.14060	3½
24	27.57	1.95	6.69	7.67	−6.79	14.23	7.44	2.10531	2½
	29.52	29.58	37.11	21.26	. .	. .	21.02	2.14928	2½
26	27.96	41 0 0.79	7.51	6.50	−6.82	13.86	7.04	2.10394	2½
	28.75	138 55 31.48	36.03	20.15	. .	. .	20.68	2.14823	2½
31	29.16	40 59 57.87	10.20	7.52	−5.87	13.58	7.71	2.10622	1
	27.03	138 55 34.96	35.24	19.26	. .	. .	19.45	2.14442	1
Nov. 1	29.43	57.29	10.83	5.96	−6 74	12.20	5.46	2.09850	3½
	26.72	38.30	32.53	19.45	. .	. .	18.94	2.14283	3
6	30.55	54.39	11.44	7.10	−6.63	14.20	7.57	2.10575	3½
	24.94	37.20	34.24	20.37	. .	. .	20.83	2.14869	3½
22	34.08	44.75	15.77	8.19	−7.01	15.18	8.17	2.10779	4
	18.83	44.88	30.89	22.21	. .	. .	22.19	2.15287	4
23	34.29	44.15	14.80	7.57	−6.65	14.81	8.16	2.10775	1½
	18.44	46.22	28.58	20.87	. .	. .	21.46	2.15063	1½
Dec. 1	36.19	39.40	17.83	5.06	−6.71	12.44	5.73	2.09944	3½
	15.59	55.72	22.11	18.48	. .	. .	19.15	2.14348	3½
Dec. 6	−31 11 37.05	40 59 36.57	383 27 18.63	2 5.94	−6.76	2 12.30	2 5.54	2.09878	3
	+72 11 13.62	138 55 58.83	244 31 19.80	2 19.47	. .	. .	2 19.06	2.14320	3

Stars No. { 202 / 911 l. c.

Date.	δ	A′ / B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	−29° 22′ 41.″82	46° 45′ 47.″94	381° 38′ 33.″52	1′ 54.″42	+0.″41	1′ 53.″73	1′ 54.″14	2.05744	4
	+76 8 29.76	133 10 24.60	248 28 8.92	1 53.59	. .	. .	1 53.32	2.05431	4
24	43.29	43.91	34.17	56.62	+0.41	55.74	56.15	2.06502	1½
	27.20	24.51	9.66	55.79	. .	. .	55.33	2.06194	1½
26	43.65	42.74	34.48	55.96	+0.61	55.32	55.93	2.06420	3
	26.39	26.61	7.87	54.73	. .	. .	54.71	2.05960	3
31	44.81	39.78	38.77	55.28	+0.02	55.02	55.04	2.06085	2
	24.59	30.18	7.59	55.23	. .	. .	55.00	2.06070	2
Nov. 1	45.08	39.16	37.27	55.39	+0.57	54.18	54.75	2.05975	3½
	24.24	32.48	4.79	54.24	. .	. .	53.61	2.05542	3½
6	46.27	36.07	38.12	56.18	+0.46	55.73	56.19	2.06517	4½
	22.34	32.37	5.75	55.25	. .	. .	55.27	2.06172	4½
22	50.04	26.13	38 41.61	57.60	+0.65	57.22	57.87	2.07140	4
	16.17	39.42	28 2.19	56.29	. .	. .	56.57	2.06659	4
23	50.26	25.52	38 42.19	56.75	+0.58	55.94	56.52	2.06640	2
	15.78	42.59	27 59.60	55.59	. .	. .	55.36	2.06205	2
Dec. 1	52.36	20.71	45.22	54.52	+0.54	54.15	54.69	2.05953	3
	13.07	50.99	54.23	53.43	. .	. .	53.61	2.05542	3
Dec. 6	−29 22 53.40	46 45 17.80	381 38 46.31	1 55.33	+0.56	1 54.74	1 55.30	2.06183	3
	+76 8 11.20	133 10 52.71	248 27 53.60	1 54.21	. .	. .	1 54.18	2.05759	3

Stars No. { 179 / 944 l. e.

Date.	δ	A′ / B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	−32° 49′ 23.″05	41° 44′ 33.″76	385° 4′ 52.″53	2′ 16.″28	+6.″92	2′ 9.″81	2′ 16.″73	2.13586	4
	+74 33 56.81	138 11 6.62	246 53 45.91	2 2.43	. .	. .	2 2.89	2.08952	4
24	24.64	29.72	53.88	19.03	+7.03	11.99	19.02	2.14308	2
	54.36	7.29	46.59	4.97	. .	. .	4.96	2.09677	2
26	25.03	28.53	53.92	17.83	+6.95	10.71	17.66	2.13881	3
	53.56	10.04	43.88	3.92	. .	. .	3.76	2.09258	3
31	26.29	25.50	58.65	17.40	+6.98	9.96	16.94	2.13653	1
	51.79	14.57	44.08	3.43	. .	. .	2.98	2.08983	1
Nov. 1	26.57	24.88	57.55	17.19	+6.93	9.25	16.18	2.13411	4
	51.45	16.61	40.94	3.32	. .	. .	2.32	2.08750	4
6	27.80	21.78	4 57.65	18.29	+7.00	11.41	18.41	2.14117	4
	49.58	15.40	53 42.25	4.29	. .	. .	4.41	2.09485	4
22	31.67	11.73	5 2.00	19.68	+7.02	12.69	19.71	2.14523	4
	43.40	22.88	53 39.12	5.63	. .	. .	5.67	2.09923	4
23	31.90	11.10	1.52	18.69	+6.97	11.81	18.78	2.14233	2
	43.00	25.18	36.34	4.75	. .	. .	4.84	2.09635	2
Dec. 1	34.01	6.21	4.93	16.12	+6.87	10.06	16.93	2.13650	3
	40.22	33.67	31.26	2.37	. .	. .	3.19	2.09057	3
Dec. 6	−32 49 35.02	41 44 3.26	385 5 6.26	2 17.11	+6.92	2 9.84	2 16.76	2.13596	3
	+74 33 38.28	138 11 37.05	246 53 29.19	2 3.26	. .	. .	2 2.92	2.08962	3

Stars No. { **179** / **976 l. c.** }

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−32° 49′ 23″.05	39° 22′ 8″.73	385° 4′ 52″.53	2′ 16″.28	−0″.98	2′ 17″.28	2′ 16″.30	2.13450	3½
	+72 11 31.78	140 33 16.70	244 31 35.83	2 18.24	. .	. .	2 18.26	2.14070	3½
24	24.64	4.88	53.88	19.03	−1.11	19.17	18.06	2.14007	2½
	29.52	16.77	37.11	21.26	. .	. .	20.28	2.14700	2½
26	25.03	3.72	53.92	17.83	−1.16	19.19	18.03	2.13997	3½
	28.75	17.89	36.03	20.15	. .	. .	20.35	2.14721	3½
31	26.29	0.74	58.65	17.40	−0.93	17.92	16.99	2.13669	½
	27.03	23.41	35.24	19.26	. .	. .	18.85	2.14255	½
Nov. 1	26.57	22 0.15	57.55	17.19	−1.13	17.41	16.28	2.13443	4
	26.72	33 25.02	32.53	19.45	. .	. .	18.54	2.14157	4
6	27.80	21 57.14	4 57.65	18.29	−1.04	19.72	18.68	2.14201	3½
	24.94	33 23.41	31 34.24	20.37	. .	. .	20.76	2.14848	3½
22	31.67	47.16	5 2.00	19.68	−1.26	20.86	19.60	2.14488	4½
	18.83	31.11	31 30.89	22.21	. .	. .	22.12	2.15265	4½
23	31.90	46.54	1.52	18.69	−1.09	20.26	19.17	2.14355	2
	18.44	32.94	28.58	20.87	. .	. .	21.35	2.15030	2
Dec. 1	34.01	41.58	4.93	16.12	−1.18	17.75	16.57	2.13535	3
	15.59	42.82	22.11	18.48	. .	. .	18.93	2.14280	3
Dec. 6	−32 49 35.02	39 21 38.60	385 5 6.26	2 17.11	−1.18	2 17.52	2 16.34	2.13462	3
	+72 11 13.62	140 33 46.36	244 31 19.90	2 19.47	. .	. .	2 18.70	2.14208	3

Stars No. { **202** / **944 l. c.** }

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−29° 22′ 41″.82	45° 11′ 14″.99	381° 38′ 33″.52	1′ 54″.42	−4″.00	1′ 58″.70	1′ 54″.70	2.05956	4
	+74 33 56.81	134 44 47.61	246 53 45.91	2 2.43	. .	. .	2 2.70	2.08884	4
24	43.29	11.07	34.17	56.62	−4.17	2 0.67	56.50	2.06633	1½
	54.36	47.58	46.59	4.97	. .	. .	4.84	2.09635	1½
26	43.65	9.91	34.48	55.96	−3.98	1 59.74	55.76	2.06356	3
	53.56	50.60	43.88	3.92	. .	. .	3.72	2.09244	3
31	44.81	6.98	38.77	55.28	−4.02	59.16	55.14	2.06123	1½
	51.79	54.69	44.08	3.43	. .	. .	3.18	2.09054	1½
Nov. 1	45.08	6.37	37.27	55.39	−3.96	1 58.60	54.64	2.05934	4
	51.45	56.33	40.94	3.32	. .	. .	2.56	2.08835	4
6	46.27	11 3.31	38.12	56.18	−4.05	2 0.41	56.36	2.06580	4
	49.58	44 55.87	42.25	4.29	. .	. .	4.46	2.09503	4
22	50.04	10 53.36	41.61	57.60	−4.01	2.07	58.06	2.07210	4½
	43.40	45 2.49	39.12	5.63	. .	. .	6.08	2.10065	4½
23	50.26	52.74	42.19	56.75	−4.00	2 0.70	56.70	2.06707	2
	43.00	5.85	36.34	4.75	. .	. .	4.70	2.09587	2
Dec. 1	52.36	47.86	45.22	54.52	−3.92	1 59.09	55.17	2.06134	3
	40.22	13.96	31.26	2.37	. .	. .	3.01	2.08994	3
Dec. 6	−29 22 53.40	45 10 44.88	381 38 46.31	1 55.33	−3.96	1 59.00	1 55.04	2.06085	3
	+74 33 38.28	134 45 17.12	246 53 29.19	2 3.26	. .	. .	2 2.96	2.08976	3

Stars No. { **290** / **944 l. c.**

Date.	δ	A′ / B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	−30° 45′ 48.44″	43° 48′ 8.37″	383° 1′ 31.74″	2′ 2.27″	−0.08″	2′ 2.90″	2′ 2.82″	2.08927	3½
	+74 33 56.81	136 7 45.83	246 53 45.91	2 2.43	. .	. .	2 2.98	2.08983	3½
24	49.93	4.43	32.02	5.21	+0.12	5.07	5.19	2.09757	2½
	54.36	45.43	46.59	4.97	. .	. .	4.95	2.09674	2½
26	50.28	3.28	32.90	3.96	+0.02	3.85	3.87	2.09297	3½
	53.56	49.02	43.88	3.92	. .	. .	3.83	2.09283	3½
31									
Nov. 1	51.67	48 0.78	34.48	3.73	+0.20	2.84	3.04	2.09005	3½
	51.45	7 53.54	40.94	3.32	. .	. .	2.64	2.08863	3½
6	52.95	47 56.63	36.39	4.28	. .	4.61	4.61	2.09555	4
	49.58	7 54.14	42.25	4.29	. .	. .	4.61	2 09555	4
22	57.11	47 46.29	41.83	5.36	−0.13	5.50	5.37	2.09819	4
	43.40	8 2.71	39.12	5.63	. .	. .	5.63	2.09909	4
23									
Dec. 1	45 59.73	40.49	45.01	2.70	+0.16	2.88	3.04	2.09005	4
	33 40.22	13.75	31.26	2.37	. .	. .	2.72	2.08891	4
Dec. 6	−30 46 1.03	43 47 37.25	383 1 45.72	2 3.51	+0 12	2 3.16	2 3.28	2.09089	3
	+74 33 38.28	136 8 16.53	246 53 29.19	2 3.26	. .	. .	2 3.04	2.09005	3

Stars No. { **202** / **1084 l. c.**

Date.	δ	A′ / B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	−29° 22′ 41.82″	46° 45′ 16.28″	381° 38′ 33.52″	1′ 54.42″	+0.54″	1′ 53.56″	1′ 54.10″	2.05729	3½
	+76 7 58.10	133 10 56.59	248 27 36.93	1 53.33	. .	. .	1 53.02	2.05315	3½
24	43.29	12.92	34.17	56.62	+0.34	56.11	56.45	2.06614	2
	56.21	54.86	39.31	55.93	. .	. .	55.77	2.06360	2
26	43.65	45 11.89	34.48	55.96	+0.48	55.03	55.51	2.06262	3½
	55.54	10 58.04	36.44	54.99	. .	. .	54.55	2.05899	3½
31									
Nov. 1	45.08	45 8.64	37.27	55.39	+0.40	54.74	55.14	2.06123	3½
	53.72	11 1.88	35.39	54.57	. .	. .	54.34	2.05820	3½
6	46.27	45 5.87	38.12	56.18	+0.53	55.39	55.92	2.06416	3½
	52.14	11 3.34	34.78	55.12	. .	. .	54.86	2.06017	3½
22	50.04	44 56.40	41.61	57.60	+0.67	57.18	57.85	2.07133	4½
	46.44	11 9.24	32.37	56.26	. .	. .	56.51	2.06636	4½
23	50.26	55.79	42.19	56.75	+0.68	56.49	57.17	2.06882	2
	46.05	11.23	30.96	55.38	. .	. .	55.81	2.06375	2
Dec. 1	52.36	50.89	45.22	54.52	+0.44	54.42	54.86	2.06017	2½
	43.25	20.26	24.96	53.63	. .	. .	53.98	2.05683	2½
Dec. 6	−29 22 53.40	46 44 47.91	381 38 46.31	1 55.33	+0.44	1 54.76	1 55.20	2.06145	2½
	+76 7 41.31	133 11 22.57	248 27 23.74	1 54.44	. .	. .	1 54.32	2.05812	2½

Stars No. { 202
1045 l. e.

Date.	δ	A′ B	C′	r′	½ (r′ₑ − r′ₐ)	90° − ½ (A′ + B)	r	log r	p
1899.									
Oct. 17	−29° 22′ 41″.82 +75 59 21.84	46° 36′ 40″.02 133 19 32.25	381° 38′ 33″.52 248 19 1.27	1′ 54″.42 1 54.10	+0″.16 . .	1′ 53″.86 . .	1′ 54″.02 1 53.70	2.05698 2.05576	3½ 3½
24	43.29 20.01	36.72 30.41	34.17 3.76	56.62 56.77	−0.07 . .	56.43 . .	56.36 56.50	2.06580 2.06633	2½ 2½
26	43.65 19.36	35.71 34.12	38 34.48 19 0.36	55.96 55.73	+0.11 . .	55.08 . .	55.19 54.97	2.06141 2.06058	3½ 3½
31									
Nov. 1	45.08 17.56	32.48 37.43	38 37.27 18 59.84	55.39 55.39	. . . .	55.04 . .	55.04 55.04	2.06085 2.06085	3½ 3½
6	46.27 16.03	29.76 38.97	38.12 59.15	56.18 55.91	+0.13 . .	55.63 . .	55.76 55.50	2.06356 2.06258	3½ 3½
22	50.04 10.42	20.38 45.67	41.61 55.94	57.60 57.00	+0.30 . .	56.97 . .	57.27 56.67	2 06919 2.06696	4½ 4½
23	50.26 10.02	19.76 48.28	42.19 53.91	56.75 56.20	+0.27 . .	55.93 . .	56.20 55.66	2.06521 2.06318	2 2
Dec. 1	52.36 7.24	14.88 56.60	45.22 48.62	54.52 54.43	+0.04 . .	54.26 . .	54.30 54.22	2.05805 2.05774	2½ 2½
Dec. 6	−29 22 53.40 +75 59 5.31	46 36 11.91 133 19 58.23	381 38 46.31 248 18 48.08	1 55.33 1 55.22	+0.05 . .	1 54.93 . .	1 54.98 1 54.88	2.06062 2.06024	2½ 2½

Stars No. { 210
1002 l. e.

Date.	δ	A′ B	C′	r′	½ (r′ₑ − r′ₐ)	90° − ½ (A′ + B)	r	log r	p
1899.									
Oct. 17	−43° 26′ 56″.56 +62 54 41.78	19° 27′ 45″.22 160 23 2.77	395° 39′ 52″.65 235 16 49.88	4′ 51″.09 4 23.33	+13″.88 . .	4′ 36″.00 . .	4′ 49″.88 4 22.12	2.46222 2.41850	3 3
24	58.41 39.74	27 41.33 22 55.92	50.61 54.69	57.61 29.26	+14.17 . .	41.37 . .	55.54 27.20	2.47062 2.42684	1½ 1½
26	26 58.88 54 39.03	27 40.15 23 1.27	52.17 50.90	55.20 27.39	+13.90 . .	39.29 . .	53.19 25.39	2.46715 2.42389	3 3
31									
Nov. 1	27 0.62 54 37.11	36.49 9.36	57.76 48.40	53.71 25.93	+13.89 . .	37.07 . .	50.96 23.18	2.46383 2.42025	3 [illegible]
6	2.09 35.44	33.35 9.01	57.01 48.00	55.69 27.44	+14.12 . .	38.82 . .	52.94 24.70	2.46678 2.42275	3½ 3½
22	6.70 29.54	22.84 9.26	39 58.04 16 48.78	59.67 30.77	+14.45 . .	43.95 . .	58.40 29.50	2.47480 2.43056	[illegible] [illegible]
23									
Dec. 1	9.45 26.30	16.85 30.18	40 8.09 16 37.91	51.61 23.94	+13.83 . .	36.48 . .	50.31 22.65	2.46286 2.41938	2½ 2½
Dec. 6	−43 27 10.70 +62 54 24.32	19 27 13.62 160 23 30.41	395 40 6.93 235-16 36.52	4 53.73 4 25.88	+13.92 . .	4 37.98 . .	4 51.90 4 24.06	2.46523 2.42170	3 3

Stars No. { 210 / 1050 l. c.

Date.	δ	A′ / B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	log r	p
1899.									
Oct. 17	−43° 26′ 56″.56	18° 17′ 42″.84	395° 39′ 52″.65	4′ 51″.09	−1″.85	4′ 51″.18	4′ 49″.33	2.46139	3
	+61 44 39.40	161 32 34.80	234 7 17.85	4 54.79	. .	. .	4 53.03	2.46691	3
24	58.41	39.24	50.61	4 57.61	−2.11	58.15	4 56.04	2.47135	2
	37.65	24.46	26.15	5 1.84	. .	. .	5 0.26	2.47750	2
26	26 58.88	38.13	52.17	4 55.20	−1.92	55.98	4 54.06	2.46844	3
	44 37.01	29.90	22.27	4 59.05	. .	. .	4 57.90	2.47407	3
31									
Nov. 1	27 0.62	34.65	57.76	53.71	−2.25	53.42	51.17	2.46415	3
	44 35.27	38.50	19.26	58.22	. .	. .	55.67	2.47081	3
6	2.09	31.68	57.01	4 55.69	−1.97	4 55.82	4 53.85	2.46813	3½
	33.77	36.67	20.34	4 59.64	. .	. .	4 57.79	2.47391	3½
22	6.70	17 21.52	39 58.04	4 59.67	−1.44	5 0.25	4 58.81	2.47539	4
	28.22	32 37.98	7 20.06	5 2.56	. .	. .	5 1.69	2.47956	4
23									
Dec. 1	9.45	17 15.60	40 8.09	4 51.61	−2.06	4 51.98	4 49.92	2.46228	2½
	25.05	33 0.44	7 7.65	4 55.73	. .	. .	4 54.04	2.46841	2½
Dec. 6	−43 27 10.70	18 17 12.41	395 40 6.93	4 53.73	−2.19	4 53.71	4 51.52	2.46467	2
	+61 44 23.11	161 33 0.17	234 7 6.76	4 58.12	. .	. .	4 55.90	2.47115	2

Stars No. { 216 / 1002 l. c.

Date.	δ	A′ / B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	$90° - \frac{1}{2}(A' + B)$	r	log r	p
1889.									
Oct. 17	−41° 59′ 3″.57	20° 55′ 38″.21	394° 12′ 37″.28	4′ 13″.38	−4″.97	4′ 17″.19	4′ 12″.22	2.40178	3
	+62 54 41.78	158 55 47.40	235 16 49.88	4 23.33	. .	. .	4 22.16	2.41857	3
24	5.41	34.33	35.29	19.01	−5.12	22.53	17.41	2.41062	2
	39.74	40.60	54.69	29.26	. .	. .	27.65	2.42757	2
26	5.86	33.17	36.09	16.87	−5.26	20.82	15.56	2.40749	3½
	39.03	45.19	50.90	27.39	. .	. .	26.08	2.42501	3½
31									
Nov. 1	7.57	29.54	40.63	15.63	−5.15	19.11	13.96	2.40476	3
	37.11	52.23	48.40	25.93	. .	. .	24.26	2.42203	3
6	9.05	26.39	40.03	17.28	−5.08	20.79	15.71	2.40775	3½
	35.44	52.03	48.00	27.44	. .	. .	25.87	2.42467	3½
22	13.65	15.89	42.60	20.81	−4.98	25.64	20.66	2.41607	4
	29.54	53.82	48.78	30.77	. .	. .	30.62	2.43236	4
23	13.92	55 15.23	43.89	18.28	−4.87	23.62	18.75	2.41288	2
	29.15	55 57.52	46.37	28.02	. .	. .	28.49	2.42893	2
Dec. 1	16.42	55 9.88	50.87	13.82	−5.06	18.53	13.47	2.40393	3½
	26.30	56 12.96	37.91	23.94	. .	. .	23.59	2.42093	3½
Dec. 6	−41 59 17.69	20 55 6.63	394 12 51.95	4 15.68	−5.10	4 18.97	4 13.87	2.40461	3½
	+62 54 24.32	158 56 15.43	235 16 36.52	4 25.88	. .	. .	4 24.07	2.42172	3½

Stars No. { **216** / **1050 l. e.**

Date.	δ	A′ B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−41° 59′ 3″.57 +61 44 39.40	19° 45′ 35″.83 160 5 19.43	394° 12′ 37″.28 234 7 17.85	4′ 13″.38 4 54.79	−20″.70 . .	4′ 32″.37 . .	4′ 11″.67 4 53.07	2.40083 2.46697	3 3
24	5.41 37.65	32.24 9.14	35.29 26.15	4 19.01 5 1.84	−21.41 . .	39.31 . .	4 17.90 5 0.72	2.41145 2.47816	2½ 2½
26	5.86 37.01	32.15 13.82	36.09 22.27	4 16.87 4 59.05	−21.09 . .	37.01 . .	4 15.92 4 58.10	2.40810 2.47436	3½ 3½
31									
Nov. 1	7.57 35.27	27.70 21.37	40.63 19.26	15.63 58.22	−21.29 . .	35.46 . .	14.17 56.75	2.40512 2.47239	3 3
6	9.05 33.77	24.72 19.69	40.03 20.34	4 17.28 4 59.64	−21.18 . .	37.79 . .	4 16.61 4 58.97	2.40927 2.47563	3½ 3½
22	13.65 28.22	14.57 22.54	42.60 20.06	4 20.81 5 2.56	−20.87 . .	41.44 . .	4 20.57 5 2.31	2.41592 2.48045	4 4
23	13.92 27.83	13.91 28.87	43.89 15.02	4 18.28 5 0.41	−21.06 . .	38.61 . .	4 17.55 4 59.67	2.41086 2.47664	2 2
Dec. 1	16.42 25.05	8.63 43.22	50.87 7.65	4 13.82 4 55.73	−20.95 . .	34.07 . .	13.12 55.02	2.40333 2.46985	3½ 3½
Dec. 6	−41 59 17.69 +61 44 23.11	19 45 5.42 160 5 45.19	394 12 51.95 234 7 6.76	4 15.68 4 58.12	−21.22 . .	4 34.69 . .	4 13.47 4 55.91	2.40393 2.47116	2 2

Stars No. { **228** / **1014 l.e.**

Date.	δ	A′ B	C′	r′	$\frac{1}{2}(r'_s - r'_n)$	90° − ½ (A′+B)	r	log r	p
1899.									
Oct. 17	−50° 42′ 52″.77 +55 2 8.05	4° 19′ 15″.28 175 13 1.61	402° 46′ 6″.46 227 33 4.85	14′ 45″.74 13 22.47	+41″.63 . .	13′ 51″.55 . .	14′ 33″.18 13 9.92	2.94110 2.89758	3 3
24	42 54.78 2 6.18	19 11.40 12 13.94	45 50.85 33 36.91	15 8.82 13 42.17	+43.32 . .	14 17.33 . .	15 0.65 13 34.01	2.95456 2.91063	2 2
26									
31									
Nov. 1									
6									
22	43 3.76 1 56.49	18 52.74 12 16.72	45 42.27 33 25.55	15 18.14 13 49.62	+44.26 . .	14 25.27 . .	15 9.53 13 41.01	2.95882 2.91435	4½ 4½
23									
Dec. 1	6.73 53.35	18 46.62 13 16.23	46 16.66 33 0.43	14 49.50 13 25.50	+42.00 . .	13 58.57 . .	14 40.57 13 16.57	2.94476 2.90122	3 3
Dec. 6	−50 43 8.11 +55 1 51.41	4 18 43.30 175 13 11.75	402 46 15.88 227 33 4.13	14 58.42 13 33.42	+42.50 . .	14 2.47 . .	14 44.97 13 19.97	2.94693 2.90307	3 3

Stars No. { 270, 1002 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′ + B)	r	log r	p
1899. Oct. 17	−42° 32′ 14.″26	20° 22′ 27.″52	394° 45′ 34.″08	4′ 26.″46	+ 1.″56	4′ 24.″14	4′ 25.″70	2.42439	3
	+62 54 41.78	159 28 44.20	235 16 49.88	4 23.33	. .	. .	4 22.58	2.41926	3
24	16.06	23.68	32.60	32.68	+ 1.71	29 20	30.91	2.43282	3
	39.74	37.91	54.69	29.26	. .	. .	27.49	2.42731	3
26	16.50	22.53	33.68	30.40	+ 1.50	27.34	28.84	2.42949	1½
	39.03	42.78	50.90	27.39	. .	. .	25.84	2.42462	1½
31									
Nov. 1	18.16	18.95	37.46	29.40	+ 1.73	25.99	27.72	2.42768	3
	37.11	49.06	48.40	25.93	. .	. .	24.26	2.42203	3
6	19.66	15.78	38.83	30.69	+ 1.62	26.69	28.31	2.42864	3
	35.44	50.83	48.00	27.44	. .	. .	25.07	2.42336	3
22	24.43	22 5.11	40.86	33.52	+ 1.37	31.40	32.77	2.43580	3½
	29.54	28 52.08	48.78	30.77	. .	. .	30.03	2.43141	3½
23									
Dec. 1	27.37	21 58.93	49.00	27 17	+ 1.61	24.99	26.60	2.42586	3
	26.30	29 11.09	37.91	23.94	. .	. .	23.38	2.42058	3
Dec. 6	−42 32 28.80	20 21 55.52	394 45 49.27	4 29.22	+ 1.67	4 25.86	4 27.53	2.42737	3
	+62 54 24.32	159 29 12.75	235 16 36.52	4 25.88	. .	. .	4 24.19	2.42192	3

Stars No. { 297, 1002 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′ + B)	r	log r	p
1899. Oct. 17	−42° 3′ 2.″95	20° 51′ 38.″83	394° 16′ 33.″95	4′ 14.″82	− 4.″25	4′ 18.″55	4′ 14.″30	2.40535	3
	+62 54 41.78	158 59 44.07	235 16 49.88	4 23.33	. .	. .	4 22.80	2.41962	3
24	4.68	35.06	32.17	21.09	− 4.08	23.73	19.65	2.41439	2½
	39.74	37.48	54.69	29.26	. .	. .	27.81	2.42783	2½
26	5.09	34.94	34.38	18.34	− 4.52	20.79	16.27	2.40870	3
	39.03	43.48	50.90	27.39	. .	. . .	25.31	2.42375	3
31									
Nov. 1	6.67	31.44	37.63	18.02	− 3.95	19.61	15.66	2.40766	3
	37.11	49.23	48.40	25.93	. .	. .	23.56	2.42088	3
6									
22	12.92	20 51 17.62	43.90	21.29	− 4.74	23.63	18.89	2.41311	4½
	29.54	158 59 55.12	48.78	30.77	. .	. .	28.37	2.42873	4½
23									
Dec. 1	15.88	20 51 10.52	48.94	15.76	− 4.09	19.22	15.13	2.40676	3½
	26.30	159 0 11.03	37.91	23.94	. .	. .	23.31	2.42047	3½
Dec. 6	−42 3 17.38	20 51 6.94	394 16 48.72	4 17.54	− 4.17	4 20.43	4 16.26	2.40868	3
	+62 54 24.32	159 0 12.20	233 16 36.52	4 25.88	. .	. .	4 24.60	2.42259	3

Stars No. { 270 / 1050 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′ + B)	r	log r	p
1899.									
Oct. 17	−42° 32′ 14″.26	19° 12′ 25″.14	394° 45′ 34″.08	4′ 26″.46	−14″.16	4′ 39″.31	4′ 25″.15	2.42349	3
	+61 44 39.40	160 38 16.23	234 7 17.85	4 54.79	. .	. .	4 53.47	2.46756	3
24	16.06	21.59	32.60	32.68	−14.58	45.98	31.40	2.43361	3½
	37.65	6.45	26.15	61.84	. .	. .	60.56	2.47793	3½
26	16.50	20.51	33.68	30.40	−14.32	44.04	29.72	2.43091	1½
	37.01	11.41	22.27	59.05	. .	. .	58.36	2.47474	1½
31									
Nov. 1	18.16	17.11	37.46	29.40	−14.41	42.34	27.93	2.42802	3
	35.27	18.20	19.26	58.22	. .	. .	56.75	2.47239	3
6	19.66	14.11	38.83	30.69	−14.47	43.70	29.23	2.43012	3
	33.77	18.49	20.34	59.64	. .	. .	58.17	2.47446	3
22	24.43	12 3.79	40.86	33.52	−14.52	47.70	33.18	2.43645	3½
	28.22	38 20.80	20.06	62.56	. .	. .	62.22	2.48042	3½
23									
Dec. 1	27.37	11 57.68	49.00	27.17	−14.28	40.48	26.20	2.42521	3
	25.05	38 41.35	7.65	55.73	. .	. .	54.76	2.46947	3
Dec. 6	−42 32 28.80	19 11 54.31	394 45 49.27	4 29.22	−14.45	4 41.59	4 27.14	2.42674	2
	+61 44 23.11	160 38 42.51	234 7 6.76	4 58.12	. .	. .	4 56.04	2.47135	2

Stars No. { 290 / 1084 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′ + B)	r	log r	p
1899.									
Oct. 17	−30° 45′ 48″.44	45° 22′ 9″.66	383° 1′ 31″.74	2′ 2″.27	+ 4″.47	1′ 57″.76	2′ 2″.23	2.08718	3
	+76 7 58.10	134 33 54.81	248 27 36.93	1 53.33	. .	. .	1 53.29	2.05419	3
24	49.93	6.28	32.02	5.21	+ 4.64	2 0.50	5.14	2.09740	3
	56.21	52.71	39.31	55.93	. .	. .	55.86	2.06393	3
26	50.28	5.26	32.90	3.96	+ 4.48	1 59.14	3.62	2.09209	4
	55.54	56.46	36.44	54.99	. .	. .	54.66	2.05941	4
31									
Nov. 1	51.67	22 2.05	34.48	3.73	+ 4.58	59.43	4.01	2.09346	2
	53.72	33 59.09	35.39	54.57	. .	. .	54.85	2.06013	3
6	52.95	21 59.19	36.39	4.28	+ 4.58	1 59.60	4.18	2.09405	3½
	52.14	34 1.61	34.78	55.12	. .	. .	55.02	2.06077	3½
22	57.11	49.33	41.83	5.36	+ 4.55	2 0.60	5.15	2.09743	4
	46.44	9.46	32.37	56.26	. .	. .	56.05	2.06464	4
23									
Dec. 1	45 59.73	43.52	45.01	2.70	+ 4.53	1 58.21	2.74	2.08899	3½
	7 43.25	20.05	24.96	53.63	. .	. .	53.68	2.05568	3½
Dec. 6	−30 46 1.03	45 21 40.28	383 1 45.72	2 3.51	+ 4.53	1 58.87	2 3.40	2.09131	2½
	+76 7 41.31	134 34 21.98	248 27 23.74	1 54.44	. .	. .	1 54.34	2.05820	2½

Stars No. { 290 / 1045 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′ + B)	r	log r	p
1899.									
Oct. 17	−30° 45′ 48″.44	45° 13′ 33″.40	383° 1′ 31″.74	2′ 2″.27	+ 4″.08	1′ 58″.06	2′ 2″.14	2.08686	3
	+75 59 21.84	134 42 30.47	248 19 1.27	1 54.10	. .	. .	1 53.98	2.05683	3
24	49.93	30.08	32.02	5.21	+ 4.22	2 0.83	5.05	2.09708	3½
	20.01	28.26	3.76	56.77	. .	. .	56.61	2.06674	3½
26	50.28	29.08	1 32.90	3.96	+ 4.11	1 59.19	3.30	2.09096	4
	19.36	32.54	19 0.36	55.73	. .	. .	55.08	2.06100	4
31									
Nov. 1	51.67	25.89	1 34.48	3.73	+ 4.17	59.73	3.90	2.09307	3
	17.56	34.64	18 59.84	55.39	. .	. .	55.56	2.06281	3
6	52.95	24.08	36.39	4.28	+ 4.18	1 59.34	3.52	2.09174	3½
	16.03	37.24	59.15	55.91	. .	. .	55.16	2.06130	3½
22	57.11	13.31	41.83	5.36	+ 4.18	2 0 40	4.58	2.09545	4
	10.42	45.89	55.94	57.00	. .	. .	56.22	2.06528	4
23									
Dec. 1	45 59.73	7.51	45.01	2.70	+ 4.13	1 57.55	1.68	2.08522	3½
	59 7.24	56.39	48.62	54.43	. .	. .	53.42	2.05469	3½
Dec. 6	−30 46 1.03	45 13 4.28	383 1 45.72	2 3.51	+ 4.14	1 59.04	2 3.18	2.09054	2½
	+75 59 5.31	134 42 57.64	248 18 48.08	1 55.22	. .	. .	1 54.90	2.06032	2½

Stars No. { 279 / 1050 l. c.

Date.	δ	A′ / B	C′	r′	½ (r′ₛ − r′ₙ)	90° − ½ (A′ + B)	r	log r	p
1899.									
Oct. 17	−42° 3′ 2″.95	19° 41′ 36″.45	394° 16′ 33″.95	4′ 14″.82	− 19″.98	4′ 33″.72	4′ 13″.74	2.40439	3
	+61 44 39.40	160 9 16.10	234 7 17.85	4 54.79	. .	. .	4 53.70	2.46790	3
24	4.68	32.97	32.17	21.09	− 20.37	40.50	4 20.13	2.41519	3
	37.65	6.02	26.15	61.84	. .	. .	5 0.87	2.47838	3
26	5.09	31.92	34.38	18.34	− 20.35	37.98	4 17.63	2.41100	3
	37,01	12.11	22.27	59.05	. .	. .	4 58.33	2.47470	3
31									
Nov. 1	6.67	28.60	37.63	18.02	− 20.10	36.51	4 16.41	2.40893	3
	35.27	18.37	19.26	58.22	. .	. .	4 56.61	2.47219	3
6									
22	12.92	15.30	43.90	21.29	− 20.63	40.43	4 19.80	2.41464	4½
	28.22	23.84	20.06	62.56	. .	. .	5 1.06	2.47865	4½
23									
Dec. 1	15.88	9.17	48.94	15.76	− 19.98	34.77	4 14.79	2.40618	3½
	25.05	41.29	7.65	55.73	. .	. .	4 54.75	2.46945	3½
Dec. 6	−42 3 17.38	19 41 5.73	394 16 48.72	4 17.54	− 20.29	4 36.15	4 15.86	2.40800	2
	+61 44 23.11	160 9 41.96	234 7 6.76	4 58.12	. .	. .	4 56.44	2.47194	2

The next table is given as a sample of the reduction for $\Delta = d\log r = d\log a$. The second column contains the logarithms of the computed refractions; the next contains the logarithms of the observed refractions; and the last, the difference between the two preceding in the sense O−C; the column p contains the weights.

In the computation of Δ the $\log r$ of the stars that have been grouped with more than one other star is taken to be the weighted mean of the logs r from all the determinations. The weight assigned to it is the mean of the weights of the individual determinations to the nearest half unit, in all cases where the number to be combined is the same for every date. In cases where different dates have different numbers combined the mean of the weights is multiplied by the square root of the number of determinations taken into account, and this product, taken to the nearest half unit, is the adopted weight.

Star No. 5.

Date.	log r′	log r	Δ	p
Oct. 17	2.60700	2.60307	−0.00393	5
24	1694	1317	377	3
26	1264	0981	283	4½
31	1731	1556	175	2
Nov. 1	1091	0712	379	4½
6	1541	1367	174	5
22	1964	1699	265	5½
23	1784	1369	415	1½
Dec. 1	0762	0467	295	4
Dec. 6	2.61216	2.60811	−0.00405	3

[p] = 38 $\qquad \Delta = -0.00310 \quad \pm 0.00018$

log [p v v] = 5.4101

The values of Δ are collected in the following table. The weight for each Δ has been derived from the probable error r of the Δ. The approximate zenith distance of each star is also given.

Star No.	Δ	r	p	Z. D.	Star No.	Δ	r	p	Z. D.
5	−0.00310	±0.00018	12	84°	174	− 28	39	3	52
9	− ·259	13	23	73	179	− 81	26	6	70
788	− 299	20	10	84	944	+ 40	31	4	68
793	− 221	14	20	72	187	− 198	23	7	78
38	− 281	29	5	84	190	+ 1275	147	1:6	16
49	+ 316	49	2	30	202	− 58	20	10	67
58	+ 48	28	5	67	210	− 263	19	11	81
66	− 317	22	8	84	976	− 13	33	4	70
76	+ 236	56	1	30	216	− 177	24	7	79
79	+ 192	62	1	34	223	− 501	37	3	88
83	+ 139	36	3	30	235	+ 238	29	5	47
89	− 26	39	3	51	1002	− 232	25	6	80
96	+ 184	45	2	35	1014	− 554	40	2	87
103	+ 424	54	1	30	1019	− 329	21	9	84
108	+ 446	44	2	29	264	+ 240	32	4	48
885	− 201	17	13	78	270	− 208	20	10	80
136	+ 281	61	1	29	1034	− 27	20	10	66
139	+ 35	27	5	68	1045	− 167	18	12	67
149	+ 293	71	1	30	1050	− 182	14	20	81
153	+ 95	61	1	29	290	− 64	13	23	68
911	− 15	22	8	66	297	−0.00235	±0.00021	·9	79
161	+ 1255	161	1:7	16					

It will be noticed that for this series also, Δ is clearly a function of the zenith distance, thus confirming the result found from the reduction of the summer series that the Constant of Refraction is a function of the zenith distance. Upon plotting these values, using the zenith distance z for abscissa, and Δ for ordinate, it is seen that a straight line, inclined about 145° to the z axis, and cutting it at $z =$ about 60° appears to represent Δ fairly well. Accordingly, assuming Z to be the zenith distance for $\Delta = 0$, we can set up an observation equation of the following type for each star,

$$\log a \doteq \log a_0 + (Z - z)\,x$$

or

$$\log a - \log a_0 = \Delta = Zx - zx = D - zx$$

where

$$D = Zx$$

and where a_0 is the Constant of Refraction of the tables used (Pulkowa).

Equations of this kind were, therefore, formed and solved for Z and x by the method of Least Squares. The details of the solution for the autumn series alone will not be given here. The result of the solution is

$$x = +0.000117 \pm 0.000011$$
$$Z = 59^{\circ}.3 \pm 8^{\circ}.7$$

The solution of the summer series alone gave

$$x = +0.000101 \pm 0.000013$$
$$Z = 56^{\circ}.6 \pm 12^{\circ}.2$$

The details of the solution for the two series combined are here given. To reduce the number of equations those nearly alike were combined into one equation, and given a weight equal to the sum of the weights of the equations combined.

This leads to the following equations of condition:

No.	a	b	n	p
1	D	$-79.63\,x$	$= -0.00191$	138
2		-29.92	$= +\ 368$	16
3		-87.10	$= -\ 352$	32
4		-67.29	$= -\ 56$	94
5		-83.57	$= -\ 296$	167
6		-88.78	$= -\ 485$	10
7		-21.47	$= +\ 385$	7
8		-57.31	$= -\ 117$	29
9		-27.15	$= -\ 103$	5
10		-62.75	$= -\ 7$	25
11		-65.34	$= +\ 41$	18
12		-72.65	$= -\ 241$	43
13		-34.86	$= +\ 187$	3
14		-51.51	$= -\ 27$	6
15		-77.83	$= -\ 200$	20
16		-15.78	$= +\ 1266$	0.31
17		-70.25	$= -\ 54$	10
18	D	$-47.65\,x$	$= +0.00239$	9

WEIGHTED OBSERVATION EQUATIONS.

No.	a	b	n
1	11.75 D	− 935.7 x	= −0.02244
2	4.00	− 119.7	= + 1472
3	5.66	− 493.0	= − 1992
4	9.70	− 652.7	= − 543
5	12.92	−1079.7	= − 3824
6	3.16	− 280.5	= − 1533
7	2.65	− 56.9	= + 1020
8	5.39	− 308.9	= − 631
9	2.24	− 60.8	= − 231
10	5.00	− 313.7	= − 35
11	4.24	− 277.0	= + 174
12	6.56	− 476.6	= − 1581
13	1.73	− 60.3	= + 324
14	2.45	− 126.2	= − 66
15	4.47	− 347.9	= − 894
16	0.56	− 8.8	= + 709
17	3.16	− 222.0	= − 171
18	3.00 D	− 142.9 x	= +0.00717

To render these more nearly homogeneous, let $10D = d$; $1000x = y$; and multiply the absolute term by 100.

WEIGHTED HOMOGENEOUS OBSERVATION EQUATIONS.

No.	a	b	n
1	1.175 d	−0.936 y	= −2.244
2	0.400	−0.120	= +1.472
3	0.566	−0.493	= −1.992
4	0.970	−0.653	= −0.543
5	1.292	−1.080	= −3.824
6	0.316	−0.280	= −1.533
7	0.265	−0.057	= +1.020
8	0.539	−0.309	= −0.631
9	0.224	−0.061	= −0.231
10	0.500	−0.314	= −0.035
11	0.424	−0.277	= +0.174
12	0.656	−0.477	= −1.581
13	0.173	−0.060	= +0.324
14	0.245	−0.126	= −0.066
15	0.447	−0.348	= −0.894
16	0.056	−0.009	= +0.709
17	0.316	−0.222	= −0.171
18	0.300 d	−0.143	= +0.717

Combining these by the method of Least Squares we obtain the following

NORMAL EQUATIONS.

$$+6.325d - 4.637y = -10.389$$
$$-4.637 \quad + 3.520 \quad = + \;8.916$$

The solution of these gives

$$x = +0.000108 \pm 0.000009$$
$$Z = 58^{\circ}.1 \pm 7^{\circ}.8$$

It is evident, therefore, that the Constant of Refraction used in the Pulkowa tables needs not only a correction, but a correction for every zenith distance. In other words, the formulæ from which the refractions are computed, and hence the theory also, need to be modified. Or, the formulæ may be retained unchanged and the desired agreement between observation and computation may be obtained by correcting the tables used (Pulkowa) not by a constant amount, but by an amount represented in magnitude by

$$\Delta\,\mu = +\,0.000108\,(58^{\circ} - z)$$

where z is the zenith distance in degrees. The table for μ + log tg$z$ given hereafter is the Pulkowa table corrected by this amount.

The result of this investigation leads to an interesting question, viz., Could not the different values of the Constant of Refraction determined by various observers be brought into harmony by reducing them to the same zenith distance

by this formula? The average zenith distance of the stars used in these determinations are undoubtedly different in every case, and this may account for the varying results. This is an investigation which the writer hopes to undertake as soon as the data and the time are available.

The reductions for latitude and a readjustment of the declinations of the stars used in these two series, based upon the observations made, will form the subject matter of a later paper.

In his preliminary reductions of his observations of the PIAZZI southern stars TUCKER (*Publications of the Lick Observatory*, Vol. VI) has used the Pulkowa refractions. In his final reductions, he has corrected the declinations for errors in refraction by the amounts exhibited below

δ	Z. D.	$\Delta\delta$
°		″
— 20	57	+ 0.07
— 25	62	+ 0.14
— 30	67	+ 0.24
— 35	72	+ 0.39
— 40	77	+ 0.65
— 45	82	+ 2.44

Applying to these AUWERS' corrections (*Abhandlungen zu den A. N.* No. 7) we would have, for the same declinations, the amounts exhibited in the second column below.

Z. D.	$\Delta\delta$	$\Delta\delta$	A – C	p. e.
°	″	″	′	″
57	+ 0.07	— 0.02	+ 0.09	± 0.25
62	+ 0.12	+ 0.10	+ 0.02	0.26
67	+ 0.19	+ 0.29	— 0.10	0.37
72	+ 0.41	+ 0.60	— 0.19	0.49
77	+ 0.90	+ 1.15	— 0.25	0.77
82	+ 2.84	+ 2.29	+ 0.55	± 1.00

In the third column are the corrections which would have resulted by applying the correction, found on page 198, to the Pulkowa tables. The fourth column is the difference between the second and third (in the sense AUWERS–CRAWFORD). The fifth column contains the probable errors of a single observation given by TUCKER (Vol. VI *Publication of the Lick Observatory*) for stars observed at zenith distances indicated in the first column. This table shows conclusively that had this series of PIAZZI stars been reduced originally with the refractions found from this investigation, any error in the observed positions due to errors of refraction would have been much less than the probable error of the observation itself.

REFRACTION TABLES.

REFRACTION TABLES.

In order that all the refraction tables may be conveniently together the three tables giving the terms B, γ and T published in Volume I of these *Publications* are given again here as Tables I, II, III respectively. Table IV gives A, λ and σ. Table V gives i. [Tables IV and V are reprinted from the Pulkowa Tables.] Table VI is the Pulkowa table for $\mu + \log \mathrm{tg} z$ corrected by

$$\Delta \mu = + 0.000108 \, (58^\circ - z)$$

where z is in degrees. The refractions are then to be computed from these tables by the formula

$$\log \mathrm{Refr.} = \mu + \log \mathrm{tg} z + \mathrm{A} \, (\mathrm{B} + \mathrm{T}) + \lambda \gamma - \sigma i.$$

R. T. C.

TABLE I.

Values of the Refraction Term B.

(Units of the fifth decimal place.)

Bar. Inches.	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	With left hand col.
											20 × Bar.
25.0	- 7330	- 7313	- 7296	- 7278	- 7261	- 7243	- 7226	- 7209	- 7191	- 7174	500
.1	7157	7140	7122	7105	7088	7070	7053	7036	7019	7001	502
.2	6984	6967	6950	6933	6915	6898	6881	6864	6847	6829	504
.3	6812	6795	6778	6761	6744	6726	6709	6692	6675	6658	506
.4	6641	6624	6607	6590	6573	6555	6538	6521	6504	6487	508
.5	- 6470	- 6453	- 6436	- 6419	- 6402	- 6385	- 6368	- 6351	- 6334	- 6317	510
.6	6300	6283	6266	6249	6232	6215	6199	6182	6165	6148	512
.7	6131	6114	6097	6080	6063	6046	6030	6013	5996	5979	514
.8	5962	5945	5929	5912	5895	5878	5861	5845	5828	5811	516
.9	5794	5777	5761	5744	5727	5711	5694	5677	5660	5644	518
26.0	- 5627	- 5610	- 5594	- 5577	- 5560	- 5543	- 5527	- 5510	- 5493	- 5477	520
.1	5460	5444	5427	5410	5394	5377	5360	5344	5327	5311	522
.2	5294	5278	5261	5244	5228	5211	5195	5178	5162	5145	524
.3	5129	5112	5096	5079	5063	5046	5030	5013	4997	4980	526
.4	4964	4947	4931	4915	4898	4882	4865	4849	4832	4816	528
.5	- 4800	- 4783	- 4767	- 4751	- 4734	- 4718	- 4701	- 4685	- 4669	- 4652	530
.6	4636	4620	4603	4587	4571	4555	4538	4522	4506	4489	532
.7	4473	4457	4441	4424	4408	4392	4376	4359	4343	4327	534
.8	4311	4295	4278	4262	4246	4230	4214	4197	4181	4165	536
.9	4149	4133	4117	4101	4084	4068	4052	4036	4020	4004	538
27.0	- 3988	- 3972	- 3956	- 3940	- 3923	- 3907	- 3891	- 3875	- 3859	- 3843	540
.1	3827	3811	3795	3779	3763	3747	3731	3715	3699	3683	542
.2	3667	3651	3635	3619	3603	3587	3572	3556	3540	3524	544
.3	3508	3492	3476	3460	3444	3428	3413	3397	3381	3365	546
.4	3349	3333	3317	3302	3286	3270	3254	3238	3222	3207	548
.5	- 3191	- 3175	- 3159	- 3144	- 3128	- 3112	- 3096	- 3080	- 3065	- 3049	550
.6	3033	3018	3002	2986	2970	2955	2939	2923	2908	2892	552
.7	2876	2860	2845	2829	2814	2798	2782	2767	2751	2735	554
.8	2720	2704	2689	2673	2657	2642	2626	2610	2595	2579	556
.9	2564	2548	2533	2517	2502	2486	2470	2455	2439	2424	558
28.0	- 2408	- 2393	- 2377	- 2362	- 2346	- 2331	- 2315	- 2300	- 2285	- 2269	560
With right hand col.	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	20 × Bar.

N. B.—The left hand column and top argument give the whole argument for a barometer reading in inches; the right hand column and bottom argument give the entry for 20 times the barometer reading in inches.

TABLE II.

VALUES OF THE REFRACTION TERM γ.

(Units of the fifth decimal place.)

t. Fahr.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0°	+4412	+4318	+4222	+4128	+4032	+3938	+3846	+3750	+3658	+3562	9	19	28	37	47	56	66	75	84
10	3470	3376	3284	3190	3099	3008	2914	2823	2730	2640	9	18	28	37	46	55	64	74	83
20	2547	2457	2365	2275	2184	2094	2004	1913	1824	1734	9	18	27	36	45	54	63	72	81
30	1645	1554	1466	1378	1288	1201	1111	1023	934	847	9	18	27	35	44	53	62	71	80
40	+ 759	+ 672	+ 586	+ 497	+ 411	+ 324	+ 238	+ 150	+ 65	− 22	9	17	26	35	43	52	61	69	78
50	− 107	193	279	− 364	− 450	− 534	− 620	− 704	− 789	− 873	9	17	26	34	43	51	60	68	77
60	957	1042	1125	1209	1293	1377	1460	1543	1626	1708	8	17	25	33	42	50	58	67	75
70	1791	1873	1956	2038	2121	2202	2285	2366	2446	2528	8	16	24	33	41	49	57	65	73
80	2609	2691	2771	2852	2932	3014	3093	3172	3253	3334	8	16	24	32	40	48	56	64	72
90	−3414	−3493	−3573	−3652	3731	−3810	3889	−3967	−4046	−4124	8	16	23	31	39	47	55	63	70

TABLE III.

VALUES OF REFRACTION TERM T.

(Units of the fifth decimal place.)

t'. (Fahr.)	0	1	2	3	[illegible]	5	6	7	8	9
0°	+125	+121	+117	+113	+109	+105	+101	+ 98	+ 94	+ 90
10	+ 86	+ 82	+ 78	+ 74	+ 70	+ 66	+ 62	+ 58	+ 55	+ 51
20	+ 47	+ 43	+ 39	+ 35	+ 31	+ 27	+ 23	+ 20	+ 16	+ 12
30	+ 8	+ 4	0	− 4	− 8	− 12	− 16	− 20	− 23	− 27
40	− 31	− 35	− 39	− 43	− 47	− 51	− 55	− 58	− 62	− 66
50	− 70	− 74	− 78	− 82	− 86	− 90	− 93	− 97	−101	−105
60	−109	−113	−117	−121	−124	−128	−132	−136	−140	−144
70	−148	−152	−156	−160	−164	−168	−171	−175	−179	−183
[illegible]	−187	−191	−195	−199	−202	−206	−210	−214	−218	−221
90	−225	−229	−233	−237	−241	−244	−248	−252	−256	−260

TABLE IV.

Values of the Refraction Terms λ, A and σ.

z	λ	A	σ
45°	1.0018		
50	1.0022		
55	1.0029		
60	1.0044		
61	1.0047		
62	1.0051		
63	1.0055		
64	1.0059		
65	1.0064		
66	1 0070		
67	1.0077		
68	1.0085		
69	1.0093		
70	1.0103		
71	1.0115		
72	1.0130		
73	1.0147		
74	1.0166		
75 0′	1.0188		
75 30	1.0200		
76 0	1.0216		
76 30	1.0235		
77 0	1.0253	1.0029	
77 30	1.0271	1.0030	
78 0	1.0293	1.0033	
78 30	1.0318	1.0035	
79 0	1.0344	1.0038	
79 30	1.0374	1.0041	
80 0	1.0409	1.0044	0.00019
10	1.0421		
20	1.0433		
30	1.0447	1.0048	0.00022
40	1.0461		
50	1.0475		
81 0	1.0491	1.0052	0.00025
10	1.0508		
20	1.0525		
30	1.0542	1.0057	0.00029
40	1.0561		
50	1.0580		
82 0	1.0600	1.0063	0.00035
10	1.0622		
20	1.0645		
30	1.0669	1.0070	0.00045
40	1.0694		
50	1.0720		
83 0	1.0747	1.0078	0.00057
10	1.0776		
20	1.0807		
30	1.0839	0.0087	0.00073
40	1.0874		
50	1.0911		
84 0	1.0949	1.0098	0.00091
10	1.0990		
20	1.1034		
30	1.1080	1.0112	0.00116
40	1.1128		
50	1.1180		
85 0	1.1235	1.0127	0.00146

z	λ	A	σ
85° 0′	1.1235	1.0127	0.00146
5	1264		
10	1294		
15	1325		
20	1357	141	170
25	1390		
30	1424		
35	1459		
40	1495	156	202
45	1532		
50	1571		
55	1611		
86 0	1.1652	1.0172	0.00241
5	1695		
10	1740	181	264
15	1786		
20	1833	192	290
25	1882		
30	1934	203	320
35	1987		
40	2040	214	352
45	2095		
50	2153	227	386
55	2214		
87 0	1.2277	1.0241	0.00421
2	2303		
4	2329	247	437
6	2356		
8	2383	254	453
10	2410		
12	2438	261	471
14	2466		
16	2495	268	489
18	2524		
20	2554	275	509
22	2584		
24	2615	282	529
26	2646		
28	2677	290	550
30	2708		
32	2740	297	572
34	2772		
36	2805	305	594
38	2839		
40	2873	313	617
42	2908		
44	2942	321	641
46	2978		
48	3014	328	666
50	3051		
52	3088	338	693
54	3125		
56	3163	347	721
87 58	3202		
88 0	1.3241	1.0357	0.00749

TABLE VI.

$\mu + \log \text{tang}\, z$.

z	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	z
0'	$-\infty$	0.0083	0.3094	0.4856	0.6108	0.7080	0.7875	0.8550	0.9135	0.9653	0'
1	8.230	155	130	880	126	094	888	560	144	661	1
2	.531	226	166	904	144	109	900	571	153	669	2
3	.707	295	201	928	162	123	912	581	162	678	3
4	8.832	0.0364	0.3236	0.4952	.06180	0.7138	0.7924	0.8591	0.9172	0.9686	4
5	8.929	0.0431	0.3271	0.4975	0.6198	0.7152	0.7936	0.8602	0.9181	0.9694	5
6	9.008	497	306	0.4999	215	166	948	612	190	702	6
7	.075	563	341	0.5022	233	180	960	622	199	710	7
8	.133	627	375	045	250	195	972	632	208	718	8
9	9.184	0.0691	0.3408	0.5068	0.6268	0.7209	0.7983	0.8643	0.9217	0.9726	9
10	9.230	0.0753	0.3442	0.5091	0.6285	0.7223	0.7995	0.8653	0.9226	0.9734	10
11	.271	815	475	114	303	237	0.8007	663	235	742	11
12	.309	875	508	137	320	251	019	673	244	750	12
13	.344	935	541	160	338	265	031	683	253	758	13
14	9.376	0.0994	0.3574	0.5182	0.6355	0.7279	0.8042	0.8694	0.9262	0.9766	14
15	9.406	0.1053	0.3606	0.5205	0.6372	0.7293	0.8054	0.8704	0.9271	0.9774	15
16	.434	110	638	227	389	307	066	714	279	782	16
17	.460	167	670	249	406	320	077	724	288	790	17
18	.485	223	702	271	423	334	089	734	297	798	18
19	9.509	0.1279	0.3733	0.5293	0.6440	0.7348	0.8101	0.8744	0.9306	0.9806	19
20	9.531	0.1333	0.3764	0.5315	0.6457	0.7362	0.8112	0.8754	0.9315	0.9814	20
21	.552	387	795	336	473	375	124	764	324	822	21
22	.572	440	826	358	490	389	135	774	332	829	22
23	.592	493	856	380	507	402	147	784	341	836	23
24	9.610	0.1545	0.3887	0.5401	0.6523	0.7416	0.8158	0.8794	0.9350	0.9844	24
25	9.628	0.1596	0.3917	0.5422	0.6540	0.7429	0.8169	0.8803	0.9359	0.9852	25
26	.645	647	947	443	556	443	181	813	367	860	26
27	.661	698	0.3976	464	572	456	192	823	376	868	27
28	.677	747	0.4006	485	589	469	203	833	384	876	28
29	9.692	0.1796	0.4035	0.5506	0.6605	0.7483	0.8215	0.8843	0.9392	0.9883	29
30	9.707	0.1845	0.4064	0.5527	0.6621	0.7496	0.8226	0.8852	0.9401	0.9891	30
31	.721	893	093	548	637	509	237	862	410	899	31
32	.735	940	122	568	653	522	248	872	418	907	32
33	.749	0.1987	150	589	669	535	259	881	427	914	33
34	9.762	0.2034	0.4179	0.5609	0.6685	0.7549	0.8271	0.8890	0.9436	0.9922	34
35	9.774	0.2080	0.4207	0.5630	0.6701	0.7562	0.8282	0.8900	0.9444	0.9930	35
36	.786	125	235	650	717	575	292	910	453	937	36
37	.798	170	263	670	733	588	303	919	461	945	37
38	.810	215	290	690	748	601	314	929	470	953	38
39	9.821	0.2259	0.4317	0.5710	0.6764	0.7613	0.8325	0.8938	0.9478	0.9961	39
40	9.832	0.2303	0.4345	0.5730	0.6780	0.7625	0.8336	0.8948	0.9487	0.9968	40
41	.843	346	372	749	795	638	347	957	495	976	41
42	.853	389	399	769	811	651	358	967	504	983	42
43	.864	431	426	789	826	664	369	976	512	991	43
44	9.874	0.2473	0.4452	0.5808	0.6841	0.7677	0.8379	0.8986	0.9520	0.9999	44
45	9.883	0.2515	0.4479	0.5828	0.6856	0.7689	0.8390	0.8995	0.9529	1.0006	45
46	.893	556	505	847	871	702	401	0.9005	537	014	46
47	.902	597	531	866	886	714	412	014	546	021	47
48	.911	637	557	885	902	727	422	024	554	029	48
49	9.920	0.2677	0.4583	0.5904	0.6917	0.7740	0.8433	0.9033	0.9562	1.0036	49
50	9.929	0.2717	0.4608	0.5922	0.6932	0.7752	0.8444	0.9043	0.9571	1.0044	50
51	.938	756	634	941	947	765	455	052	579	051	51
52	.946	795	659	960	962	777	465	061	587	059	52
53	.954	834	685	979	977	790	476	071	595	066	53
54	9.962	0.2872	0.4709	0.5997	0.6991	0.7802	0.8486	0.9080	0.9604	1.0074	54
55	9.970	0.2909	0.4734	0.6016	0.7006	0.7814	0.8497	0.9089	0.9612	1.0081	55
56	.978	947	758	034	021	827	508	098	620	089	56
57	.986	0.2984	783	053	036	839	518	107	629	096	57
58	9.994	0.3021	808	071	050	851	529	117	637	104	58
59	0.0011	0.3058	0.4832	0.6089	0.7065	0.7863	0.8539	0.9126	0.9645	1.0111	59
60	0.0083	0.3094	0.4856	0.6108	0.7080	0.7875	0.8550	0.9135	0.9653	1.0118	60

TABLE VI.

$\mu + \log \text{tang}\, z$.

z	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	z
0′	1.0118	1.0540	1.0928	1.1286	1.1618	1.1929	1.2223	1.2500	1.2764	1.3014	0′
1	126	547	934	291	623	934	227	505	768	018	1
2	133	554	940	297	628	940	232	509	772	023	2
3	140	561	946	303	634	945	237	514	776	027	3
4	148	567	952	309	639	950	242	518	781	031	4
5	155	574	959	314	644	955	246	523	785	035	5
6	162	581	965	319	650	960	251	527	789	039	6
7	170	587	971	325	655	965	256	532	794	043	7
8	177	594	977	330	660	970	261	536	798	047	8
9	1.0184	1.0601	1.0983	1.1336	1.1666	1.1975	1.2266	1.2541	1.2802	1.3051	9
10	192	1.0607	1.0989	1.1342	1.1671	1.1980	1.2270	1.2545	1.2806	055	10
11	199	614	1.0995	348	676	985	275	550	811	059	11
12	206	620	1.1001	353	682	990	280	554	815	063	12
13	213	626	007	359	687	1.1995	284	559	819	067	13
14	221	633	013	365	692	1.2000	289	563	823	072	14
15	228	640	019	370	698	005	294	567	828	076	15
16	235	646	025	376	703	010	299	572	832	080	16
17	242	653	031	382	708	015	303	576	836	084	17
18	249	659	037	387	714	020	308	581	840	088	18
19	1.0256	1.0666	1.1043	1.1393	1.1719	1.2025	1.2313	1.2585	1.2845	1.3092	19
20	1.0263	1.0673	1.1049	1.1399	1.1724	1.2030	1.2317	1.2590	1.2849	1.3096	20
21	270	679	055	404	729	034	322	594	853	100	21
22	277	686	061	410	735	039	327	599	857	104	22
23	284	692	067	415	740	044	331	603	861	108	23
24	291	699	073	421	745	049	336	607	866	112	24
25	298	705	079	427	750	054	341	612	870	116	25
26	305	712	086	432	756	059	345	616	874	120	26
27	312	718	092	438	761	064	350	621	878	124	27
28	319	725	098	443	766	069	355	625	883	128	28
29	1.0327	1.0731	1.1104	1.1449	1.1771	1.2074	1.2359	1.2630	1.2887	1.3132	29
30	1.0334	1.0738	1.1109	1.1454	1.1776	1.2079	1.2364	1.2634	1.2891	1.3136	30
31	341	744	115	460	782	084	369	638	895	140	31
32	348	751	121	466	787	089	373	643	899	144	32
33	355	757	127	471	792	093	378	647	904	147	33
34	362	763	133	477	797	098	382	652	908	151	34
35	369	770	139	482	803	103	387	656	912	155	35
36	376	776	145	488	808	108	392	660	916	159	36
37	383	783	151	493	813	113	396	665	920	163	37
38	390	789	157	499	818	118	401	669	924	167	38
39	1.0397	1.0795	1.1163	1.1504	1.1823	1.2123	1.2405	1.2673	1.2928	1.3171	39
40	1.0404	1.0802	1.1169	1.1510	1.1828	1.2128	1.2410	1.2678	1.2932	1.3175	40
41	411	808	175	515	833	132	415	682	936	179	41
42	418	815	181	521	839	137	419	687	940	183	42
43	424	821	187	526	844	142	424	691	944	187	43
44	431	827	192	532	849	147	428	695	948	191	44
45	438	834	198	537	854	152	433	699	952	195	45
46	445	840	204	543	859	157	437	703	957	199	46
47	452	846	210	548	864	161	442	707	961	203	47
48	459	853	216	554	869	166	447	712	965	207	48
49	1.0466	1.0859	1.1222	1.1559	1.1874	1.2171	1.2451	1.2716	1.2969	1.3211	49
50	1.0473	1.0865	1.1228	1.1565	1.1879	1.2176	1.2455	1.2720	1.2973	1.3215	50
51	479	871	233	570	885	181	459	725	977	219	51
52	486	878	239	575	890	186	464	729	981	223	52
53	493	884	245	581	895	190	468	733	986	227	53
54	500	890	251	586	900	194	473	738	990	230	54
55	507	897	257	592	904	199	477	742	994	234	55
56	514	903	262	597	909	204	482	746	1.2998	238	56
57	520	909	268	602	914	208	486	750	1.3002	242	57
58	527	915	274	608	919	213	491	755	006	246	58
59	1.0534	1.0921	1.1280	1.1613	1.1924	1.2218	1.2496	1,2759	1.3010	1.3250	59
60	1.0540	1.0928	1.1286	1.1618	1.1929	1.2223	1.2500	1.2764	1.3014	1.3254	60

TABLE VI.

$\mu + \log \text{tang } z$.

z	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	z
0′	1.3254	1.3484	1.3705	1.3919	1.4125	1.4325	1.4519	1.4707	1 4891	1.5070	0′
1	258	488	709	922	128	328	522	710	894	073	1
2	262	492	713	926	132	331	525	713	897	076	2
3	266	495	716	929	135	335	528	716	900	079	3
4	270	499	720	933	139	338	531	719	903	082	4
5	274	503	724	936	142	341	534	722	906	085	5
6	278	507	727	940	145	345	537	725	909	088	6
7	282	511	731	943	149	348	540	728	912	091	7
8	286	514	734	947	152	351	543	732	915	094	8
9	1.3289	1.3518	1.3738	1.3950	1.4156	1.4354	1.4547	1.4735	1.4918	1.5097	9
10	1.3293	1.3522	1.3742	1.3954	1.4159	1.4357	1.4550	1.4738	1.4921	1.5100	10
11	297	526	745	957	162	360	553	741	924	103	11
12	301	529	749	961	165	363	556	744	927	106	12
13	305	533	753	964	168	367	559	747	930	109	13
14	309	537	756	968	172	370	563	750	933	112	14
15	313	540	760	971	175	373	566	753	936	115	15
16	317	544	763	974	178	376	569	756	939	118	16
17	321	548	767	977	182	380	572	760	942	121	17
18	325	552	770	981	185	383	575	763	945	124	18
19	1.3328	1.3555	1.3774	1.3984	1.4188	1.4386	1.4579	1.4766	1.4948	1.5127	19
20	1.3332	1.3559	1.3778	1.3988	1.4192	1.4390	1.4582	1.4769	1.4951	1.5129	20
21	336	563	781	991	195	393	585	772	954	132	21
22	340	567	784	995	198	396	588	775	957	135	22
23	344	570	788	1.3998	202	399	591	778	960	138	23
24	348	574	791	1.4002	205	403	594	781	963	141	24
25	352	578	795	005	209	406	597	784	966	144	25
26	356	582	798	009	212	409	601	787	969	147	26
27	359	585	802	012	215	412	604	790	972	150	27
28	363	588	805	015	219	416	607	794	975	153	28
29	1 3367	1.3592	1.3809	1.4019	1.4222	1.4419	1.4610	1.4797	1.4978	1.5156	29
30	1.3370	1.3595	1.3812	1.4022	1.4225	1.4422	1.4613	1.4800	1.4981	1.5159	30
31	374	599	816	026	229	425	617	803	984	162	31
32	378	603	820	029	232	429	620	806	987	165	32
33	381	606	823	032	235	432	623	809	990	168	33
34	385	610	827	036	239	435	626	812	993	171	34
35	389	614	830	040	242	438	629	815	996	174	35
36	393	618	834	043	245	442	632	818	1.4999	177	36
37	397	621	838	046	249	445	636	821	1.5002	180	37
38	401	625	841	050	252	448	639	824	005	183	38
39	1.3404	1.3629	1.3845	1.4053	1.4255	1.4451	1.4642	1.4827	1.5008	1.5186	39
40	1.3408	1.3632	1.3848	1 4057	1.4259	1.4455	1.4645	1.4830	1.5011	1.5188	40
41	412	636	852	060	262	458	648	834	014	191	41
42	416	640	855	064	265	461	651	837	017	194	42
43	420	643	859	067	269	464	654	840	020	197	43
44	423	647	862	070	272	467	658	843	023	200	44
45	427	651	866	074	275	471	661	846	026	203	45
46	431	654	870	077	279	474	664	849	029	206	46
47	435	658	873	081	282	477	667	852	032	209	47
48	439	662	877	084	285	480	670	855	035	211	48
49	1.3443	1.3665	1.3880	1.4088	1.4288	1.4484	1.4673	1.4858	1.5038	1.5214	49
50	1.3446	1.3669	1.3884	1.4091	1.4292	1.4487	1.4676	1.4861	1.5040	1.5217	50
51	450	673	887	094	295	490	680	864	043	220	51
52	454	676	891	098	298	493	683	867	046	223	52
53	458	680	894	101	302	496	686	870	049	226	53
54	462	684	898	105	305	500	689	873	052	228	54
55	465	687	901	108	308	503	692	875	055	231	55
56	469	691	905	111	312	506	695	878	058	234	56
57	473	695	908	115	315	509	698	881	061	237	57
58	477	698	912	118	318	513	701	885	064	240	58
59	1.3481	1.3702	1.3915	1.4122	1.4322	1.4516	1.4704	1.4888	1.5067	1.5243	59
60	1.3484	1.3705	1.3919	1.4125	1.4325	1.4519	1.4707	1.4891	1.5070	1.5246	60

TABLE VI.

$\mu + \log \tan z$.

z	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°	z
0′	1.5246	1.5418	1.5587	1.5753	1.5917	1.6078	1.6237	1.6394	1.6550	1.6705	0′
1	49	21	90	56	19	81	40	397	53	07	1
2	52	24	93	59	22	83	42	400	55	10	2
3	55	27	96	62	25	86	45	02	58	12	3
4	58	30	598	64	28	89	48	05	61	15	4
5	61	32	601	67	30	91	50	08	63	17	5
6	63	35	04	70	33	94	53	10	66	19	6
7	66	38	07	73	36	97	56	13	68	22	7
8	69	41	10	75	38	099	58	15	71	24	8
9	1.5272	1.5444	1.5612	1.5778	1.5941	1.6102	1.6261	1.6418	1.6574	1.6727	9
10	1.5275	1.5447	1.5615	1.5781	1.5944	1.6105	1.6264	1.6421	1.6576	1.6729	10
11	78	50	18	84	47	07	66	23	78	32	11
12	81	52	21	86	49	10	69	25	80	34	12
13	84	55	24	89	52	13	72	28	83	37	13
14	87	58	26	92	55	16	74	30	86	40	14
15	90	61	29	95	58	18	77	33	88	42	15
16	92	64	32	797	60	21	79	35	91	45	16
17	95	67	35	800	63	23	81	38	93	47	17
18	298	70	38	03	66	26	84	41	96	50	18
19	1.5301	1.5472	1.5640	1.5806	1.5968	1.6129	1.6287	1.6443	1.6598	1.6752	19
20	1.5304	1.5475	1.5643	1.5808	1.5971	1.6131	1.6289	1.6446	1.6601	1.6755	20
21	07	78	46	11	74	34	92	48	04	58	21
22	10	81	49	14	76	36	94	51	06	60	22
23	13	84	51	17	79	39	297	54	09	63	23
24	16	87	54	19	81	41	300	56	11	65	24
25	19	89	57	22	84	44	02	59	14	68	25
26	21	92	60	25	87	47	05	62	17	70	26
27	24	95	63	28	90	49	08	64	19	73	27
28	27	498	65	30	92	52	10	67	22	76	28
29	1.5330	1.5501	1.5668	1.5833	1.5994	1.6155	1.6313	1.6469	1.6624	1.6778	29
30	1.5333	1.5504	1.5671	1.5835	1.5997	1.6157	1.6315	1.6472	1.6627	1.6781	30
31	36	06	74	38	1.6000	60	18	75	30	83	31
32	39	09	76	40	02	63	21	77	32	86	32
33	42	12	78	43	05	65	23	80	35	88	33
34	45	15	81	46	08	68	26	82	37	91	34
35	47	18	84	48	11	71	29	85	40	94	35
36	50	21	87	51	13	73	31	88	43	96	36
37	53	23	90	54	16	76	34	90	45	799	37
38	56	26	92	57	19	79	37	93	48	801	38
39	1.5359	1.5528	1.5695	1.5859	1.6021	1.6181	1.6339	1.6495	1.6650	1.6804	39
40	1.5362	1.5531	1.5698	1.5862	1.6024	1.6184	1.6342	1.6498	1.6653	1.6806	40
41	65	34	701	65	27	87	44	501	56	09	41
42	68	37	03	68	30	89	47	03	58	12	42
43	70	39	06	70	32	92	50	06	61	14	43
44	72	42	09	73	35	94	52	09	63	1.6817	44
45	75	45	12	76	38	197	55	11	66	19	45
46	78	48	14	79	40	200	58	14	68	22	46
47	81	51	17	81	43	02	60	16	71	24	47
48	84	53	20	84	46	05	63	19	74	27	48
49	1.5387	1.5556	1.5723	1.5887	1.6048	1.6208	1.6365	1.6521	1.6676	1.6830	49
50	1.5390	1.5559	1.5726	1.5889	1.6051	1.6210	1.6368	1.6524	1.6679	1.6832	50
51	92	62	28	92	54	13	71	27	81	35	51
52	95	65	31	95	56	16	73	30	84	37	52
53	398	67	34	898	59	18	76	32	86	40	53
54	401	70	37	900	62	21	79	35	89	42	54
55	04	73	39	03	64	24	81	37	92	45	55
56	07	76	42	06	67	26	84	40	94	47	56
57	10	79	45	09	70	29	87	42	97	50	57
58	12	82	48	11	73	32	89	45	699	53	58
59	1.5415	1.5584	1.5750	1.5914	1.6075	1.6234	1.6392	1.6548	1.6702	1.6855	59
60	1.5418	1.5587	1.5753	1.5917	1.6078	1.6237	1.6394	1.6550	1.6705	1.6857	60

REFRACTION TABLES.

In order that all the refraction tables may be conveniently together the three tables giving the terms B, γ and T published in Volume I of these *Publications* are given again here as Tables I, II, III respectively. Table IV gives A, λ and σ. Table V gives i. [Tables IV and V are reprinted from the Pulkowa Tables.] Table VI is the Pulkowa table for $\mu + \log \operatorname{tg} z$ corrected by

$$\Delta \mu = +0.000108\ (58^\circ - z)$$

where z is in degrees. The refractions are then to be computed from these tables by the formula

$$\log \text{Refr.} = \mu + \log \operatorname{tg} z + A\ (B + T) + \lambda \gamma - \sigma i.$$

R. T. C.

TABLE I.

Values of the Refraction Term B.

(Units of the fifth decimal place.)

Bar. Inches.	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09	With left hand col.
											20 × Bar.
25.0	-7330	-7313	-7296	-7278	-7261	-7243	-7226	-7209	-7191	-7174	500
.1	7157	7140	7122	7105	7088	7070	7053	7036	7019	7001	502
.2	6984	6967	6950	6933	6915	6898	6881	6864	6847	6829	504
.3	6812	6795	6778	6761	6744	6726	6709	6692	6675	6658	506
.4	6641	6624	6607	6590	6573	6555	6538	6521	6504	6487	508
.5	-6470	-6453	-6436	-6419	-6402	-6385	-6368	-6351	-6334	-6317	510
.6	6300	6283	6266	6249	6232	6215	6199	6182	6165	6148	512
.7	6131	6114	6097	6080	6063	6046	6030	6013	5996	5979	514
.8	5962	5945	5929	5912	5895	5878	5861	5845	5828	5811	516
.9	5794	5777	5761	5744	5727	5711	5694	5677	5660	5644	518
26.0	-5627	-5610	-5594	-5577	-5560	-5543	-5527	-5510	-5493	-5477	520
.1	5460	5444	5427	5410	5394	5377	5360	5344	5327	5311	522
.2	5294	5278	5261	5244	5228	5211	5195	5178	5162	5145	524
.3	5129	5112	5096	5079	5063	5046	5030	5013	4997	4980	526
.4	4964	4947	4931	4915	4898	4882	4865	4849	4832	4816	528
.5	-4800	-4783	-4767	-4751	-4734	-4718	-4701	-4685	-4669	-4652	530
.6	4636	4620	4603	4587	4571	4555	4538	4522	4506	4489	532
.7	4473	4457	4441	4424	4408	4392	4376	4359	4343	4327	534
.8	4311	4295	4278	4262	4246	4230	4214	4197	4181	4165	536
.9	4149	4133	4117	4101	4084	4068	4052	4036	4020	4004	538
27.0	-3988	-3972	-3956	-3940	-3923	-3907	-3891	-3875	-3859	-3843	540
.1	3827	3811	3795	3779	3763	3747	3731	3715	3699	3683	542
.2	3667	3651	3635	3619	3603	3587	3572	3556	3540	3524	544
.3	3508	3492	3476	3460	3444	3428	3413	3397	3381	3365	546
.4	3349	3333	3317	3302	3286	3270	3254	3238	3222	3207	548
.5	-3191	-3175	-3159	-3144	-3128	-3112	-3096	-3080	-3065	-3049	550
.6	3033	3018	3002	2986	2970	2955	2939	2923	2908	2892	552
.7	2876	2860	2845	2829	2814	2798	2782	2767	2751	2735	554
.8	2720	2704	2689	2673	2657	2642	2626	2610	2595	2579	556
.9	2564	2548	2533	2517	2502	2486	2470	2455	2439	2424	558
28.0	-2408	-2393	-2377	-2362	-2346	-2331	-2315	-2300	-2285	-2269	560
With right hand col.	0.0	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	20 × Bar.

N. B.—The left hand column and top argument give the whole argument for a barometer reading in inches; the right hand column and bottom argument give the entry for 20 times the barometer reading in inches.

TABLE II.

Values of the Refraction Term γ.

(Units of the fifth decimal place.)

t. Fahr.	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
0°	+4412	+4318	+4222	+4128	+4032	+3938	+3846	+3750	+3658	+3562	9	19	28	37	47	56	66	75	84
10	3470	3376	3284	3190	3099	3008	2914	2823	2730	2640	9	18	28	37	46	55	64	74	83
20	2547	2457	2365	2275	2184	2094	2004	1913	1824	1734	9	18	27	36	45	54	63	72	81
30	1645	1554	1466	1378	1288	1201	1111	1023	934	847	9	18	27	35	44	53	62	71	80
40	+ 759	+ 672	+ 586	+ 497	+ 411	+ 324	+ 238	+ 150	+ 65	− 22	9	17	26	35	43	52	61	69	78
50	− 107	− 193	− 279	− 364	− 450	− 534	− 620	− 704	− 789	− 873	[illegible]	17	26	34	43	51	60	68	77
60	957	1042	1125	1209	1293	1377	1460	1543	1626	1708	8	17	25	33	42	50	58	67	75
70	1791	1873	1956	2038	2121	2202	2285	2366	2446	2528	8	16	24	33	41	49	57	65	73
80	2609	2691	2771	2852	2932	3014	3093	3172	3253	3334	8	16	24	32	40	48	56	64	72
90	−3414	−3493	−3573	−3652	−3731	−3810	3889	−3967	−4046	−4124	8	16	23	31	39	47	55	63	70

TABLE III.

Values of Refraction Term T.

(Units of the fifth decimal place.)

t'. (Fahr.)	0	1	2	3	[illegible]	5	6	7	8	9
0°	+125	+121	+117	+113	+109	+105	+101	+ 98	+ 94	+ 90
10	+ 86	+ 82	+ 78	+ 74	+ 70	+ 66	+ 62	+ 58	+ 55	+ 51
20	+ 47	+ 43	+ 39	+ 35	+ 31	+ 27	+ 23	+ 20	+ 16	+ 12
30	+ 8	+ 4	0	− 4	− 8	− 12	− 16	− 20	− 23	− 27
40	− 31	− 35	− 39	− 43	− 47	− 51	− 55	− 58	− 62	− 66
50	− 70	− 74	− 78	− 82	− 86	− 90	− 93	− 97	−101	−105
60	−109	−113	−117	−121	−124	−128	−132	−136	−140	−144
70	−148	−152	−156	−160	−164	−168	−171	−175	−179	−183
80	−187	−191	−195	−199	−202	−206	−210	−214	−218	−221
90	−225	−229	−233	−237	−241	−244	−248	−252	−256	−260

TABLE IV.

Values of the Refraction Terms λ, A and σ.

z	λ	A	σ
45°	1.0018		
50	1.0022		
55	1.0029		
60	1.0044		
61	1.0047		
62	1.0051		
63	1.0055		
64	1.0059		
65	1.0064		
66	1 0070		
67	1.0077		
68	1.0085		
69	1.0093		
70	1.0103		
71	1.0115		
72	1.0130		
73	1.0147		
74	1.0166		
75 0'	1.0188		
75 30	1.0200		
76 0	1.0216		
76 30	1.0235		
77 0	1.0253	1.0029	
77 30	1.0271	1.0030	
78 0	1.0293	1.0033	
78 30	1.0318	1.0035	
79 0	1.0344	1.0038	
79 30	1.0374	1.0041	
80 0	1.0409	1.0044	0.00019
10	1.0421		
20	1.0433		
30	1.0447	1.0048	0.00022
40	1.0461		
50	1.0475		
81 0	1.0491	1.0052	0.00025
10	1.0508		
20	1.0525		
30	1.0542	1.0057	0.00029
40	1.0561		
50	1.0580		
82 0	1.0600	1.0063	0.00035
10	1.0622		
20	1.0645		
30	1.0669	1.0070	0.00045
40	1.0694		
50	1.0720		
83 0	1.0747	1.0078	0.00057
10	1.0776		
20	1.0807		
30	1.0839	0.0087	0.00073
40	1.0874		
50	1.0911		
84 0	1.0949	1.0098	0.00091
10	1.0990		
20	1.1034		
30	1.1080	1.0112	0.00116
40	1.1128		
50	1.1180		
85 0	1.1235	1.0127	0.00146

z	λ	A	σ
85° 0'	1.1235	1.0127	0.00146
5	1264		
10	1294		
15	1325		
20	1357	141	170
25	1390		
30	1424		
35	1459		
40	1495	156	202
45	1532		
50	1571		
55	1611		
86 0	1.1652	1.0172	0.00241
5	1695		
10	1740	181	264
15	1786		
20	1833	192	290
25	1882		
30	1934	203	320
35	1987		
40	2040	214	352
45	2095		
50	2153	227	386
55	2214		
87 0	1.2277	1.0241	0.00421
2	2303		
4	2329	247	437
6	2356		
8	2383	254	453
10	2410		
12	2438	261	471
14	2466		
16	2495	268	489
18	2524		
20	2554	275	509
22	2584		
24	2615	282	529
26	2646		
28	2677	290	550
30	2708		
32	2740	297	572
34	2772		
36	2805	305	594
38	2839		
40	2873	313	617
42	2908		
44	2942	321	641
46	2978		
48	3014	328	666
50	3051		
52	3088	338	693
54	3125		
56	3163	347	721
87 58	3202		
88 0	1.3241	1.0357	0.00749

TABLE IV.

Values of the Refraction Terms λ, A and σ.

z	λ	A	σ	z	λ	A	σ
88° 0′	1.3241	1.0357	0.00749	89° 15′	1.5218		
2	3280			16	5252	1.0609	0.01582
4	3321	367	778	17	5287		
6	3362			18	5322		1613
8	3403	376	810	19	5357		
10	3445			20	5392	628	1644
12	3487	387	842	21	5427		
14	3531			22	5463		1675
16	3575	397	876	23	5500		
18	3620			24	5537	648	1709
20	3665	409	912	25	5574		
22	3710			26	5611		1744
24	3757	420	948	27	5648		
26	3805			28	5685	669	1779
28	3853	431	986	29	5723		
30	3902			30	5762		1813
32	3951	443	1026	31	5801		
34	4002			32	5840	690	1849
36	4053	455	1067	33	5879		
38	4107			34	5918		1886
40	4161	468	1111	35	5957		
42	4214			36	5997	712	1923
44	4268	482	1155	37	6037		
46	4323			38	6077		1961
48	4378	496	1201	39	6117		
50	4433			40	6158	734	1999
52	4490	511	1249	41	6199		
54	4549			42	6241		2038
56	4608	526	1300	43	6283		
88 58	4669			44	6325	758	2079
89 0	1.4729	1.0541	0.01352	45	6367		
1	4760			46	6410		2119
2	4791		1379	47	6453		
3	4823			48	6497	782	2161
4	4855	558	1407	49	6541		
5	4887			50	6585		2203
6	4919		1435	51	6630		
7	4951			52	6675	807	2246
8	4984	575	1464	53	6720		
9	5017			54	6765		2288
10	5050		1493	55	6811		
11	5083			56	6857	833	2332
12	5116	1.0592	1522	57	6904		
13	5150			58	6951		2377
14	5184		0.01552	89 59	6998		
89 15	1.5218			90 0	1.7046	1.0859	0.02424

TABLE V.

Values of the Refraction Term i.

Jan. 15	−0.34	July 15	+0.33
Feb. 15	−0.27	Aug. 15	+0.30
Mar. 15	−0.05	Sept. 15	+0.19
Apr. 15	+0.08	Oct. 15	−0.16
May 15	+0.20	Nov. 15	−0.33
June 15	+0.26	Dec. 15	−0.37

TABLE VI.

$\mu + \log \tan z$.

z	0°	1°	2°	3°	4°	5°	6°	7°	8°	9°	z
0'	$-\infty$	0.0083	0.3094	0.4856	0.6108	0.7080	0.7875	0.8550	0.9135	0.9653	0'
1	8.230	155	130	880	126	094	888	560	144	661	1
2	.531	226	166	904	144	109	900	571	153	669	2
3	.707	295	201	928	162	123	912	581	162	678	3
4	8.832	0.0364	0.3236	0.4952	.06180	0.7138	0.7924	0.8591	0.9172	0.9686	4
5	8.929	0.0431	0.3271	0.4975	0.6198	0.7152	0.7936	0.8602	0.9181	0.9694	5
6	9.008	497	306	0.4999	215	166	948	612	190	702	6
7	.075	563	341	0.5022	233	180	960	622	199	710	7
8	.133	627	375	045	250	195	972	632	208	718	8
9	9.184	0.0691	0.3408	0.5068	0.6268	0.7209	0.7983	0.8643	0.9217	0.9726	9
10	9.230	0.0753	0.3442	0.5091	0.6285	0.7223	0.7995	0.8653	0.9226	0.9734	10
11	.271	815	475	114	303	237	0.8007	663	235	742	11
12	.309	875	.508	137	320	251	019	673	244	750	12
13	.344	935	541	160	338	265	031	683	253	758	13
14	9.376	0.0994	0.3574	0.5182	0.6355	0.7279	0.8042	0.8694	0.9262	0.9766	14
15	9.406	0.1053	0.3606	0.5205	0.6372	0.7293	0.8054	0.8704	0.9271	0.9774	15
16	.434	110	638	227	389	307	066	714	279	782	16
17	.460	167	670	249	406	320	077	724	288	790	17
18	.485	223	702	271	423	334	089	734	297	798	18
19	9.509	0.1279	0.3733	0.5293	0.6440	0.7348	0.8101	0.8744	0.9306	0.9806	19
20	9.531	0.1333	0.3764	0.5315	0.6457	0.7362	0.8112	0.8754	0.9315	0.9814	20
21	.552	387	795	336	473	375	124	764	324	822	21
22	.572	440	826	358	490	389	135	774	332	829	22
23	.592	493	856	380	507	402	147	784	341	836	23
24	9.610	0.1545	0.3887	0.5401	0.6523	0.7416	0.8158	0.8794	0.9350	0.9844	24
25	9.628	0.1596	0.3917	0.5422	0.6540	0.7429	0.8169	0.8803	0.9359	0.9852	25
26	.645	647	947	443	556	443	181	813	367	860	26
27	.661	698	0.3976	464	572	456	192	823	376	868	27
28	.677	747	0.4006	485	589	469	203	833	384	876	28
29	9.692	0.1796	0.4035	0.5506	0.6605	0.7483	0.8215	0.8843	0.9392	0.9883	29
30	9.707	0.1845	0.4064	0.5527	0.6621	0.7496	0.8226	0.8852	0.9401	0.9891	30
31	.721	893	093	548	637	509	237	862	410	899	31
32	.735	940	122	568	653	522	248	872	418	907	32
33	.749	0.1987	150	589	669	535	259	881	427	914	33
34	9.762	0.2034	0.4179	0.5609	0.6685	0.7549	0.8271	0.8890	0.9436	0.9922	34
35	9.774	0.2080	0.4207	0.5630	0.6701	0.7562	0.8282	0.8900	0.9444	0.9930	35
36	.786	125	235	650	717	575	292	910	453	937	36
37	.798	170	263	670	733	588	303	919	461	945	37
38	.810	215	290	690	748	601	314	929	470	953	38
39	9.821	0.2259	0.4317	0.5710	0.6764	0.7613	0.8325	0.8938	0.9478	0.9961	39
40	9.832	0.2303	0.4345	0.5730	0.6780	0.7625	0.8336	0.8948	0.9487	0.9968	40
41	.843	346	372	749	795	638	347	957	495	976	41
42	.853	389	399	769	811	651	358	967	504	983	42
43	.864	431	426	789	826	664	369	976	512	991	43
44	9.874	0.2473	0.4452	0.5808	0.6841	0.7677	0.8379	0.8986	0.9520	0.9999	44
45	9.883	0.2515	0.4479	0.5828	0.6856	0.7689	0.8390	0.8995	0.9529	1.0006	45
46	.893	556	505	847	871	702	401	0.9005	537	014	46
47	.902	597	531	866	886	714	412	014	546	021	47
48	.911	637	557	885	902	727	422	024	554	029	48
49	9.920	0.2677	0.4583	0.5904	0.6917	0.7740	0.8433	0.9033	0.9562	1.0036	49
50	9.929	0.2717	0.4608	0.5922	0.6932	0.7752	0.8444	0.9043	0.9571	1.0044	50
51	.938	756	634	941	947	765	455	052	579	051	51
52	.946	795	659	960	962	777	465	061	587	059	52
53	.954	834	685	979	977	790	476	071	595	066	53
54	9.962	0.2872	0.4709	0.5997	0.6991	0.7802	0.8486	0.9080	0.9604	1.0074	54
55	9.970	0.2909	0.4734	0.6016	0.7006	0.7814	0.8497	0.9089	0.9612	1.0081	55
56	.978	947	758	034	021	827	508	098	620	089	56
57	.986	0.2984	783	053	036	839	518	107	629	096	57
58	9.994	0.3021	808	071	050	851	529	117	637	104	58
59	0.0011	0.3058	0.4832	0.6089	0.7065	0.7863	0.8539	0.9126	0.9645	1.0111	59
60	0.0083	0.3094	0.4856	0.6108	0.7080	0.7875	0.8550	0.9135	0.9653	1.0118	60

TABLE VI.

$\mu + \log \text{tang } z$.

z	10°	11°	12°	13°	14°	15°	16°	17°	18°	19°	z
0′	1.0118	1.0540	1.0928	1.1286	1.1618	1.1929	1.2223	1.2500	1.2764	1.3014	0′
1	126	547	934	291	623	934	227	505	768	018	1
2	133	554	940	297	628	940	232	509	772	023	2
3	140	561	946	303	634	945	237	514	776	027	3
4	148	567	952	309	639	950	242	518	781	031	4
5	155	574	959	314	644	955	246	523	785	035	5
6	162	581	965	319	650	960	251	527	789	039	6
7	170	587	971	325	655	965	256	532	794	043	7
8	177	594	977	330	660	970	261	536	798	047	8
9	1.0184	1.0601	1.0983	1.1336	1.1666	1.1975	1.2266	1.2541	1.2802	1.3051	9
10	192	1.0607	1.0989	1.1342	1.1671	1.1980	1.2270	1.2545	1.2806	055	10
11	199	614	1.0995	348	676	985	275	550	811	059	11
12	206	620	1.1001	353	682	990	280	554	815	063	12
13	213	626	007	359	687	1.1995	284	559	819	067	13
14	221	633	013	365	692	1.2000	289	563	823	072	14
15	228	640	019	370	698	005	294	567	828	076	15
16	235	646	025	376	703	010	299	572	832	080	16
17	242	653	031	382	708	015	303	576	836	084	17
18	249	659	037	387	714	020	308	581	840	088	18
19	1.0256	1.0666	1.1043	1.1393	1.1719	1.2025	1.2313	1.2585	1.2845	1.3092	19
20	1.0263	1.0673	1.1049	1.1399	1.1724	1.2030	1.2317	1.2590	1.2849	1.3096	20
21	270	679	055	404	729	034	322	594	853	100	21
22	277	686	061	410	735	039	327	599	857	104	22
23	284	692	067	415	740	044	331	603	861	108	23
24	291	699	073	421	745	049	336	607	866	112	24
25	298	705	079	427	750	054	341	612	870	116	25
26	305	712	086	432	756	059	345	616	874	120	26
27	312	718	092	438	761	064	350	621	878	124	27
28	319	725	098	443	766	069	355	625	883	128	28
29	1.0327	1.0731	1.1104	1.1449	1.1771	1.2074	1.2359	1.2630	1.2887	1.3132	29
30	1.0334	1.0738	1.1109	1.1454	1.1776	1.2079	1.2364	1.2634	1.2891	1.3136	30
31	341	744	115	460	782	084	369	638	895	140	31
32	348	751	121	466	787	089	373	643	899	144	32
33	355	757	127	471	792	093	378	647	904	147	33
34	362	763	133	477	797	098	382	652	908	151	34
35	369	770	139	482	803	103	387	656	912	155	35
36	376	776	145	488	808	108	392	660	916	159	36
37	383	783	151	493	813	113	396	665	920	163	37
38	390	789	157	499	818	118	401	669	924	167	38
39	1.0397	1.0795	1.1163	1.1504	1.1823	1.2123	1.2405	1.2673	1.2928	1.3171	39
40	1.0404	1.0802	1.1169	1.1510	1.1828	1.2128	1.2410	1.2678	1.2932	1.3175	40
41	411	808	175	515	833	132	415	682	936	179	41
42	418	815	181	521	839	137	419	687	940	183	42
43	424	821	187	526	844	142	424	691	944	187	43
44	431	827	192	532	849	147	428	695	948	191	44
45	438	834	198	537	854	152	433	699	952	195	45
46	445	840	204	543	859	157	437	703	957	199	46
47	452	846	210	548	864	161	442	707	961	203	47
48	459	853	216	554	869	166	447	712	965	207	48
49	1.0466	1.0859	1.1222	1.1559	1.1874	1.2171	1.2451	1.2716	1.2969	1.3211	49
50	1.0473	1.0865	1.1228	1.1565	1.1879	1.2176	1.2455	1.2720	1.2973	1.3215	50
51	479	871	233	570	885	181	459	725	977	219	51
52	486	878	239	575	890	186	464	729	981	223	52
53	493	884	245	581	895	190	468	733	986	227	53
54	500	890	251	586	900	194	473	738	990	230	54
55	507	897	257	592	904	199	477	742	994	234	55
56	514	903	262	597	909	204	482	746	1.2998	238	56
57	520	909	268	602	914	208	486	750	1.3002	242	57
58	527	915	274	608	919	213	491	755	006	246	58
59	1.0534	1.0921	1.1280	1.1613	1.1924	1.2218	1.2496	1.2759	1.3010	1.3250	59
60	1.0540	1.0928	1.1286	1.1618	1.1929	1.2223	1.2500	1.2764	1.3014	1.3254	60

TABLE VI.

$\mu + \log \operatorname{tang} z$.

z	20°	21°	22°	23°	24°	25°	26°	27°	28°	29°	z
0′	1.3254	1.3484	1.3705	1.3919	1.4125	1.4325	1.4519	1.4707	1 4891	1.5070	0′
1	258	488	709	922	128	328	522	710	894	073	1
2	262	492	713	926	132	331	525	713	897	076	2
3	266	495	716	929	135	335	528	716	900	079	3
4	270	499	720	933	139	338	531	719	903	082	4
5	274	503	724	936	142	341	534	722	906	085	5
6	278	507	727	940	145	345	537	725	909	088	6
7	282	511	731	943	149	348	540	728	912	091	7
8	286	514	734	947	152	351	543	732	915	094	8
9	1.3289	1.3518	1.3738	1.3950	1.4156	1.4354	1.4547	1.4735	1.4918	1.5097	9
10	1.3293	1.3522	1.3742	1.3954	1.4159	1.4357	1.4550	1.4738	1.4921	1.5100	10
11	297	526	745	957	162	360	553	741	924	103	11
12	301	529	749	961	165	363	556	744	927	106	12
13	305	533	753	964	168	367	559	747	930	109	13
14	309	537	756	968	172	370	563	750	933	112	14
15	313	540	760	971	175	373	566	753	936	115	15
16	317	544	763	974	178	376	569	756	939	118	16
17	321	548	767	977	182	380	572	760	942	121	17
18	325	552	770	981	185	383	575	763	945	124	18
19	1.3328	1.3555	1.3774	1.3984	1.4188	1.4386	1.4579	1.4766	1.4948	1.5127	19
20	1.3332	1.3559	1.3778	1.3988	1.4192	1.4390	1.4582	1.4769	1.4951	1.5129	20
21	336	563	781	991	195	393	585	772	954	132	21
22	340	567	784	995	198	396	588	775	957	135	22
23	344	570	788	1.3998	202	399	591	778	960	138	23
24	348	574	791	1.4002	205	403	594	781	963	141	24
25	352	578	795	005	209	406	597	784	966	144	25
26	356	582	798	009	212	409	601	787	969	147	26
27	359	585	802	012	215	412	604	790	972	150	27
28	363	588	805	015	219	416	607	794	975	153	28
29	1 3367	1.3592	1.3809	1.4019	1.4222	1.4419	1.4610	1.4797	1.4978	1.5156	29
30	1.3370	1.3595	1.3812	1.4022	1.4225	1.4422	1.4613	1.4800	1.4981	1.5159	30
31	374	599	816	026	229	425	617	803	984	162	31
32	378	603	820	029	232	429	620	806	987	165	32
33	381	606	823	032	235	432	623	809	990	168	33
34	385	610	827	036	239	435	626	812	993	171	34
35	389	614	830	040	242	438	629	815	996	174	35
36	393	618	834	043	245	442	632	818	1.4999	177	36
37	397	621	838	046	249	445	636	821	1.5002	180	37
38	401	625	841	050	252	448	639	824	005	183	38
39	1.3404	1.3629	1.3845	1.4053	1.4255	1.4451	1.4642	1.4827	1.5008	1.5186	39
40	1.3408	1.3632	1.3848	1 4057	1.4259	1.4455	1.4645	1.4830	1.5011	1.5188	40
41	412	636	852	060	262	458	648	834	014	191	41
42	416	640	855	064	265	461	651	837	017	194	42
43	420	643	859	067	269	464	654	840	020	197	43
44	423	647	862	070	272	467	658	843	023	200	44
45	427	651	866	074	275	471	661	846	026	203	45
46	431	654	870	077	279	474	664	849	029	206	46
47	435	658	873	081	282	477	667	852	032	209	47
48	439	662	877	084	285	480	670	855	035	211	48
49	1.3443	1.3665	1.3880	1.4088	1.4288	1.4484	1.4673	1.4858	1.5038	1.5214	49
50	1.3446	1.3669	1.3884	1.4091	1.4292	1.4487	1.4676	1.4861	1.5040	1.5217	50
51	450	673	887	094	295	490	680	864	043	220	51
52	454	676	891	098	298	493	683	867	046	223	52
53	458	680	894	101	302	496	686	870	049	226	53
54	462	684	898	105	305	500	689	873	052	228	54
55	465	687	901	108	308	503	692	875	055	231	55
56	469	691	905	111	312	506	695	878	058	234	56
57	473	695	908	115	315	509	698	881	061	237	57
58	477	698	912	118	318	513	701	885	064	240	58
59	1.3481	1.3702	1.3915	1.4122	1.4322	1.4516	1.4704	1.4888	1.5067	1.5243	59
60	1.3484	1.3705	1.3919	1.4125	1.4325	1.4519	1.4707	1.4891	1.5070	1.5246	60

TABLE VI.

$\mu + \log \text{tang}\ z$.

z	30°	31°	32°	33°	34°	35°	36°	37°	38°	39°	z
0'	1.5246	1.5418	1.5587	1.5753	1.5917	1.6078	1.6237	1.6394	1.6550	1.6705	0'
1	49	21	90	56	19	81	40	397	53	07	1
2	52	24	93	59	22	83	42	400	55	10	2
3	55	27	96	62	25	86	45	02	58	12	3
4	58	30	598	64	28	89	48	05	61	15	4
5	61	32	601	67	30	91	50	08	63	17	5
6	63	35	04	70	33	94	53	10	66	19	6
7	66	38	07	73	36	97	56	13	68	22	7
8	69	41	10	75	38	099	58	15	71	24	8
9	1.5272	1.5444	1.5612	1.5778	1.5941	1.6102	1.6261	1.6418	1.6574	1.6727	9
10	1.5275	1.5447	1.5615	1.5781	1.5944	1.6105	1.6264	1.6421	1.6576	1.6729	10
11	78	50	18	84	47	07	66	23	78	32	11
12	81	52	21	86	49	10	69	25	80	34	12
13	84	55	24	89	52	13	72	28	83	37	13
14	87	58	26	92	55	16	74	30	86	40	14
15	90	61	29	95	58	18	77	33	88	42	15
16	92	64	32	797	60	21	79	35	91	45	16
17	95	67	35	800	63	23	81	38	93	47	17
18	298	70	38	03	66	26	84	41	96	50	18
19	1.5301	1.5472	1.5640	1.5806	1.5968	1.6129	1.6287	1.6443	1.6598	1.6752	19
20	1.5304	1.5475	1.5643	1.5808	1.5971	1.6131	1.6289	1.6446	1.6601	1.6755	20
21	07	78	46	11	74	34	92	48	04	58	21
22	10	81	49	14	76	36	94	51	06	60	22
23	13	84	51	17	79	39	297	54	09	63	23
24	16	87	54	19	81	41	300	56	11	65	24
25	19	89	57	22	84	44	02	59	14	68	25
26	21	92	60	25	87	47	05	62	17	70	26
27	24	95	63	28	90	49	08	64	19	73	27
28	27	498	65	30	92	52	10	67	22	76	28
29	1.5330	1.5501	1.5668	1.5833	1.5994	1.6155	1.6313	1.6469	1.6624	1.6778	29
30	1.5333	1.5504	1.5671	1.5835	1.5997	1.6157	1.6315	1.6472	1.6627	1.6781	30
31	36	06	74	38	1.6000	60	18	75	30	83	31
32	39	09	76	40	02	63	21	77	32	86	32
33	42	12	78	43	05	65	23	80	35	88	33
34	45	15	81	46	08	68	26	82	37	91	34
35	47	18	84	48	11	71	29	85	40	94	35
36	50	21	87	51	13	73	31	88	43	96	36
37	53	23	90	54	16	76	34	90	45	799	37
38	56	26	92	57	19	79	37	93	48	801	38
39	1.5359	1.5528	1.5695	1.5859	1.6021	1.6181	1.6339	1.6495	1.6650	1.6804	39
40	1.5362	1.5531	1.5698	1.5862	1.6024	1.6184	1.6342	1.6498	1.6653	1.6806	40
41	65	34	701	65	27	87	44	501	56	09	41
42	68	37	03	68	30	89	47	03	58	12	42
43	70	39	06	70	32	92	50	06	61	14	43
44	72	42	09	73	35	94	52	09	63	1.6817	44
45	75	45	12	76	38	197	55	11	66	19	45
46	78	48	14	79	40	200	58	14	68	22	46
47	81	51	17	81	43	02	60	16	71	24	47
48	84	53	20	84	46	05	63	19	74	27	48
49	1.5387	1.5556	1.5723	1.5887	1.6048	1.6208	1.6365	1.6521	1.6676	1.6830	49
50	1.5390	1.5559	1.5726	1.5889	1.6051	1.6210	1.6368	1.6524	1.6679	1.6832	50
51	92	62	28	92	54	13	71	27	81	35	51
52	95	65	31	95	56	16	73	30	84	37	52
53	398	67	34	898	59	18	76	32	86	40	53
54	401	70	37	900	62	21	79	35	89	42	54
55	04	73	39	03	64	24	81	37	92	45	55
56	07	76	42	06	67	26	84	40	94	47	56
57	10	79	45	09	70	29	87	42	97	50	57
58	12	82	48	11	73	32	89	45	699	53	58
59	1.5415	1.5584	1.5750	1.5914	1.6075	1.6234	1.6392	1.6548	1.6702	1.6855	59
60	1.5418	1.5587	1.5753	1.5917	1.6078	1.6237	1.6394	1.6550	1.6705	1.6857	60

TABLE VI.

$\mu + \log \tan z$.

z	40°	41°	42°	43°	44°	45°	46°	47°	48°	49°	z
0′	1.6857	1.7009	1.7160	1.7311	1.7462	1.7612	1.7762	1.7913	1.8063	1.8215	0′
1	59	12	63	14	64	15	65	15	66	17	1
2	62	14	66	16	67	17	67	18	69	20	2
3	64	17	68	19	69	20	70	20	71	22	3
4	67	19	71	21	72	22	72	23	74	25	4
5	70	22	73	24	74	25	75	25	76	28	5
6	72	24	76	26	77	27	77	28	79	30	6
7	75	27	78	29	79	30	80	30	81	33	7
8	77	29	81	31	82	32	82	33	84	35	8
9	1.6880	1.7032	1.7183	1.7334	1.7484	1.7635	1.7785	1 7935	1.8086	1.8238	9
10	1.6882	1.7034	1.7186	1.7337	1.7487	1.7637	1.7787	1.7938	1.8089	1.8240	10
11	85	37	88	39	89	40	90	40	91	43	11
12	87	39	91	42	92	42	93	43	94	45	12
13	90	42	93	44	94	45	95	46	96	48	13
14	92	45	96	47	497	47	798	48	099	50	14
15	95	47	198	49	500	50	800	51	102	53	15
16	898	50	201	52	02	52	03	53	04	56	16
17	900	52	03	54	05	55	05	56	06	58	17
18	03	55	06	57	07	57	08	58	09	60	18
19	1.6905	1.7057	1.7209	1.7359	1.7510	1.7660	1.7810	1.7961	1.8112	1.8262	19
20	1.6908	1.7060	1.7211	1.7362	1.7512	1.7662	1.7813	1.7963	1.8114	1.8265	20
21	10	62	14	64	15	65	15	66	17	67	21
22	13	65	16	67	17	68	18	68	19	70	22
23	16	68	19	69	20	70	20	71	21	72	23
24	18	70	21	72	22	73	23	73	23	75	24
25	21	73	24	74	25	75	25	76	26	78	25
26	23	75	26	77	27	78	28	78	28	80	26
27	26	78	29	79	30	80	30	81	31	83	27
28	28	80	31	82	32	83	33	83	33	85	28
29	1.6931	1.7083	1.7234	1.7385	1.7535	1.7685	1.7835	1.7985	1.8136	1.8288	29
30	1.6933	1.7085	1.7236	1.7387	1.7537	1.7688	1.7837	1.7988	1.8139	1.8290	30
31	36	88	39	90	40	90	39	90	41	93	31
32	38	90	41	92	42	92	42	93	44	95	32
33	41	93	44	95	45	94	45	95	46	298	33
34	44	95	47	397	47	97	47	1.7998	49	300	34
35	46	098	49	400	50	699	50	1.8000	51	03	35
36	49	101	52	02	53	702	52	03	54	06	36
37	51	03	54	05	55	04	55	05	56	08	37
38	54	06	57	07	58	07	57	08	59	11	38
39	1.6956	1.7108	1.7259	1.7410	1.7560	1.7709	1.7860	1.8010	1.8161	13	39
40	1.6959	1.7111	1.7262	1.7412	1.7562	1.7712	1.7862	1.8013	1.8164	1.8316	40
41	61	13	64	15	64	14	65	15	66	18	41
42	64	16	67	17	67	17	67	18	69	21	42
43	67	18	69	19	69	19	70	20	72	23	43
44	69	21	72	21	72	22	72	23	74	26	44
45	72	23	74	24	74	24	75	25	77	29	45
46	74	26	77	26	77	27	77	28	79	31	46
47	77	28	79	29	79	29	80	31	82	34	47
48	79	31	81	31	82	32	82	33	84	36	48
49	1.6982	1.7133	1.7284	1.7434	1.7584	1.7735	1.7885	1.8036	1.8187	1.8339	49
50	1.6984	1.7135	1.7286	1.7437	1.7587	1.7737	1.7887	1.8038	1.8189	1.8341	50
51	87	38	89	39	89	40	90	41	92	44	51
52	90	40	91	42	92	42	92	43	95	46	52
53	92	43	94	44	94	45	95	46	197	49	53
54	95	45	96	47	97	47	897	48	200	52	54
55	97	48	299	49	599	50	900	51	02	54	55
56	6999	50	301	52	602	52	03	53	05	57	56
57	1.7001	53	04	54	04	55	05	56	07	59	57
58	04	55	06	57	07	57	08	58	10	62	58
59	1.7006	1.7158	1.7309	1.7459	1.7610	1.7760	1.7910	1.8061	1.8212	1.8364	59
60	1.7009	1.7160	1.7311	1.7462	1.7612	1.7762	1.7913	1.8063	1.8215	1.8367	60

TABLE VI.

$\mu + \log \text{tang}\ z$.

z	50°	51°	52°	53°	54°	55°	56°	57°	58°	59°	z
0′	1.8367	1.8520	1.8674	1.8829	1.8985	1.9144	1.93044	1.94672	1.96325	1.98005	0′
1	69	22	76	31	88	46	072	700	353	033	1
2	72	25	79	33	90	49	099	728	381	062	2
3	75	28	82	36	93	52	126	755	408	090	3
4	77	30	84	39	96	54	153	782	436	118	4
5	80	33	87	41	1.8998	57	180	809	464	146	5
6	82	35	89	44	1.9001	60	207	837	492	174	6
7	85	37	91	47	04	62	234	865	520	203	7
8	87	39	94	49	06	65	261	892	548	232	8
9	1.8390	1.8542	1.8696	1.8852	1.9009	1.9168	1.93288	1.94919	1.96575	1.98260	9
10	1.8392	1.8545	1.8699	1.8854	1.9012	1.9170	1.93315	1.94946	1.96603	1.98288	10
11	94	47	701	57	14	73	342	1.94973	631	316	11
12	97	50	04	60	17	76	369	1.95001	659	345	12
13	399	52	07	62	19	78	396	028	687	373	13
14	402	55	09	65	22	81	423	055	714	402	14
15	04	58	12	68	25	84	450	083	742	430	15
16	07	60	14	70	27	86	477	111	770	458	16
17	09	63	17	73	30	89	504	138	798	487	17
18	12	65	20	75	33	92	531	166	826	515	18
19	1.8415	1.8568	1.8722	1.8878	1.9035	1.9195	1.93558	1.95193	1.96854	1.98544	19
20	1.8417	1.8570	1.8725	1.8881	1.9038	1.9197	1.93585	1.95220	1.96881	1.98573	20
21	20	73	27	83	41	200	612	248	909	601	21
22	22	76	30	86	43	03	639	275	937	629	22
23	25	78	33	88	46	05	666	303	965	658	23
24	27	81	35	91	49	08	693	330	1.96993	687	24
25	30	83	38	94	51	11	720	357	1.97021	715	25
26	33	86	40	96	54	13	747	385	049	744	26
27	35	88	43	899	57	16	774	413	077	772	27
28	38	91	46	902	59	19	802	441	105	800	28
29	1.8440	1.8594	1.8748	1.8904	1.9062	1.9221	1.93829	1.95468	1.97133	1.98829	29
30	1.8443	1.8596	1.8751	1.8907	1.9065	1.9224	1.93856	1.95495	1.97161	1.98857	30
31	45	599	53	10	67	27	883	522	189	886	31
32	48	601	56	12	70	30	910	550	217	915	32
33	50	04	59	15	73	32	937	578	245	944	33
34	53	07	61	17	75	35	965	606	273	1.98972	34
35	56	09	64	20	78	38	1.93992	634	301	1.99000	35
36	58	12	67	23	81	40	1.94019	661	329	029	36
37	61	14	69	25	83	43	046	688	357	058	37
38	63	17	72	28	86	46	073	716	385	086	38
39	1.8466	1.8620	1.8774	1.8931	1.9089	1.9248	1.94100	1.95743	1.97413	1.99115	39
40	1.8468	1.8622	1.8777	1.8933	1.9091	1.9251	1.94127	1.95771	1.97441	1.99143	40
41	71	25	80	36	94	54	154	798	469	171	41
42	74	27	82	39	96	57	182	825	497	200	42
43	76	30	85	41	099	59	209	853	525	229	43
44	79	32	87	44	102	62	236	881	554	258	44
45	81	35	90	46	05	65	263	909	582	287	45
46	84	38	93	49	07	67	290	937	610	315	46
47	87	40	95	52	10	70	317	964	638	344	47
48	89	43	798	54	12	72	345	1.95991	666	372	48
49	1.8492	1.8645	1.8800	1.8957	1.9115	1.9274	1.94372	1.96019	1.97694	1.99401	49
50	1.8494	1.8648	1.8803	1.8960	1.9117	1.9277	1.94400	1.96047	1.97723	1.99430	50
51	97	51	06	62	20	80	427	075	751	459	51
52	499	53	08	65	22	82	454	102	779	487	52
53	502	56	11	68	25	85	481	130	807	516	53
54	04	58	14	70	28	88	508	158	835	545	54
55	07	61	16	72	30	91	536	186	864	574	55
56	10	63	19	74	33	93	564	214	892	603	56
57	12	66	22	77	36	96	591	242	920	632	57
58	15	69	24	80	38	299	618	269	948	660	58
59	1.8517	1.8671	1.8827	1.8982	1.9141	1.9301	1.94645	1.96297	1.97977	1.99689	59
60	1.8520	1.8674	1.8829	1.8985	1.9144	1.9304	1.94672	1.96325	1.98005	1.99718	60

TABLE VI.

$\mu + \log \text{tang}\, z$.

z	60°	61°	62°	63°	64°	65°	66°	67°	68°	69°	z
0′	1.99718	2.01462	2.03244	2.05068	2.06938	2.08860	2.10837	2.12874	2.14979	2.17160	0′
1	747	492	274	099	2.06970	893	871	909	2.15015	197	1
2	776	521	304	130	2.07002	925	904	943	051	234	2
3	804	550	334	160	033	957	937	2.12977	087	270	3
4	833	580	364	191	064	2.08990	2.10971	1.13012	122	307	4
5	862	609	394	222	096	2.09023	2.11005	047	158	344	5
6	891	639	425	253	128	055	038	082	194	381	6
7	920	669	455	284	160	088	072	116	230	419	7
8	949	698	485	315	192	121	105	151	266	456	8
9	1.99977	2.01727	2.03515	2.05345	2.07224	2.09153	2.11138	2.13185	2.15302	2.17493	9
10	2.00006	2.01757	2.03545	2.05376	2.07255	2.09186	2.11172	2.13220	2.15337	2.17530	10
11	035	786	575	407	286	219	206	254	373	568	11
12	064	816	605	438	318	251	239	289	409	605	12
13	093	845	635	469	350	284	273	324	445	642	13
14	121	875	665	500	382	316	306	358	481	679	14
15	150	904	695	531	414	349	340	393	517	716	15
16	179	934	726	562	445	382	374	428	553	754	16
17	208	963	756	593	477	415	408	463	589	792	17
18	237	2.01993	787	624	509	448	442	498	625	829	18
19	2.00266	2.02023	2.03817	2.05655	2.07541	2.09481	2.11476	2.13533	2.15661	2.17866	19
20	2.00295	2.02052	2.03847	2.05686	2.07573	2.09513	2.11509	2.13568	2.15697	2.17903	20
21	324	082	877	717	605	546	543	603	733	941	21
22	353	111	908	749	637	579	577	638	769	2.17979	22
23	382	141	938	780	669	612	611	673	805	2.18016	23
24	411	171	968	811	701	645	645	708	842	054	24
25	440	200	2.03998	842	733	677	678	742	878	091	25
26	469	230	2.04029	873	765	710	712	777	915	128	26
27	498	259	059	904	797	743	746	812	951	166	27
28	528	289	090	935	829	776	780	847	2.15987	204	28
29	2.00557	2.02319	2.04120	2.05966	2.07861	2.09809	2.11814	2.13883	2.16023	2.18242	29
30	2.00586	2.02348	2.04150	2.05997	2.07893	2.09842	2.11848	2.13918	2.16059	2.18280	30
31	615	378	181	2.06028	925	874	881	953	096	317	31
32	644	407	211	059	957	907	916	2.13988	133	355	32
33	673	437	242	090	2.07989	940	950	2.14023	169	393	33
34	702	467	272	122	2.08021	2.09973	2.11984	058	205	431	34
35	731	496	302	153	053	2.10006	2.12018	093	242	469	35
36	760	526	332	184	085	038	052	128	278	506	36
37	789	555	363	215	117	071	086	163	315	544	37
38	819	585	393	247	150	105	120	199	351	582	38
39	2.00848	2.02615	2.04424	2 06278	2.08182	2.10138	2.12154	2.14235	2.16387	2.18620	39
40	2.00877	2.02645	2.04454	2.06309	2.08214	2.10171	2 12188	2.14270	2.16423	2.18658	40
41	906	675	484	340	246	204	222	305	460	696	41
42	935	705	515	372	278	237	256	340	497	734	42
43	964	735	546	403	310	270	291	376	534	772	43
44	2.00993	765	577	435	343	304	325	411	571	811	44
45	2.01023	795	607	466	375	337	359	447	608	849	45
46	052	825	638	497	407	370	393	482	644	887	46
47	081	855	668	528	439	403	427	517	680	925	47
48	110	885	699	560	471	436	462	552	717	2.18963	48
49	2.01139	2.02915	2.04730	2.06591	2.08504	2.10470	2.12496	2 14588	2.16754	2.19001	49
50	2.01169	2.02945	2.04761	2.06623	2.08536	2.10504	2.12530	2.14623	2.16791	2.19039	50
51	198	2.02975	792	655	568	537	565	659	828	078	51
52	227	2.03004	822	686	600	570	599	694	864	115	52
53	256	034	852	718	633	603	633	729	901	153	53
54	286	064	883	749	665	637	668	765	938	192	54
55	315	094	914	781	698	670	702	801	2.16975	230	55
56	345	124	945	813	730	704	737	836	2.17012	269	56
57	374	154	2.04976	844	763	737	771	872	049	307	57
58	403	184	2.05006	875	795	770	805	907	086	345	58
59	2.01433	2.03214	2.05037	2.06907	2.08828	2.10804	2.12840	2.14943	2.17123	2.19384	59
60	2.01462	2.03244	2.05068	2.06938	2.08860	2.10837	2.12874	2.14979	2.17160	2.19422	60

TABLE VI.

μ + log tang z.

z	70°	71°	72°	73°	74°	75°	76°	77°	78°	79°	z
0′	2.19422	2.21778	2.24236	2.26811	2.29517	2.32367	2.35383	2 38591	2.42015	2.45693	0′
1	461	818	278	855	564	415	435	646	074	757	1
2	500	858	320	900	610	464	487	701	133	821	2
3	538	898	362	944	656	512	539	756	192	884	3
4	2.19576	2.21938	2.24404	2.26988	2.29702	2.32561	2.35591	2.38811	2.42251	2.45948	4
5	2.19615	2.21978	2.24446	2.27032	2.29749	2.32610	2.35643	2.38867	2.42310	2.46012	5
6	654	2.22019	489	076	795	660	695	922	370	076	6
7	693	059	531	120	842	709	747	2.38978	430	141	7
8	731	099	573	165	889	758	799	2.39034	490	205	8
9	2.19769	2.22139	2.24615	2.27209	2.29935	2.32807	2.35851	2.39089	2.42550	2.46269	9
10	2.19808	2.22180	2.24657	2.27253	2.29981	2.32857	2.35904	2.39144	2.42610	2.46333	10
11	847	220	699	297	2.30028	906	2.35956	200	670	398	11
12	886	261	742	342	075	2.32956	2.36009	256	730	462	12
13	925	301	784	386	122	2.33005	061	312	790	527	13
14	2.19963	2.22341	2.24826	2.27431	2.30168	2.33054	2.36113	2.39368	2.42850	2.46591	14
15	2.20002	2.22382	2.24869	2.27475	2.30215	2.33104	2.36166	2.39424	2.42910	2.46656	15
16	041	423	912	519	262	154	218	480	2.42970	721	16
17	080	464	954	564	309	203	271	536	2.43030	786	17
18	119	505	2.24997	609	356	253	324	593	090	851	18
19	2.20158	2.22545	2.25039	2.27654	2.30403	2.33303	2.36377	2.39649	2.43150	2.46917	19
20	2.20197	2.22585	2.25081	2.27699	2.30450	2.33352	2.36430	2.39706	2.43210	2.46982	20
21	236	626	124	744	497	402	483	762	271	2.47047	21
22	275	667	167	788	545	452	536	818	332	112	22
23	314	708	209	833	592	502	590	875	393	178	23
24	2.20353	2.22749	2.25252	2.27878	2.30639	2.33552	2.36643	2.39932	2.43454	2.47243	24
25	2.20392	2.22789	2.25294	2.27923	2.30685	2.33602	2.36696	2.39989	2.43515	2 47308	25
26	431	830	337	2.27968	733	652	749	2.40046	576	374	26
27	470	871	380	2.28013	780	702	802	103	637	440	27
28	510	912	423	057	827	752	855	159	698	506	28
29	2.20549	2.22953	2.25466	2.28102	2.30875	2.33803	2.36909	2.40216	2.43759	2.47572	29
30	2.20588	2.22994	2.25509	2.28148	2.30922	2.33853	2.36963	2.40273	2.43820	2.47638	30
31	627	2.23035	551	193	2.30967	903	2.37016	330	881	704	31
32	666	076	594	238	2.31017	2.33953	069	387	2.43943	770	32
33	706	117	637	283	065	2.34004	123	444	2.44005	836	33
34	2.20745	2.23158	2.25680	2.28328	2.31113	2.34054	2.37176	2.40501	2.44066	2.47903	34
35	2.20785	2.23200	2.25723	2.28374	2.31161	2.34105	2.37230	2.40559	2.44128	2.47970	35
36	824	241	766	419	208	155	283	616	189	2.48036	36
37	863	282	809	464	256	205	337	674	251	103	37
38	903	323	852	510	304	256	391	732	313	170	38
39	2.20943	2.23364	2.25896	2.28555	2.31352	2.34307	2.37445	2.40790	2.44375	2.48237	39
40	2.20983	2.23405	2.25939	2.28600	2.31400	2.34358	2.37499	2.40847	2.44437	2.48304	40
41	2.21022	446	2.25982	645	447	408	553	904	499	371	41
42	061	487	2.26025	691	495	459	607	2.40962	561	439	42
43	101	529	069	736	543	510	661	2.41020	624	506	43
44	2.21141	2.23570	2.26112	2.28782	2.31592	2.34561	2.37715	2.41078	2.44686	2.48574	44
45	2.21181	2.23612	2.26156	2.28828	2.31640	2.34612	2.37770	2.41136	2.44749	2.48641	45
46	220	653	199	873	688	663	824	194	811	709	46
47	259	694	242	919	736	714	878	252	873	776	47
48	299	736	286	2.28965	784	765	933	311	936	844	48
49	2.21339	2.23778	2.26329	2.29011	2.31833	2.34817	2.37988	2.41370	2.44999	2.48912	49
50	2.21379	2.23819	2.26373	2.29057	2.31881	2.34868	2.38042	2.41428	2.45061	2.48980	50
51	419	861	417	103	930	919	096	487	124	2.49048	51
52	458	902	460	148	2.31978	2.34970	150	545	187	116	52
53	498	944	5C4	195	2.32026	2.35021	205	603	250	184	53
54	2.21538	2.23986	2.26548	2.29241	2.32075	2.35073	2.38260	2.41662	2.45313	2.49253	54
55	2.21578	2.24028	2.26592	2.29287	2.32124	2.35125	2.38315	2.41721	2.45376	2.49322	55
56	618	070	636	333	173	177	370	780	440	390	56
57	658	112	680	379	221	228	425	838	503	459	57
58	698	153	723	425	269	279	480	896	566	5 7	58
59	2.21738	2.24194	2.26767	2.29471	2.32318	2.35331	2.38535	2.41955	2.45629	2.49596	59
60	2.21778	2.24236	2.26811	2.29517	2.32367	2.35383	2.38591	2.42015	2.45693	2.49665	60

TABLE VI.

$\mu + \log \tan z$.

z	80°	81°	82°	83°	84°	85°	86°	87°	88°	89°	z
0′	2.49665	2.53986	2.58722	2.63958	2.69809	2.76427	2.84003	2.92802	3.03197	3.15661	0
1	734	2.54061	805	2.64050	2.69913	545	139	2.92961	3.03388	3.15890	1
2	803	136	888	142	2.70017	663	275	2.93121	3.03579	3.16120	2
3	872	211	2.58971	234	120	781	412	2.93280	3.03769	3.16350	3
4	2.49942	2.54286	2.59054	2.64327	2.70224	2.76900	2.84548	2.93440	3.03960	3.16581	4
5	2.50011	2.54362	2.59137	2.64420	2.70328	2.77018	2.84685	2.93600	3.04151	3.16813	5
6	081	438	221	513	432	137	823	2.93761	3.04343	3.17046	6
7	150	514	305	607	537	257	2.84961	2.93923	3.04536	3.17279	7
8	220	591	389	701	642	377	2.85099	2.94086	2.04729	3.17513	8
9	2.50289	2.54667	2.59473	2.64794	2.70747	2.77497	2.85238	2.94249	3.04921	2.17747	9
10	2.50359	2.54743	2.59557	2.64888	2.70852	2.77617	2.85376	2.94412	3.05114	3.17982	10
11	429	820	642	2.64982	2.70958	737	515	2.94576	3.05309	3.18218	11
12	499	897	726	2.65077	2.71064	858	655	2.94740	3.05506	3.18455	12
13	569	2.54974	811	171	170	2.77979	795	2.94905	3.05703	3.18693	13
14	2.50639	2.55051	2.59895	2.65265	2.71276	2.78099	2.85935	2.95070	3.05900	3.18932	14
15	2.50709	2.55128	2.59980	2.65360	2.71382	2.78221	2.86076	2.95235	3.06097	3.19172	15
16	779	205	2.60066	455	489	344	217	2.95401	3.06295	3.19413	16
17	850	282	151	551	597	466	358	2.95567	3.06494	3.19655	17
18	921	359	236	646	704	589	500	2.95734	3.06693	3.19898	18
19	2.50992	2.55437	2.60321	2.65742	2.71812	2.78712	2.86642	2.95902	3.06893	3.20142	19
20	2.51063	2.55514	2.60406	2.65837	2.71919	2.78834	2.86785	2.96070	3.07093	3.20385	20
21	134	592	492	2.65933	2.72027	2.78957	2.86928	2.96238	3.07294	3.20629	21
22	205	669	578	2.66029	135	2.79081	2.87071	2.96406	3.07495	3.20874	22
23	276	747	664	125	244	205	214	2.46574	3.07697	3.21120	23
24	2.51347	2.55826	2.60750	2.66221	2.72353	2.79330	2.87358	2.96743	3.07899	3.21368	24
25	2.51418	2.55904	2.60836	2.66317	2.72461	2.29454	2.87502	2.96913	3.08101	3.21615	25
26	489	2.55982	2.60923	414	570	579	647	2.97083	3.08305	3.21864	26
27	561	2.56060	2.61010	511	680	704	792	2.97254	3.08510	3.22113	27
28	633	138	097	609	790	830	2.87937	2.97426	3.08716	3.22364	28
29	2.51705	2.56217	2.61185	2.66706	2.72900	2.79955	2.88083	2.97598	3.08923	3.22616	29
30	2.51777	2.56296	2.61272	2.66803	2.73010	2.80081	2.88229	2.97771	3.09131	3.22868	30
31	849	376	359	900	120	207	376	2.97944	3.09338	3.23120	31
32	922	455	446	2.66997	231	333	523	2.98117	4.09546	3.23373	32
33	2.51994	534	534	2.67095	342	460	671	2.98291	3.09755	3.23627	33
34	2.52067	2.56613	2.61622	2.67193	2.73453	2.80587	2.88818	2.98466	3.09965	3.23882	34
35	2.52140	2.56693	2.61710	2.67291	2.73565	2.80714	3.88966	2.98642	3.10175	3.24139	35
36	212	772	798	389	676	841	2.89115	2.98817	3.10386	3.24396	36
37	284	852	886	488	788	2.80969	264	2.48993	3.10597	3.24654	37
38	357	2.56932	2.61974	587	2.73900	2.81097	414	2.99170	3.10809	3.24913	38
39	2.52430	2.57012	2.62063	2.67687	2.74012	2.81226	2.89554	2.99348	3.11023	3.25173	39
40	2.52503	2.57092	2.62152	2.67786	2.74125	2.81355	2.89714	2.99526	3.11238	3.25434	40
41	576	172	240	885	237	484	2.89864	2.99705	3.11452	3.25695	41
42	649	252	329	2.67984	350	614	2.90015	2.99885	3.11667	3.25957	42
43	723	333	418	2.68084	463	744	167	3.00066	3.11882	3.26220	43
44	2.52797	2.57413	2.62508	2.68184	2.74577	2.81874	2.90319	3.00246	3.12098	3.26484	44
45	2.52870	2.57494	2.62598	2.68284	2.74691	2.82005	2.90471	3.00427	3.12315	3.26749	45
46	2.52944	575	687	384	805	136	623	3.00608	3.12532	3.27016	46
47	2.53017	656	776	484	2.74918	267	776	3.00789	3.12750	3.27284	47
48	091	737	866	585	2.75032	398	2.90929	3.00970	3.12969	3.27552	48
49	2.53165	2.57819	2.62956	2.68686	2.75147	2.82530	2.91083	3.01152	3.13189	3.27821	49
50	2.53239	2.57900	2.63046	2.68787	2.75262	2.82663	2.91237	3.01334	3.13411	3.28091	50
51	313	2.57982	137	888	377	795	392	3.01517	3.13633	3.28362	51
52	387	2.58063	227	2.68989	492	2.82927	547	3.01700	3.13855	3.28634	52
53	462	146	317	2.69091	608	2.83060	702	3.01884	3.14078	3.28907	53
54	2.53536	2.58228	2.63408	2.69193	2.75724	2.83194	2.91858	3.02070	3.14302	3.29181	54
55	2.53611	2.58310	2.63499	2.69295	2.75841	2.83328	2.92014	3.02256	3.14527	3.29456	55
56	686	393	591	398	2.75958	463	171	3.02443	3.14753	3.29732	56
57	761	475	682	501	2.76075	598	328	3.02630	3.14979	3.30009	57
58	835	557	773	603	192	732	485	3.02817	3.15206	3.30286	58
59	2.53910	2.58639	2.63865	2.69706	2.76309	2.83867	2.92643	3.03006	3.15433	3.30564	59
60	2.53986	2.58722	2.63958	2.69809	2.76427	2.84003	2.92802	3.03197	3.15661	3.30842	60

SHORT METHODS OF DETERMINING ORBITS

SECOND PAPER.

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Section of the American Mathematical Society. In this case the solution of the orbit without assumption regarding the eccentricity was indeterminate by any known method. But in this connection it was shown that the *Short Method* readily permits of the determination of the limits of the period and eccentricity within which, in more or less indeterminate cases, the observations may be satisfied, and hints were given for the computation of a parabola when the upper limit of the period or semimajor axis is found to be very large. With the application of these principles the *Short Method* gave parabolic orbits of comet *a* 1904 for the different sets of observations under consideration in accordance with the parabolic orbits deduced by the usual methods. Hence, even in this exceptional case, the *Short Method* in a modified form has yielded results as accurate as may be desired.

Concerning LAPLACE'S original method,[1] BAUSCHINGER quotes LAGRANGE[2] as having reached the conclusion that analytically it constitutes the simplest solution of the problem, but that in practice it does not afford corresponding advantages, because the differential coefficients could not be determined with the necessary accuracy. POINCARÉ, on the contrary, in a paper entitled *Sur la Détermination des Orbits par la Méthode de Laplace*[3] has recently shown that in the first hypothesis LAPLACE'S method is superior to that of GAUSS, inasmuch as for unequal intervals LAPLACE'S error in the geocentric distance ρ is only of the second order as compared to GAUSS'S error of the first order.

If we examine further into the question in dispute, it appears that of late a very important factor has been left out of consideration in judging the efficiency of the methods which are based on LAPLACE. All published improvements of the same are methods of determining orbits without assumption regarding the eccentricity. They ought, therefore, to be compared only with similar methods, as POINCARÉ has done. The preliminary orbits of comets derived, for instance, by the *Short Method* in its original form (Part 1) are not always comparable to parabolic orbits derived from the same observations by OLBERS'S method. If the physical orbit of a comet happen to be nearly parabolic, then a preliminary orbit derived from a short arc by OLBERS'S method will approximate the true orbit more or less closely. If the physical orbit differ considerably from a parabola, then a preliminary orbit derived by parabolic methods is based on erroneous assumptions and will fail to represent subsequent observations. The *Short Method* in its original form (Part 1) produces a preliminary orbit without assumption regarding the eccentricity. The degree of approximation of this orbit to the physical orbit depends on the accuracy with which the observations, in the nature of things, permit of the determination of a general orbit. If the range of practical solutions within which the observations may be represented be limited, then the *Short*

[1] LAPLACE, *Méc. Cél.*, t. I, première partie, livre II, chap. IV.

[2] LAGRANGE, *Oeuvres*, t. XIV, p. 108.

[3] *Bulletin Astronomique*, t. XXIII.

the necessity of introducing an additional observation. The method of determining circular orbits outlined in these pages is, therefore, not intended to lessen the numerical work required by the usual methods, but it has a far more extended range of usefulness.

The comet orbits computed at the Students' Observatory on the basis of the principles just outlined, demonstrate that the accuracy of the differential coefficients of the geocentric coördinates is such as to lead to orbits which agree more closely with the true orbits than orbits computed from similar arcs by other methods. Even a mere cursory[1] examination of the orbits, contained in *Lick Observatory Bulletins* 73 and thereafter, reveals that, with but one exception, they approximate the final orbits as closely as those computed from much longer arcs by other methods, in fact in general from arcs twice the length. The material for this comparison may be derived from the *A. N.* The final orbits are probably not as yet available for any of these comets, but the orbit computed from the longest geocentric arc will serve the purpose of comparison equally well. It is safe to conclude on the basis of these results that if, formerly, discrepancies from the true orbits have been observed in orbits derived by LAPLACEAN methods, the explanation therefor must be sought not in the insufficiency of the differential coefficients, but in the range of practical solutions, for it is safe to assume that no orbit has been published that does not actually satisfy the observations on which it is based. This range becomes highly restricted if an assumption be made regarding the eccentricity.

Orbits which are indeterminate are of two classes. They may be indeterminate, on the one hand, because the observations can actually be represented by a variety of orbits; on the other hand, because the adopted methods of solution may lead to an indeterminate result, although the orbit may be well defined by the given observations. Thus, for example, a solution by GAUSS's method becomes indeterminate if the three observations lie in a great circle; but this geometrical condition in itself does not in the least affect the accuracy of an orbit derived by LAPLACEAN methods, as for example by the *Short Method.* In dealing with cases of the first class, recourse must be had to an assumption regarding the eccentricity. For the second class, for which general methods are applicable, and for determinate cases, the solutions based on LAPLACEAN methods give in the first hypothesis results of greater accuracy than solutions by GAUSS'S method, as shown above by practical experience as well as analytically by reference to POINCARÉ'S investigation and my own.

The foregoing considerations apply to the first hypothesis in preliminary orbits, which is always based on moderate arcs. To discuss the relative merits in the first hypothesis of various methods on the basis of arcs of *any* length would be of purely theoretical interest and of no practical value, as moderate arcs are always available. In the study of the practical problem of determining preliminary orbits

[1] Compare, also, CRAWFORD'S conclusions in Part 5

coördinates in the methods based on LAPLACE. The results of the first hypothesis are used for the determination of more approximate values of these quantities, which then form the basis of the second hypothesis, and so forth. Where ratios of triangles are used, the convergence of the successive hypotheses depends in the main on the intervals, on r, and $\frac{dr}{dt}$ and may become very slow. If the highest accuracy be aimed at, as in v. OPPOLZER'S general method, the computation is very laborious. Thus, in v. OPPOLZER'S theoretically very elegant and precise method, it is necessary, as I have mentioned in my paper *On the General Applicability*, etc., to derive hypothesis after hypothesis, to perform a series of trials for the distances in each hypothesis, and to perform for each triangular ratio a series of approximations in passing from one hypothesis to the next. GIBBS'S vector method, although the most accurate method of all in the first hypothesis, but unfortunately requiring too large an amount of numerical work, does not readily lend itself for application to a second hypothesis, and POINCARÉ'S recent presentation of LAPLACE'S method has not as yet been cast in a suitable form for practical application.

Instead of forming a second hypothesis in the usual sense by correcting the geocentric velocities and accelerations on the basis of the first solution, it is here proposed, as in Part I, to determine at once from differential relations such corrections to the geocentric distance and to the velocities in the heliocentric rectangular coördinates,—which, together with the geocentric coördinates at the middle date, practically, are the elements of the orbit,—as will remove whatever residuals the first solution may leave in the first and third geocentric places. This mode of approximation is not comparable to the method of forming successive hypotheses, but obviously leads directly to the required result. Hitherto it has been applied to the solution of more or less definitive orbits from normal places, with the aid of the differential coefficients of the geocentric coördinates with respect to the geometrical elements. For this purpose very elegant solutions have been arranged, chief among which is that by BAUSCHINGER.[1] It may safely be asserted that, with proper adaptation to the computation of strictly parabolic and circular preliminary orbits so as to include cases in which a general solution is unwarranted, the methods of differential correction will lead more directly to the final result than the method of successive hypotheses in the usual sense.

The advantages of the method of differential correction, over the method of successive hypotheses in improving an orbit, are exactly the same as the advantages of the method of differential correction over the ordinary trials in the successive approximations for the distances in OLBERS'S and GAUSS'S methods, etc. In the course of the ordinary trials, the final values of the distances of one trial form the initial values in the next trial, etc. In the method of differential correction, such corrections to the initial values of one trial are derived differentially from

[1] *Veröffentlichungen des Kgl. Recheninstituts*, Berlin, No. 23.

the differences between the initial and final values in the same trial, as will produce an agreement of the initial and final values in the next trial. But the number of approximations required by the ordinary trials is, in general, far in excess of that required by the method of differential correction.

Another simplification involved in the direct solution and the differential correction proposed here is that the consideration of various topics essential to the older methods becomes unnecessary, as, for instance, the consideration of the ratio of sector to triangle, etc.

If it had been deemed desirable or essential, formulæ on the plan of successive hypotheses might have been set up without difficulty. For this purpose it is not necessary to determine the corrections to the initial values of the velocities and accelerations from the results of the first solution. Instead, an expression of $\sigma = \rho \cos \delta$ may be derived in which account is taken of the third and higher derivatives expressed in terms of the coördinates and their lower derivatives. The actual derivation of these expressions of the higher derivatives is somewhat complicated, although theoretically quite simple, but the resulting formula for σ admits of a second hypothesis as soon as r and $r' = \frac{1}{k}\frac{dr}{dt}$ become known in the first hypothesis. In this manner an expression for x has been derived on page 268, formula (4a), for the purpose of discussing the degree of approximation of the only hypothesis made in the *Short Method*. The formula, however, also serves the purpose of indicating the principles on which we would proceed in making a second hypothesis. For the purpose in hand, it was deemed unnecessary to take account of derivatives of the geocentric coördinates higher than the third.

POINCARÉ'S discussion of LAPLACE'S method is the most elegant treatment of that subject yet attained. His choice of the direction cosines, their velocities, and accelerations as fundamental data leads to a symmetry in the developments affording a clear perspective of the method and permitting of easy determination of its accuracy in relation to other methods. His discussion is also full of suggestions from a practical point of view. If none of these have been adopted here it is due to the fact that the formulæ presented below had already been applied to numerous cases prior to the publication of POINCARÉ'S paper and had proved entirely sufficient.

With reference to the corrections for parallax and aberration, POINCARÉ points out that they can be dealt with in a simpler manner than has been done in Parts 1–3. This was recognized soon after the publication of the *Short Method* in 1902 and was referred to in my *Notes on the Short Method of Determining Orbits*, etc., presented at the 1902 meeting of the Astronomical and Astrophysical Society of America.[1] The details of the elimination referred to are given on page 233 *et seq*.

With reference to the question of multiple solutions it is shown, among other things, on page 291, that there never can exist more than one physical parabolic

[1] *Publications*, Vol. 1, pp. 194–195.

solution. By applying the criteria on page 288 the computer may detect the existence of additional mathematical solutions, of which there may be two. By following the directions outlined on page 291 he may then discard the two fictitious out of the three mathematical parabolic solutions with the aid of the tables at the end of this paper. It is, indeed, one of the chief advantages of the method outlined here that the possibility of adopting a fictitious for the true solution is eliminated in the course of the computation. Further advantages are that the feasibility of a parabolic solution may be readily tested; that transference may be made to a general solution with little additional labor, and that the degree of accuracy of the adopted solution, which determines the range of practical solutions, is readily ascertained. While, however, there never can exist more than one solution when an assumption is made regarding the eccentricity, two solutions may sometimes occur when no such assumption is made; but at the present stage of the investigation it is found possible to distinguish the mathematical from the physical solution only in certain cases; that is to say, in the case of a general solution two orbits may sometimes result, satisfying the three given observations within the errors of observation and the degree of accuracy of the direct solution.

To assist the computer in the application of the methods derived here and in Part 1, the various formulæ for the direct computation of an orbit, with or without assumption regarding the eccentricity, and for the derivation of an orbit on the basis of a previous approximation, are collected in a "Synopsis of Formulæ" at the end of this paper.

Acknowledgment is due to Professor CRAWFORD for many valuable suggestions, particularly as regards the arrangement of the formulæ in a convenient form for computation.

Geneva, Switzerland, April, 1909.

THE INITIAL GEOCENTRIC VELOCITIES AND ACCELERATIONS.

The geocentric velocities and accelerations derived from three places at short intervals as in formulæ II, page 15, are practically always sufficiently accurate to serve as the basis of an orbit computation. But, if the geocentric motion be very irregular, as is sometimes the case for comets, or if it be desired to base the preliminary orbit from the start on a longer arc, then it may become expedient to deduce the velocities and accelerations from more than three places. For long arcs or intervals, however, it is preferable to derive initial values of the fundamental data σ, x', y', z' from the existing approximate orbits.

If more than three places be made the basis of the computation, the initial geocentric velocities and accelerations may be determined graphically or analytically.

The graphical determination of the velocities and accelerations is accomplished in the usual manner by drawing smooth curves for the right ascensions and declinations, then tabulating these coördinates from the curves for equidistant intervals and finally computing their first and second derivatives by the formulæ of numerical differentiation.

The analytical determination of the velocities and accelerations from any number of observations is readily accomplished by successive approximation.

In either case, it is essential that the observed or apparent places are expressed in right ascension and declination, and are reduced to the mean equinox at the beginning of the year in the usual manner by including the aberration of the fixed stars, while the dates remain uncorrected for planetary aberration until the geocentric distances shall have been determined.

We select as epoch the time of an observation lying about the middle of the available range of dates. As in Part I, we shall call this date the normal or zero date t_0, and designate the corresponding right ascension and declination by α_0 and δ_0. Ultimately the epoch will be the true date of this observation. Any other of the available right ascensions may then be represented by TAYLOR'S theorem as follows:

$$\alpha_i = \alpha_0 + \frac{(t_i - t_0)}{1}\alpha' + \frac{(t_i - t_0)^2}{2}\alpha'' + \frac{(t_i - t_0)^3}{6}\alpha''' + \ldots\ldots \tag{1}$$

where

$$\alpha^{m} = \frac{\partial^{m}\alpha}{\partial t^{m}},$$

with a similar series for the declinations. The numerical values of the differential coefficients of α and δ at the epoch with respect to the time are to be determined from the observed values α_i and δ_i.

The number of coefficients that can be determined is restricted by the number of positions available in a preliminary series of observations. The number of differential coefficients which is necessary and sufficient in a given case depends on the nature of the geocentric motion. The number is sufficient if intermediate places not used in the derivation of the differential coefficients are represented by (1) within the accuracy of the observations.

The number of observations necessary to determine n differential coefficients is $n+1$. The computation of these differential coefficients by a solution of n simultaneous equations of the form (1) would be very laborious without a convenient method of procedure, even if, as is generally the case, the number of differential coefficients to be determined is limited.

In the first approximation we confine ourselves to the determination of the first two differential coefficients from the first, last, and the normal observation of the series. This is accomplished in the same manner as the velocities and accelerations in α and δ are determined in formulæ II, page 15. The unit of time in these formulæ is $1/k$ mean solar days, where k^2 is the solar constant of attraction. Inasmuch as this unit can be introduced more conveniently later, we shall for the present choose the mean solar day as the unit of time in determining the successive derivatives. Hence, we let $k = 1$ in II, page 15. Let the velocities and accelerations so determined from three observations be denoted by

$$\alpha'_0, \quad \delta'_0, \quad \alpha''_0, \quad \delta''_0 \tag{2a}$$

If determined from five observations by formula (1) the velocities and accelerations may be written

$$\alpha'_0 + \partial_1\alpha'_0, \quad \delta'_0 + \partial_1\delta'_0, \quad \alpha''_0 + \partial_1\alpha''_0, \quad \delta''_0 + \partial_1\delta''_0 \tag{2b}$$

and if from seven observations

$$\alpha'_0 + \partial_1\alpha'_0 + \partial_2\alpha'_0, \quad \delta'_0 + \partial_1\delta'_0 + \partial_2\delta'_0, \quad \alpha''_0 + \partial_1\alpha''_0 + \partial_2\alpha''_0, \quad \delta''_0 + \partial_1\delta''_0 + \partial_2\delta''_0 \tag{2c}$$

and so forth. For the true values α', δ', α'', δ'' of the velocities and accelerations at the normal date we shall have

$$\begin{aligned} \alpha' &= \alpha'_0 + \partial_1\alpha'_0 + \partial_2\alpha'_0 + \ldots \\ \alpha'' &= \alpha''_0 + \partial_1\alpha''_0 + \partial_2\alpha''_0 + \ldots \end{aligned} \tag{2d}$$

with similar expressions for δ' and δ''. In the successive approximations of the velocities and accelerations, the higher differential coefficients are neglected, viz., differential coefficients higher than the second in the first approximation, higher than the fourth in the second, and so forth. In general, we may denote a differential coefficient of any order resulting from the first approximation by α^m_0; from the second by $\alpha^m_0 + \partial_1\alpha^m_0$ from the third by $\alpha^m_0 + \partial_1\alpha^m_0 + \partial_2\alpha^m_0$ *a. s. f.*, so that

$$\alpha^m = \alpha^m_0 + \partial_1\alpha^m_0 + \partial_2\alpha^m_0 + \partial_3\alpha^m + \ldots,$$

and similarly for δ^m.

Hence, $\alpha'''_0 = \alpha^{iv}_0 = \alpha^{v}_0 = \alpha^{vi}_0 = \ldots = 0$

and $\partial_1\alpha^{v}_0 = \partial_1\alpha^{vi}_0 = \ldots = 0$, *a. s. f.*

For the second approximation, based on five observations, we have from (1) and (2)

$$\alpha_i = \alpha_0 + (t_i - t_0)[\alpha_0' + \partial_1\alpha_0'] + \frac{(t_i - t_0)^2}{2}[\alpha_0'' + \partial_1\alpha_0''] + \frac{(t_i - t_0)^3}{6}\partial_1\alpha_0''' + \frac{(t_i - t_0)^4}{24}\partial_1\alpha_0^{\text{iv}}, \quad (3)$$

$i = 1, 2, 4, 5$; $\alpha_0 = \alpha_3$, where the i are taken in the order of time.
Let us suppose that the first approximation was based on

$$\alpha_1, \quad \alpha_3, \quad \alpha_5.$$

Then putting

$$\left.\begin{aligned}
&\frac{\alpha_i - [\alpha_0 + (t_i - t_0)\alpha_0' + \frac{(t_i - t_0)^2}{2}\alpha_0'']}{t_i - t_0} = \partial\alpha_i = n_{i,\alpha} \\
&w_\alpha = \partial_1\alpha_0^{\text{iv}}; \qquad a_i = \frac{(t_i - t_0)^3}{4!} \\
&x_\alpha = \partial_1\alpha_0'''; \qquad b_i = \frac{(t_i - t_0)^2}{3!} \\
&y_\alpha = \partial_1\alpha_0''; \qquad c_i = \frac{t_i - t_0}{2!} \\
&z_\alpha = \partial_1\alpha_0'; \qquad d_i = 1,
\end{aligned}\right\} \quad (4)$$

we have the following system of equations

$$\left.\begin{aligned}
a_1w_\alpha + b_1x_\alpha + c_1y_\alpha + z_\alpha &= 0 & \text{(a)}\\
a_2w_\alpha + b_2x_\alpha + c_2y_\alpha + z_\alpha &= n_{2,\alpha} & \text{(b)}\\
a_4w_\alpha + b_4x_\alpha + c_4y_\alpha + z_\alpha &= n_{4,\alpha} & \text{(c)}\\
a_5w_\alpha + b_5x_\alpha + c_5y_\alpha + z_\alpha &= 0 & \text{(d)}
\end{aligned}\right\} \quad (5)$$

In the same way we shall have for the declinations

$$\left.\begin{aligned}
a_1w_\delta + b_1x_\delta + c_1y_\delta + z_\delta &= 0 & \text{(a)}\\
a_2w_\delta + b_2x_\delta + c_2y_\delta + z_\delta &= n_{2,\delta} & \text{(b)}\\
a_4w_\delta + b_4x_\delta + c_4y_\delta + z_\delta &= n_{4,\delta} & \text{(c)}\\
a_5w_\delta + b_5x_\delta + c_5y_\delta + z_\delta &= 0 & \text{(d)}
\end{aligned}\right\} \quad (6)$$

As only the unknowns y and z will ultimately be required, it is not necessary to compute the w and x. The following form of solution of (5) and (6) has been arranged after GAUSS'S method of elimination. Neglecting the subscripts α and δ, equations (a) give:

$$w = -\frac{b_1}{a_1}x - \frac{c_1}{a_1}y - \frac{1}{a_1}z. \quad (7)$$

Eliminating w from (b), (c), (d), by means of (7) and letting

$$\left.\begin{aligned}
p_i - \frac{a_i}{a_1}p_1 = [p_i \cdot 1]; \quad p_i = b_i, c_i, 1 \\
i = 2, 4, 5.
\end{aligned}\right\} \quad (8)$$

we obtain

$$\left.\begin{aligned}
[b_2 \cdot 1]x + [c_2 \cdot 1]y + [1_2 \cdot 1]z &= n_2 & \text{(e)}\\
[b_4 \cdot 1]x + [c_4 \cdot 1]y + [1_4 \cdot 1]z &= n_4 & \text{(f)}\\
[b_5 \cdot 1]x + [c_5 \cdot 1]y + [1_5 \cdot 1]z &= 0 & \text{(g)}
\end{aligned}\right\} \quad (9)$$

Eliminating x from (e) and (f) by means of (g) and letting

$$\left.\begin{array}{c} [p_i \cdot 1] - \dfrac{[h_i \cdot 1]}{[h_3 \cdot 1]}[p_3 \cdot 1] = [p_i \cdot 2] \\ p_i = c_i,\ 1;\quad i = 2, 4, \end{array}\right\} \qquad (10)$$

we have the following equation in y and z

$$\left.\begin{array}{l} [c_2 \cdot 2]\, y + [1_2 \cdot 2]\, z = n_2 \\ [c_4 \cdot 2]\, y + [1_4 \cdot 2]\, z = n_4 . \end{array}\right\} \qquad (11)$$

From these

$$y = \frac{n_2 [1_4 \cdot 2] - n_4 [1_2 \cdot 2]}{[c_2 \cdot 2][1_4 \cdot 2] - [c_4 \cdot 2][1_2 \cdot 2]} \; ; \quad z = \frac{n_4 [c_2 \cdot 2] - n_2 [c_4 \cdot 2]}{[c_2 \cdot 2][1_4 \cdot 2] - [c_4 \cdot 2][1_2 \cdot 2]} \qquad (12)$$

In practice this form of solution is extremely simple. Its chief advantage consists in the identity of the coefficients for both systems of equations, (5) and (6), so that after the $[p_i \cdot 2]$ coefficients are computed, equations (12) give:

$$y_\alpha = \partial_1 \alpha_0'' , \quad z_\alpha = \partial_1 \alpha_0' , \text{ from } n_{2,\alpha} \text{ and } n_{4,\alpha} ,$$

and

$$y_\delta = \partial_1 \delta_0'' , \quad z_\delta = \partial_1 \delta_0' , \text{ from } n_{2,\delta} \text{ and } n_{4,\delta} .$$

The combined number of $[p_i \cdot 1]$ and $[p_i \cdot 2]$ coefficients to be computed is thirteen. The numerical values of

$$\partial_1 \alpha_0' ,\ \partial_1 \delta_0' ,\ \partial_1 \alpha_0'' ,\ \partial_1 \delta_0''$$

are to be substituted in (2d) together with the first approximation for the velocities and accelerations defined in (2a), in order to obtain

$$\alpha',\ \delta',\ \alpha'',\ \delta''.$$

The procedure outlined above is readily extended to any given odd number of observations. The successive approximations have been based on an odd number of observations merely on account of the symmetry of the resulting formulæ. Similar forms of solution may readily be written out for an even number of observations. The velocities and accelerations may now be expressed in the proper unit of time. If the unit of time is to be $\frac{1}{k}$ mean solar days, α' and δ' must be divided by k, and α'' and δ'' by k^2.

and still simpler procedure was proposed for the elimination of the parallax, the aberration requiring no further consideration in a preliminary orbit calculation. This procedure is as follows:

In the equations

$$\xi = \rho \cos \delta_g \cos \alpha_g = x + X; \qquad \eta = \rho \cos \delta_g \sin \alpha_g = y + Y$$
$$\zeta = \rho \sin \delta_g = z + Z. \tag{1}$$

let ξ, η, ζ, ρ, α_g, δ_g, X, Y, Z be geocentric coördinates. Let $p_\alpha^g \rho$ and $p_\delta^g \rho$ designate the parallax factors for the reduction of the observed place to the center of the Earth, expressed in seconds of arc, and let α and δ be the observed coördinates. Then we may write

$$\rho \cos (\delta + p_\delta^g) \cos (\alpha + p_\alpha^g) = x + X; \quad \rho \cos (\delta + p_\delta^g) \sin (\alpha + p_\alpha^g) = y + Y$$
$$\rho \sin (\delta + p_\delta^g) = z + Z. \tag{2}$$

or, expanding the sines and cosines and neglecting the second and higher powers of p_α^g and p_δ^g, as well as their products,

$$\rho \cos \delta \cos \alpha - \rho \cos \delta \sin \alpha \, p_\alpha^g - \rho \sin \delta \cos \alpha \, p_\delta^g = x + X$$
$$\rho \cos \delta \sin \alpha + \rho \cos \delta \cos \alpha \, p_\alpha^g - \rho \sin \delta \sin \alpha \, p_\delta^g = y + Y \tag{3}$$
$$\rho \sin \delta + \rho \cos \delta \, p_\delta^g = z + Z.$$

Since the parallax factors $p_\alpha^g \rho$ and $p_\delta^g \rho$ are known, the terms containing them may be applied as corrections to the solar coördinates. Hence, if we let

$$\Delta X = \cos \delta \sin \alpha \, p_\alpha^g \rho + \sin \delta \cos \alpha \, p_\delta^g \rho; \quad \Delta Y = - \cos \delta \cos \alpha \, p_\alpha^g \rho + \sin \delta \sin \alpha \, p_\delta^g \rho$$
$$\Delta Z = - \cos \delta \, p_\delta^g \rho, \tag{4}$$

we may write

$$(\xi) = \rho \cos \delta \cos \alpha = x + X + \Delta X; \quad (\eta) = \rho \cos \delta \sin \alpha = y + Y + \Delta Y$$
$$(\zeta) = \rho \sin \delta = z + Z + \Delta Z, \tag{5}$$

where ρ is the geocentric distance, while α and δ are referred to the observer. In computing the corrections to the solar coördinates by (4), $p_\alpha^g \rho$ and $p_\delta^g \rho$ must be multiplied by sin 1″ to reduce the corrections to the proper unit.

It will be observed that by correcting the solar coördinates by (4), the parallax is fully taken into account in computing coördinates by (5). Thus, for instance, after the value of $\sigma_0 - \rho_0 \cos \delta_{,,}$ has been derived by V, page 16, the heliocentric coördinates at the middle date, needed in VI, follow at once from (5). Similarly in the comparison between theory and observation for the first and third places no correction for parallax needs to be applied, it being contained in (5). But it remains to investigate what effect this elimination of the parallax has on the velocities and accelerations which are involved.

By successive differentiation of expressions (5) we obtain

$$(\xi)'' = (\rho \cos \delta \cos \alpha)'' = x'' + [X + \Delta X]'' \tag{6}$$

and similar expressions for $(\eta)''$ and $(\zeta)''$. If the α_g and δ_g were given at the start, the same operation might have been performed on (1). We should then have to derive the velocities and accelerations in α and δ from geocentric positions and the expressions would have been correct, so far as *this* parallax is concerned. But as the α_g and δ_g are not available at the start, it is proposed to use (6) and to introduce into these formulæ the velocities and accelerations derived by numerical differ-

with similar equations in Y and Z, where m and m_1 represent the mass of the Earth and of the Moon, respectively, and where the subscript m signifies that the coördinates are referred to the center of mass of the system Earth-Moon. The necessity of reducing the geocentric solar coördinates taken from an astronomical ephemeris to the center of mass of the system Earth-Moon, before they may be introduced for the elimination of the solar accelerations in direct orbit solutions by LAPLACEAN methods has been pointed out by BRUNS in his discussion of LAMBERT'S theorem.[1] The reduced places are referred to by BRUNS as *barycentric* places.

In the equation $\xi = x + X$, it is, of course, immaterial whether ξ and X are referred to the observer, or to the center of the Earth, or to the center of mass of Earth-Moon system, or to any other arbitrary intermediate origin, so long as ξ and X are referred to the *same* intermediate origin. The same statement holds for the equations $\xi' = x' + X'$ and $\xi'' = x'' + X''$, which give the heliocentric velocities and accelerations. But since in the development of the formulæ of the LAPLACEAN methods the solar accelerations are to be eliminated by means of the equations of motion (9a) and since these equations are rigorously true only with reference to the center of mass of the Earth-Moon system, therefore ξ and X must also be referred to the center of mass. In a rigorous solution of the orbit, equations (1) and their derivations must, therefore, be replaced by the equations

$$\xi_m = \rho_m \cos\alpha_m \cos\delta_m = x + X_m \tag{1a}$$
$$\xi'_m = (\rho_m \cos\alpha_m \cos\delta_m)' = x' + X'_m \tag{1b}$$
$$\xi''_m = (\rho_m \cos\alpha_m \cos\delta_m)'' = x'' + X''_m \tag{1c}$$

and the corresponding equations in η_m and ζ_m, where the subscript m again signifies that the coordinates, velocities, and accelerations are referred to the center of mass of the Earth-Moon system. But just as the geocentric places, so also must the barycentric places of the body remain unknown until after the geocentric distances shall have been determined.

The angle subtended at the body by the line joining the observer and the center of the Earth will hereafter be called the *geocentric parallax* and, as above, be denoted by p^g. The angle subtended at the body by the line joining the center of the Earth and the center of mass of the Earth-Moon system will be called the *barycentric parallax* and be denoted by p^m, while the angle subtended at the body by the line joining the observer and the center of mass will simply be called the *parallax* and be denoted by p. Similarly the corresponding parallaxes in α and δ will be denoted by p^g_α, p^g_δ; p^m_α, p^m_δ; and p_α, p_δ, respectively. The corresponding parallax factors will be denoted by $p^g_\alpha\rho$, $p^g_\delta\rho$; $p^m_\alpha\rho_m$, $p^m_\delta\rho_m$; and $p_\alpha\rho_m$, $p_\delta\rho_m$, respectively.

Since $p_\alpha = p^g_\alpha + p^m_\alpha$; $p_\delta = p^g_\delta + p^m_\delta$, we have, after multiplying by ρ_m and writing, with an error of the second order only of the parallax, $p^g_\alpha\rho$ for $p^g_\alpha\rho_m$; $p^g_\delta\rho$ for $p^g_\delta\rho_m$; $p_\alpha\rho_m = p^g_\alpha\rho + p^m_\alpha\rho_m$; $p_\delta\rho_m = p^g_\delta\rho + p^m_\delta\rho_m$.

It has been the rule to neglect the geocentric as well as the barycentric parallax in the first direct solution of the geocentric distances. HARZER repeats the direct solution after correcting the observations only for geocentric parallax on

[1] H. BRUNS, *der Lambert'sche Satz* [illegible]

the basis of geocentric distances resulting from the first direct solution. BRUNS, however, corrects also for barycentric parallax, before repeating the direct solution.

As stated above, on page 233, it was pointed out in Part 1, page 9, that the second direct solution might be omitted altogether, and that the small changes in the orbit resulting from the geocentric parallax corrections might be determined just as well from the observation equations as part of the final corrections of the heliocentric coördinates and velocities. This remark holds also for the barycentric parallax.

It is evident, however, that the method defined by equations (4) and (5) of eliminating the geocentric parallax in the first direct solution may at once be extended to the simultaneous elimination of the barycentric parallax. For this purpose it is only necessary to introduce in equations (4) the parallax factors $p_\alpha \rho_m$, $p_\delta \rho_m$ defined above, which correspond to a reduction of the observed α and δ to the center of mass. These *parallax factors* we have seen are equal to the algebraical sum of the corresponding *geocentric* and *barycentric parallax factors*. As stated above, hereafter this sum will simply be designated by *parallax factor*, without modification. Equations (4) will therefore be applicable to the elimination of the *parallax* or of the *geocentric parallax alone*, or of the *barycentric parallax alone*, as we substitute in that equation the corresponding factors for $p_\alpha^g \rho$ and $p_\delta^g \rho$.

Let the corrections to the solar coördinates resulting from equation (4) on the basis of $p_\alpha \rho_m$ and $p_\delta \rho_m$ be denoted by $\Delta_2 X$, $\Delta_2 Y$, and $\Delta_2 Z$; and let $\Delta_1 X$, $\Delta_1 Y$, $\Delta_1 Z$ represent the corrections which must be applied to the geocentric solar coördinates in order to reduce them to the center of mass. Then

$$\Delta X = \Delta_1 X + \Delta_2 X\,; \qquad \Delta Y = \Delta_1 Y + \Delta_2 Y\,; \qquad \Delta Z = \Delta_1 Z + \Delta_2 Z$$

are the complete corrections which must be applied to the interpolated solar coördinates for the *total elimination of the parallax* in α and δ (geocentric and barycentric), including the reduction of the solar coördinates to the center of mass.

We shall first derive $\Delta_1 X$, $\Delta_1 Y$, $\Delta_1 Z$. Let d_1 and d be the distances of the center of mass and of the Moon, respectively, from the center of the Earth, α_1, δ_1 the geocentric coördinates of the Moon, as given in the astronomical ephemerides. Then the corrections to be applied to the geocentric solar coördinates X, Y, Z are

$$\Delta_1 X = -\,d_1 \cos\delta_1 \cos\alpha_1\,; \qquad \Delta_1 Y = -\,d_1 \cos\delta_1 \sin\alpha_1\,; \qquad \Delta_1 Z = -\,d_1 \sin\delta_1\,.$$

Let π and π_1 be the values of the mean horizontal equatorial parallax of the Sun and Moon respectively, and m and m_1 the masses of the Earth and of the Moon, respectively. Then we have with more than sufficient accuracy

$$d_1 = d\,\frac{m_1}{m + m_1} = \frac{\pi}{\pi_1}\,\frac{\mu}{1+\mu}\,,$$

where μ is the ratio of the Moon's mass to the Earth's mass and where d is given in astronomical units. With HINKS'S[1] value of $\pi = 8''.807$ and NEWCOMB'S[2] values of $\pi_1 = 57'\,2''.68$ and $\mu = \frac{1}{81.45}$ we obtain

$$\frac{d_1}{\sin 1''} = 6''.437\,.$$

[1] *M. N. R. A. S.* Vol. LXIX, page 567.

[2] *The Elements of the Four Inner Planets and the Fundamental Constants of Astronomy. Supplement to the American Ephemeris and Nautical Almanac for 1897.*

The barycentric parallax factors in α and δ are given by the left hand sides of equations (15), page 11, if $\Delta_1 X$, $\Delta_1 Y$, $\Delta_1 Z$ be substituted on the right hand side for ∂x, ∂y, ∂z, respectively. In the first of these equations $\partial_1\alpha = \cos\delta\,\partial\alpha$. Hence

$$p_\alpha^m \rho_m = \frac{\sin\alpha}{\cos\delta}\, d_1 \cos\delta_1 \cos\alpha_1 - \frac{\cos\alpha}{\cos\delta}\, d_1 \cos\delta_1 \sin\alpha_1\,,$$

$$p_\delta^m \rho_m = -\sin\delta\cos\alpha\, d_1\cos\delta_1\cos\alpha_1 + \sin\delta\sin\alpha\, d_1\cos\delta_1\sin\alpha_1 - \cos\delta\, d_1\sin\delta_1\,,$$

or

$$p_\alpha^m \rho_m = d_1 \frac{\cos\delta_1}{\cos\delta} \sin(\alpha - \alpha_1)$$

$$p_\delta^m \rho_m = d_1\,[\sin\delta\cos\delta_1\cos(\alpha - \alpha_1) - \cos\delta\sin\delta_1].$$

By adding these *barycentric* to the *geocentric parallax factors* which are generally given with the observations, the required *parallax factors* $p_\alpha\rho_m$ and $p_\delta\rho_m$ may now be formed.

For equations (1a) we now write with reference to the center of mass

$$\xi_m = \rho_m \cos(\delta + p_\delta)\cos(\alpha + p_\alpha) = x + X_m\,; \quad \eta_m = \rho_m\cos(\delta + p_\delta)\sin(\alpha + p_\alpha) = y + Y_m$$
$$\zeta_m = \rho_m \sin(\delta + p_\delta) = z + Z_m\,,$$

where evidently $X_m = X + \Delta_1 X$, etc. From these equations we derive the following equations in the same manner as equations (4) to (11) were derived from equations (1)

$$\Delta_2 X = -\cos\delta\sin\alpha\, p_\alpha\rho_m + \sin\delta\cos\alpha\, p_\delta\rho_m\,; \qquad \Delta_2 Y = -\cos\delta\cos\alpha\, p_\alpha\rho_m + \sin\delta\sin\alpha\, p_\delta\rho_m$$
$$\Delta_2 Z = -\cos\delta\, p_\delta\rho_m \tag{4a}$$

$$(\xi) \quad \rho_m\cos\delta\cos\alpha = x + X_m + \Delta_2 X\,, \qquad (\eta) \quad \rho_m\cos\delta\sin\alpha = y + Y_m + \Delta_2 Y$$
$$(\zeta) \quad \rho_m \sin\delta + Z_m + \Delta_2 Z\,. \tag{5a}$$

$$(\xi)'' = \frac{X_m + \Delta_2 X - (\xi)}{r^3} + [X_m + \Delta_2 X]''\,. \tag{8a}$$

And, subject to later correction,

$$(X_m + \Delta_2 X)'' = -\frac{X_m + \Delta_2 X}{(R_m + \Delta_2 R)^3}\,, \tag{9aa}$$

where

$$R\Delta_2 R = X\Delta_2 X + Y\Delta_2 Y + Z\Delta_2 Z\,. \tag{10a}$$

$$(\xi)'' + \frac{(\xi)}{r^3} = (X_m + \Delta_2 X)\left[\frac{1}{r^3} - \frac{1}{(R_m + \Delta_2 R)^3}\right]. \tag{11a}$$

Equation (11a) is rigid except for the error committed in using the substitution (9aa) instead of (9a). The mass $m + m_1$ of the Earth-Moon system is negligible in preliminary orbit computations.

To determine the remaining error and to allow for the same, if necessary, we may write (9aa) by means of (9a)

$$\frac{X_m}{R_m^3} + (\Delta_2 X)'' = \frac{X_m}{R_m^3} - \frac{\Delta_2 X}{R_m^3} + 3\frac{X_m \Delta_2 R}{R_m^4}\,, \tag{12}$$

where powers of $\Delta_2 R$ higher than the first and the product, $\Delta_2 X\,\Delta_2 R$, are neglected. Expression (11a), therefore, requires the correction

$$(\Delta_2 X)'' + \frac{\Delta_2 X}{R_m^3} - 3\frac{X_m\Delta_2 R}{R_m^4}\,. \tag{13}$$

In this expression we may use the corrected values of the solar coördinates.

Denoting these by (X), (Y), (Z), so that

$$(X) = (X_m + \Delta_2 X)\ , \text{ etc. ;} \qquad (R) = (R_m + \Delta_2 R)\ , \tag{14}$$

we obtain for the corrected equation (11a)

$$(\xi)'' + \frac{(\xi)}{r^3} = (X)\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right] + \frac{\Delta_2 X}{(R)^3} + (\Delta_2 X)''\ , \tag{15}$$

and analogous equations in the other two coördinates.

The numerical values of the $\Delta_2 X$, $\Delta_2 Y$ and $\Delta_2 Z$ depend by (4a) on the coördinates of the body and on the parallax factors. Hence, unless the observations have been made at stations differing widely in latitude, or unless the hour angles be very different, they may vary proportionally to the intervals for the three dates. If the proportionality hold true, then $(\Delta_2 X)''_0 = (\Delta_2 Y)''_0 = (\Delta_2 Z)''_0 = 0$. But we shall first consider the case where these accelerations cannot be neglected. Then $(\Delta_2 X)''_0$, etc., must be obtained from $\Delta_2 X_{,}$, $\Delta_2 X_{,,} = \Delta_2 X_0$, and $\Delta_2 X_{,,,}$, etc., in the same manner as α''_0 or δ''_0 is obtained from $\alpha_{,}$, $\alpha_{,,} = \alpha_0$, and $\alpha_{,,,}$, etc., in II, page 15, and the last two terms of (15) become for the middle date

$$\frac{1}{k^2}\cdot\frac{\dfrac{\Delta_2 X_{,,,} - \Delta_2 X_0}{t_{,,,} - t_{,,}} - \dfrac{\Delta_2 X_0 - \Delta_2 X_{,}}{t_{,,} - t_{,}}}{t_{,,,} - t_{,}} + \frac{\Delta_2 X_0}{(R)^3}\,. \tag{16}$$

Let

$$\Delta_2 X_{,,,}\,(t_{,,} - t_{,}) + \Delta_2 X_{,}\,(t_{,,,} - t_{,,}) = \Delta_2 X_0\,(t_{,,,} - t_{,})\,(1 + d_x)\ . \tag{17}$$

Then (16) reduces to

$$\left[\frac{2d_x}{\theta_{,}\theta_{,,,}} + \frac{1}{(R)^3}\right]\Delta_2 X_0\ , \tag{18}$$

where $\theta_{,} = k\,(t_{,,,} - t_{,,})$ and $\theta_{,,,} = k\,(t_{,,} - t_{,})$. After deriving analogous expressions in the second and third coördinates and letting

$$j\cos a = \left[\frac{1}{(R)^3} + \frac{2d_x}{\theta_{,}\theta_{,,,}}\right]\Delta_2 X_0\,;\; j\sin a = \left[\frac{1}{(R)^3} + \frac{2d_y}{\theta_{,}\theta_{,,,}}\right]\Delta_2 Y_0;\; j\tan d = \left[\frac{1}{(R)^3} + \frac{2d_z}{\theta_{,}\theta_{,,,}}\right]\Delta_2 Z_0, \tag{19}$$

equations (15) become

$$(\xi)'' + \frac{(\xi)}{r^3} = (X)\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right] + j\cos a$$

$$(\eta)'' + \frac{(\eta)}{r^3} = (Y)\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right] + j\sin a$$

$$(\zeta)'' + \frac{(\zeta)}{r^3} = (Z)\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right] + j\tan d\ .$$

If, after introducing polar coördinates throughout in these equations, the indicated differentiations be performed, they may be written

$$\sigma''\cos\alpha - 2\,\sigma'\sin\alpha\alpha' - \sigma\sin\alpha\alpha'' - \sigma\cos\alpha\left[(\alpha')^2 - \frac{1}{r^3}\right] = S\cos A\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right] + j\cos a$$

$$\sigma''\sin\alpha + 2\,\sigma'\cos\alpha\alpha' + \sigma\cos\alpha\alpha'' - \sigma\sin\alpha\left[(\alpha')^2 - \frac{1}{r^3}\right] = S\sin A\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right] + j\sin a$$

$$\sigma''\tan\delta + 2\,\sigma'(\tan\delta)' + \sigma\left[(\tan\delta)'' + \frac{\tan\delta}{r^3}\right] = S\tan D\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right] + j\tan d\,,$$

where the subscripts have been dropped, as all quantities refer to the second date.

If, as usual, the first two of these equations be added, on the one hand after multiplication of the first by $\cos\alpha$ and of the second by $\sin\alpha$, on the other hand after multiplication of the first by $-\sin\alpha$ and of the second by $\cos\alpha$, we have

$$\left.\begin{aligned}
\sigma'' + \sigma\left[\frac{1}{r^3} - (\alpha')^2\right] &= S\cos(A-\alpha)\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right] + j\cos(a-\alpha) \\
2\sigma'\alpha' + \sigma\alpha'' &= S\sin(A-\alpha)\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right] + j\sin(a-\alpha) \\
\sigma''\tan\delta + 2\sigma'(\tan\delta)' + \sigma\left[\frac{\tan\delta}{r^3} + (\tan\delta)''\right] &= S\tan D\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right] + j\tan d\,.
\end{aligned}\right\}\quad(20)$$

The solution of these equations for σ and σ', which is conveniently accomplished by means of determinants, gives, after replacing the subscripts

$$\left.\begin{aligned}
\sigma_0 = \rho_m\cos\delta_0 &= \kappa\left\{\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right\} + \Delta\kappa \\
\sigma'_0 &= \lambda\left\{\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\Delta_2 R}{(R)}\right)\right\} + \Delta\lambda\,,
\end{aligned}\right\}\quad(21)$$

where

$$\left.\begin{aligned}
\Delta\kappa &= -\frac{j}{N}\left\{\left[\tan\delta_0\cos(a-\alpha_0) - \tan d\right]\alpha'_0 + \sin(a-\alpha_0)\,(\tan\delta)'_0\right\} \\
\Delta\lambda &= \frac{j}{2N}\left\{\left[\tan\delta_0\cos(a-\alpha_0) - \tan d\right]\alpha'_0 + \sin(a-\alpha_0)\left[(\alpha'_0)^2\tan\delta_0 + (\tan\delta)''_0\right]\right\},
\end{aligned}\right\}\quad(22)$$

and where N, λ, and κ are defined as in III and V, pages 15 and 16.

Divide both sides of the first of (21) by $(R)\cos\delta_0$ and let

$$z = \frac{\rho_m}{(R)}\,;\quad (m) = \frac{\kappa}{(R)^4\cos\delta_0}\left(1 + 3\frac{\Delta_2 R}{(R)}\right) + \frac{\Delta\kappa}{(R)\cos\delta_0}\,. \quad (23)$$

Then the first of (21) may be written

$$z = (m) - \frac{\kappa}{r^3(R)\cos\delta_0} = (\nu)\,. \quad (24)$$

The value of z is found from this equation by eliminating r by means of

$$r^2 = \rho_m^2 + (R)^2 - 2\rho_m(R)\cos\psi\,,$$

or if, as before, we introduce $z = \frac{\rho_m}{(R)}$, by means of

$$r^2 = (R)^2(z^2 - 2z\cos\psi + 1) = \mu(R)^2\,. \quad (25)$$

The elimination gives

$$r^6 = \mu^3(R)^6 = \frac{\kappa^2}{(R)^2\cos^2\delta_0\,(\nu)^2}\,,$$

or

$$\mu^3(\nu)^2 = \frac{\kappa^2}{(R)^8\cos^2\delta_0} = m^2\,, \quad (26)$$

where m is defined as in III, page 15. Replacing μ and (ν) by means of (25) and (24) we obtain for the equation in $z = \frac{\rho_m}{(R)}$

$$(z^2 - 2z\cos\psi + 1)^3(z - (m))^2 - m^2 = 0\,, \quad (27)$$

which is of the same form as the corresponding equation (7), page 8, except that in (7) $(m) = m$, the parallax being entirely neglected. Since (m) differs but little from m, the solution of (27) is accomplished in exactly the same manner as that of (7),

page 8, except that in equations (10) and (11), page 8, we must write (ν) for ν. Hence, if z_1 be a first approximation to z, taken from the tables at the end of this volume with $\frac{1}{m}$ or $\frac{1}{(m)}$ and ψ as arguments, then the correction $\varDelta z_1$ which must be added to z_1, so that $z_2 = z_1 + \varDelta z_1$, shall satisfy (27) is given by

$$f(z_1) = \mu^3_1\,(\nu)^2_1 - m^2 = M_1 \tag{28}$$

and

$$\varDelta z_1 = \frac{-M_1}{2\,\mu^2_1\,(\nu)_1\,[\mu_1 + 3\,(\nu)_1\,(z_1 - \cos\psi)]}. \tag{29}$$

The only additional labor, therefore, required for a complete elimination of the parallax in the determination of ρ_m consists in the computation of the auxiliary quantity (m). By (23) and by (9), page 8, we may also write in a convenient form for computation

$$(m) = m\left(1 + 3\,\frac{\varDelta_2 R}{(R)}\right) + \frac{\varDelta \varkappa}{(R)\cos\delta}. \tag{30}$$

The computation of σ''_0 may be omitted, as it is not required in the further solution of the orbit. It enters only in the expressions for $\rho_{,}$ and $\rho_{,,,}$, but these distances are needed solely for the purpose of freeing the observed intervals from the planetary aberration (*cf.* V, page 16). As it is permissible in preliminary orbit calculations to neglect the second differences of the aberration corrections, the computation of σ''_0 may be avoided. The expression for σ''_0, however, resulting from equations (20) may be written out by analogy from the corresponding formula in V, page 16, and is

$$\sigma''_0 = \frac{\sigma_0}{\varkappa}\left[(R)\cos D\cos(A - \alpha_{,,}) - \sigma_0\right] - \sigma_0\left[\frac{1}{(R)^3}\left(1 + 3\,\frac{\varDelta_2 R}{(R)}\right) - (\alpha'_0)^2\right] + j\cos(a - \alpha), \tag{31}$$

or it may be computed, if desired, from the first of equations (20) after r has become known.

Equations (21) give

$$\frac{\sigma'_0 - \varDelta\lambda}{\sigma_0 - \varDelta\varkappa} = \frac{\lambda}{\varkappa}\;;\quad \sigma'_0 = \frac{\lambda}{\varkappa}\,(\sigma_0 - \varDelta\varkappa) + \varDelta\lambda, \tag{32}$$

from which σ'_0 may be computed after σ_0 has been found. From (32) we may also write

$$\frac{\sigma'_0}{\sigma_0} = \frac{\lambda}{\varkappa} - \frac{1}{\sigma_0}\left[\frac{\lambda}{\varkappa}\,\varDelta\varkappa - \varDelta\lambda\right] = \frac{\lambda}{\varkappa} - \frac{\beta}{\sigma_0}, \tag{33a}$$

which equation is in a convenient form for use in *all* preliminary orbit solutions.

In fact, in the solution of *parabolic* orbits the values of λ, $\varkappa$, $\varDelta\lambda$, and $\varDelta\varkappa$ are not themselves required, but only the value of the ratio $\frac{\sigma'_0}{\sigma_0}$ involving the value of the ratio $\frac{\lambda}{\varkappa}$ and the value of β. Simple formulæ for these quantities may be deduced by the elimination of $\left[\frac{1}{r^3} - \frac{1}{(R)^3}\left(1 + 3\frac{\varDelta_2 R}{(R)}\right)\right]$ from the first of equations (21) by means of the second of equations (20). This elimination gives, if, for the present, the subscripts referring to the middle date be suppressed

$$\frac{\sigma'}{\sigma} = -\frac{\alpha''}{2\,\alpha'} + \frac{\frac{S}{\varkappa}}{2\,\alpha'}\sin(A - \alpha) + \frac{j}{2\,\alpha'\sigma}\sin(a - \alpha) - \frac{S}{2\,\alpha'}\sin(A - \alpha)\,\frac{\varDelta\varkappa}{\varkappa\sigma}\,\ldots$$

Replacing $\varkappa$ by its expression in III, page 15, and $\Delta\varkappa$ by (22) and letting

$$C_1 = \tan\delta \cos(A - \alpha) - \tan D\,; \qquad C_2 = \sin(A - \alpha)$$
$$c_1 = \tan\delta \cos(a - \alpha) - \tan d\,; \qquad c_2 = \sin(a - \alpha)\,,$$

we obtain

$$\frac{\sigma'}{\sigma} = -\frac{1}{2\alpha'}\,\frac{NC_2}{C_1\alpha' + C_2(\tan\delta)'} - \frac{\alpha''}{2\alpha'} - \frac{jc_2}{2\alpha'\sigma} + \frac{jC_2}{2\alpha'\sigma}\,\frac{c_1\alpha' + c_2(\tan\delta)'}{C_1\alpha' + C_2(\tan\delta)'}\,,$$

or introducing for N its expression by III, page 15, and letting $n = (\alpha')^2 \tan\delta + (\tan\delta)''$

$$\frac{\sigma'}{\sigma} = -\frac{1}{2}\,\frac{C_1\alpha'' + C_2 n}{C_1\alpha' + C_2(\tan\delta)'} + \frac{j}{2\sigma}\,\frac{c_2 C_1 - c_1 C_2}{C_1\alpha' + C_2(\tan\delta)'}\,.$$

Let

$$\Gamma = \frac{C_2}{C_1}\,, \quad \gamma = \frac{c_2}{c_1}\,, \quad \Phi = \alpha' + \Gamma(\tan\delta)'\,,$$

then

$$\frac{\sigma'}{\sigma} = -\frac{1}{2}\,\frac{\alpha'' + \Gamma n}{\Phi} + \frac{jc_2}{2\sigma}\,\frac{1 - \frac{\Gamma}{\gamma}}{\Phi}\,.$$

By identifying terms independent and dependent of the parallax in this equation and in (33a), we find

$$\frac{\lambda}{\varkappa} = -\frac{1}{2}\,\frac{\alpha'' + \Gamma n}{\Phi}\,; \quad \beta = -\frac{jc_2}{2}\,\frac{1 - \frac{\Gamma}{\gamma}}{\Phi}\,, \quad \frac{\sigma_0'}{\sigma_0} = \frac{\lambda}{\varkappa} - \frac{\beta}{\sigma}\,. \tag{33b}$$

In practice $\Delta\varkappa$, $\Delta\lambda$, and β which are given by (19), (22), and (33a, b) may generally be neglected. But if they be applied, their computation may be simplified in most cases. For, if by inspection the observed coördinates and the *parallax factors* be found to vary roughly as the time, then $d_x = d_y = d_z = 0$, or nearly so. In that case equations (19) become

$$j\cos a = \frac{\Delta X_0}{(R)^3}\,; \quad j\sin a = \frac{\Delta Y_0}{(R)^3}\,; \quad j\tan d = \frac{\Delta Z_0}{(R)^3}\,, \tag{34}$$

and by (4) we obtain the following expressions for jc_1 and jc_2:

$$j\tan\delta_{,,}\cos(a - \alpha_{,,}) - j\tan d = \frac{1}{(R)^3}\sec\delta_{,,}\,p_\delta\rho_m\,;\; j\sin(a - \alpha_{,,}) = -\frac{1}{(R)^3}\cos\delta_{,,}\,p_a\rho_m\,. \tag{35}$$

Hence equations (22) reduce to

$$\left.\begin{aligned} \Delta\varkappa &= -\frac{1}{N(R)^3}\left\{\sec\delta_{,,}\,p_\delta\rho_m\,\alpha_0' - \cos\delta_{,,}\,p_a\rho_m\,(\tan\delta)_0'\right\} \\ \Delta\lambda &= \frac{1}{2N(R)^3}\left\{\sec\delta_{,,}\,p_\delta\rho_m\,\alpha_0'' - \cos\delta_{,,}\,p_a\rho_m\right\}, \end{aligned}\right\} \tag{36a}$$

and

$$\beta = -\frac{1}{2(R)^3}\,\frac{\cos\delta_{,,}\,p_a\rho_m + \Gamma\sec\delta_{,,}\,p_\delta\rho_m}{\Phi}\,. \tag{36b}$$

In these expressions the numerical values of all quantities involved are available from previous computations.

Collecting results, we have the following directions for a *partial* and for a *complete* elimination of the parallax in σ_0 and σ_0'.

For a *partial* elimination compute (4) and (10), and retain the corrected solar coördinates throughout. In (30) let $\Delta\varkappa = 0$ and solve (27) for z. In (33a, b) let $\beta = 0$ and solve for σ_0'.

For a *complete* elimination of the parallax, proceed in the same manner by (4a), and (10a), except that β, and in general solutions also $\Delta\varkappa$, must be computed and retained, the necessary expressions being either (33a) or (33b) or simply (36a, b). σ_0'' may be neglected entirely or, if taken into account, may be computed by (31) or by the first of equations (20).

Instead of computing ΔR from (10a, b), its value may be obtained by taking the difference between (R) as derived by I, page 15, from the corrected solar coördinates or by $(R)^2 = (X)^2 + (Y)^2 + (Z)^2$ and R as interpolated from an astronomical ephemeris.

To allow for the parallax in the heliocentric velocities x_0', y_0', and z_0', we have from (5a) by differentation and omitting the subscripts referring to the middle date

$$x' = (\xi)' - X_m' - (\Delta_2 X)' = (\rho_m \cos\delta \cos\alpha)' - X' - (\Delta_1 X + \Delta_2 X)',$$

or

$$x' = (\sigma_0 \cos\alpha)' - X' - (\Delta X)' = \sigma_0 \left[\frac{\sigma_0'}{\sigma_0} \cos\alpha - \sin\alpha\alpha'\right] - X' - (\Delta X)',$$

or by (33a) or (33b),

$$\left.\begin{aligned} x' &= \sigma_0 \left[\frac{\lambda}{\varkappa} \cos\alpha - \sin\alpha\alpha'\right] - \beta\cos\alpha - X' - (\Delta X)'. \\ &\text{Similarly} \\ y' &= \sigma_0 \left[\frac{\lambda}{\varkappa} \sin\alpha + \cos\alpha\alpha'\right] - \beta\sin\alpha - Y' - (\Delta Y)' \\ z' &= \sigma_0 \left[\frac{\lambda}{\varkappa} \tan\delta' + (\tan\delta)'\right] - \beta\tan\delta - Z' - (\Delta Z)', \end{aligned}\right\} \quad (37)$$

where $(\Delta X)'$, $(\Delta Y)'$, $(\Delta Z)'$ must be obtained from $\Delta X_{,}$, $\Delta X_{,,}$, and $\Delta X_{,,,}$, etc., in the same manner as α_0' and δ_0' are obtained from $\alpha_{,}$, $\alpha_{,,}$, and $\alpha_{,,,}$, etc., in II, page 15. In the *partial* elimination of the parallax $(\Delta X)'$, $(\Delta Y)'$, and $(\Delta Z)'$ may be entirely neglected.

In all other respects the computation remains the same as in Part I.

In the comparison between theory and observation (VI, page 16), either the corrected or geocentric solar coördinates may be used. In the former case the observed places require no further correction, while in the latter the *geocentric* parallax is applied, as usual, to the observed places.

The procedure outlined above for the elimination of the parallax is extremely simple in its application and, as the quantities involved are small, requires little extra computation, even if $(\Delta X)_0''$, $(\Delta Y)_0''$, and $(\Delta Z)_0''$ be appreciable.

In correcting a preliminary orbit on the basis of a longer arc, it is generally assumed that the initial geocentric distances are known with sufficient accuracy for deriving the parallax corrections. But if the final distances involve a change in the parallax, then, although the observations as originally corrected for parallax may be properly represented, residuals remain which are equal to the difference of

the parallax corrections corresponding to the final and initial geocentric distances. In the differential formulæ developed in the First Paper (VII, page 17), any change in the parallax may be taken up in the course of the calculation by applying small corrective terms to the expressions for A_{I}, A_{III}, B_{I}, and B_{III} in formulæ (23), page 12. Let α, δ; α_g, δ_g; and α_c, δ_c be the observed, the geocentric, and the computed place, that is, the place computed from the initial fundamental data, and let ρ and $\rho + \partial\rho$ be the initial and the final geocentric distance, respectively, at either the first or third date. Then the true residuals of the initial orbit are

$$(\partial_1\alpha) = (\cos\delta\,\partial\alpha) = \cos\delta\,(\alpha_g - \alpha_c) = \cos\delta\left(\alpha + \frac{P_\alpha^0\rho}{\rho + \partial\rho} - \alpha_c\right);\ (\partial\delta) = \delta_g - \delta_c = \delta + \frac{P_\delta^0\rho}{\rho + \partial\rho} - \delta_c, \quad (38)$$

and these are the residuals which must be removed in the differential correction of the initial fundamental quantities ρ_0, x_0', y_0', and z_0'. But as the final or true geocentric distance $\rho + \partial\rho$ is not known, we may write $\frac{1}{\rho + \partial\rho} = \frac{1}{\rho} - \frac{\partial\rho}{\rho^2}$, neglecting higher powers of $\partial\rho$. The residuals (38) will then take the form

$$(\partial_1\alpha) = \cos\delta\left[\left(\alpha + \frac{P_\alpha^0\rho}{\rho}\right) - \alpha_c\right] - \cos\delta\,\frac{P_\alpha^0\rho}{\rho^2}\,\partial\rho, \quad (\partial\delta) = \left[\left(\delta + \frac{P_\delta^0\rho}{\rho}\right) - \delta_c\right] - \frac{P_\delta^0\rho}{\rho^2}\,\partial\rho. \quad (39)$$

In these expressions $\cos\delta\left[\left(\alpha + \frac{P_\alpha^0\rho}{\rho}\right) - \alpha_c\right]$ and $\left[\left(\delta + \frac{P_\delta^0\rho}{\rho}\right) - \delta_c\right]$ represent the residuals of the initial orbit on the basis of the initial parallax corrections, and if we denote these residuals by $\partial_1\alpha$ and $\partial\delta$ as on pages 11 and 12, then

$$(\partial_1\alpha) = \partial_1\alpha - \cos\delta\,\frac{P_\alpha^0\rho}{\rho^2}\,\partial\rho, \quad (\partial\delta) = \partial\delta - \frac{P_\delta^0\rho}{\rho^2}\,\partial\rho. \quad (40)$$

By replacing in (24) and (25), page 12, $\partial_1\alpha$ and $\partial\delta$ by $(\partial_1\alpha)$ and $(\partial\delta)$ as given in (40) and transferring the terms containing the parallax factors to the right-hand sides of (24) and (25), we observe that these terms, which contain the unknown $\partial\rho$ may be combined with $A\partial\rho$ and $B\partial\rho$. Hence in order to allow for a possible change in parallax, it is only necessary to write in (23)

$$\text{for } A, \quad A' = A + \cos\delta\,\frac{P_\alpha^0\rho}{\rho^2}, \quad \text{for } B, \quad B' = B + \frac{P_\delta^0\rho}{\rho^2}. \quad (41)$$

It should be observed, however, that the correction $\partial\rho$ in (40) pertains to ρ_I or ρ_{III} while $\partial\rho_0$ in (24) and (25), page 12, pertains to ρ_0. While formulæ to take account of the differences $\partial\rho_{III} - \partial\rho_0$ and $\partial\rho_0 - \partial\rho_I$ might readily be deduced, such refinement would be unwarranted.

The corrections to A and B, as defined in (41) are so easily computed that it is advisable to take them into account, unless there be reason to assume that the initial orbit is very close.

These considerations regarding a possible change in the parallax from one approximation of an orbit to the next, apply also if the parallax has been eliminated as above in the solution of the initial orbit, although this does not seem to be the case at first sight. For, the values of α_g and δ_g, occurring in (4) must correspond to the parallax corrections based on the same unknown value of ρ as that which appears in (1) to (3). Otherwise we could not identify in (3) the pro-

ducts of ρ into p_a^s and p_δ^s with the given parallax factors. This applies to all three places. This conclusion is readily verified, if we remember that in the comparison between theory and observation for the first and third places, the same residuals will result, whether we use the uncorrected solar coördinates and correct the observed places by parallaxes based on the geocentric distances resulting from the initial orbit (*cf.* VI, page 16), or whether, in accordance with the directions for eliminating the parallax, we use the corrected solar coördinates and leave the observed places as they are. Hence, even if the parallax has been eliminated in the preliminary solution, it is advisable to use the values A' and B', given in (41), instead of A and B, unless it be safe to assume that the first orbit is so close that the parallaxes are definitely determined from the initial distances.

It will be observed that for the determination of the heliocentric coördinates and velocities by VI, page 16, σ_0' is not needed. For by (37) we have

$$x_0' = a_x \sigma_0 - [X]', \text{ etc., where } a_x = \cos \alpha_{\prime\prime} \frac{\lambda}{\varkappa} - \sin \alpha_{\prime\prime}\, \alpha_0' \text{ and } [X]' = X' + (\varDelta X)' + \beta \cos \alpha, \text{ etc.}$$

Hence, beside σ_0, only the ratio $\frac{\lambda}{\varkappa}$ and β appear in the velocities. x_0', y_0', z_0' are independent of the corrections (4) applied to the solar coördinates, except in the rare cases where these corrections appear in σ_0 and $[X]'$, etc. But even then we may avoid the computation of σ_0' by using (33a,b). As far as σ_0 is concerned, the corrections enter only into the auxiliary quantity (m). Similarly $\frac{\lambda}{\varkappa} - \frac{\beta}{\sigma_0}$ may be introduced for $\frac{\sigma_0'}{\sigma}$ in the expressions for $\rho_\prime$ and $\rho_{\prime\prime\prime}$ in V, page 16.

The foregoing elimination of the parallax has a simple geometrical interpretation. Draw the arc of a circle about the body through M, the center of mass, until it intersects the line of sight. The radius of this circle is the distance of the body from M. Let P be the point of intersection. Or P may be considered the projection of M on the line of sight. Then the coördinates (ξ), (η), (ζ) in (5a) will be the coördinates of the body referred to P. The parallax p is the angle at the body subtended by $MP = \rho_m p$. The components of the parallax perpendicular to and in the body's hour circle are $\cos \delta p_a$ and p_δ, and $\overline{MP}^2 = (\rho_m \cos \delta p_a)^2 + (\rho_m p_\delta)^2$, MP being the resultant displacement of $\rho_m \cos \delta p_a$ and $\rho_m p_\delta$. The latter, lying in the hour circle, makes the angle $90° - \delta$ with the equator, while the former is parallel to the equator, so that its angles of projection on the x and y axes are $90° - \alpha$ and α, respectively. The projection of $\rho_m p_\delta$ on the equator is $\sin \delta\, \rho_m p_\delta$, and as this component lies in the hour circle, its angles of projection on the x and y axes are α and $90° - \alpha$, respectively. If the two components, $\rho_m \cos \delta p_a$ and $\rho_m p_\delta$, be thus projected upon the three axes with due regard to the algebraical signs, then we shall obtain the coördinates of P referred to M, and these will be found identical with $-\varDelta_2 X$, $-\varDelta_2 Y$, and $-\varDelta_2 Z$, as defined in (4a). Thus these corrections signify that for the purpose of eliminating the parallax and preserving the distance of the body from M, the origin of its coördinates is changed from the center of mass to its projection on the line of sight. Since the coördinates (ξ), (η), (ζ) of the body referred to P are equal to the sum of its heliocentric coördinates x, y, z,

of the barycentric coördinates X_m, Y_m, Z_m, of the Sun, and of the coordinates $\Delta_2 X$, $\Delta_2 Y$, $\Delta_2 Z$ of M referred to P, equations (4a) and (5a) might have been set up independently on the basis of these geometrical considerations. The line joining the Sun and P is (R) and ΔR is the difference between (R) and the projection of R upon it.

Reference has already been made to the fact that the planetary aberration is fully accounted for within the accuracy of the solution of the preliminary orbit by combining in the equation $\xi = x + X$, the heliocentric place of the body at the time $(t-\alpha\rho)$, with the place of the Sun and the observed or apparent place of the body at the time t, freed of the annual aberration in the reduction to the beginning of the year. The distance ρ, involved in ξ, η, ζ, is then the distance between the Earth at the time t, and the true place of the body at the time $t-\alpha\rho$. As the heliocentric coördinates and velocities are thus referred to the time $t-\alpha\rho$, or the true time, the epoch will also be the true time. The effect of the aberration is thereby eliminated from the values of x, y, z, resulting for the time $t-\alpha\rho$, provided that it can be shown that the distance ρ, as defined above and as determined in the course of the solution, is free from other errors. But it is evident that whatever errors may remain in the resulting values of σ_0 and σ'_0, and therefore, in the heliocentric coördinates and velocities derived from them, must be of the order $k\alpha$[1], the factor k being accounted for by the fact that all intervals must be expressed in the proper unit, which is $\frac{1}{k}$ mean solar days. The error involved in the solution of σ_0, σ'_0, and σ''_0, on account of neglecting third and higher derivatives in the initial determination of the geocentric velocities and accelerations α'_0, α''_0, δ'_0 and δ''_0, is of the second order if the epoch chosen be the mean of the observed dates, and of the first order, if the epoch be identified with the middle date, unless the intervals be equal, in which case it is again of the second order. If it be of the second order, then it is of the order of the product of the intervals, expressed in mean solar days, multiplied by k^2. With intervals as short as one day, the errors will therefore be of the order k^2 and larger with longer intervals. The error which may arise in the heliocentric coördinates and velocities, on account of aberration through errors in σ_0 and σ'_0, are of the order αk, as stated above, and since $\alpha k < k^2$, it is not necessary to take account of the aberration further than is provided for by the foregoing choice of coördinates.

[1] Expressions for corrections to allow for these errors have been produced by POINCARE, *Bulletin Astronomique*, t. XXIII, pp. 175-177. In these expressions $k^2 = 1$, so that, in numerical application, the aberration factor α must be multiplied by k. The errors might also be taken up in the course of the solution in a manner quite similar to that employed above in eliminating the parallax.

which applies to positive and negative values of θ. Hence if θ be always taken positive, the signs of the odd power terms must be reversed for dates before the epoch. This form is very convenient for ephemeris purposes when a large number of positions are to be computed.

If only a few positions are required, as in the comparison between theory and observation, the following expressions, arranged by PROFESSOR CRAWFORD, will prove advantageous. They contain terms inclusive of the fifth power of θ in f and g and are readily produced from the preceding equations. Let

$$\alpha = \frac{(r+r')^3}{r}\,; \qquad \beta = \frac{\theta}{(r+r')^3}\,; \qquad \gamma = \frac{\theta^2}{2r^3}\,. \tag{5}$$

Then

$$\left.\begin{aligned} f &= 1 - \gamma\left\{1 - \alpha\beta\left[1 + \frac{\beta}{3}\left(1 - \frac{15}{4}\alpha - \frac{3}{4}\frac{r}{a} - 3\alpha\beta\left[1 - \frac{7}{4}\alpha - \frac{3}{4}\frac{r}{a}\right]\right)\right]\right\}\cdots \\ g &= \theta\left\{1 - \frac{\gamma}{3}\left[1 - \frac{3}{2}\alpha\beta\left(1 + \frac{\beta}{3}\left[1 - \frac{9}{2}\left(\alpha + \frac{1}{5}\frac{r}{a}\right)\right]\right)\right]\right\}\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{6}$$

For the parabola, $a = \infty$. Hence

$$\left.\begin{aligned} f &= 1 - \gamma\left\{1 - \alpha\beta\left[1 + \frac{\beta}{3}\left(1 - \frac{15}{4}\alpha\left[1 + \frac{4}{5}\beta\left(1 - \frac{7}{4}\alpha\right)\right]\right)\right]\right\}\cdots\cdots\cdots \\ g &= \theta\left\{1 - \frac{\gamma}{3}\left[1 - \frac{3}{2}\alpha\beta\left(1 + \frac{\beta}{3}\left[1 - \frac{9}{2}\alpha\right]\right)\right]\right\}\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{7}$$

As soon as r, r', and a have been computed, the magnitude of the higher power terms in f and g may be estimated from (3) and (4). It will not be convenient to use the series if the convergence, judged from θ and r, is slow, and if $\theta^6 f_6$ and $\theta^6 g_6$ are appreciable within the last decimal of the calculation. The choice for the computation of the places then lies between the closed expressions for f and g, and the constants for the equator. But, as in preliminary orbit computations the intervals are moderate, the series will, in general, prove sufficient even for small values of r. Long intervals are used in practice only when it is desired to improve an approximate orbit, and then closed expressions for f and g and δf and δg are readily derived on the basis of the preliminary elements. The computation of the constants for the equator may be deferred until a satisfactory agreement between theory and observation has been reached by means of the f and g.

KUHNERT'S closed expressions are

$$\left.\begin{aligned} f &= 1 - \frac{2a}{r}\sin^2 g\left(1 + \frac{a}{r_0}(\cos 2g - 1)\right), \qquad 2g = E' - E \\ g &= \theta - a^{\frac{3}{2}}(2g - \sin 2g) + a^{\frac{1}{2}} r_0 \sin 2g + 2a^{\frac{1}{2}} r_0 r_0' \sin^2 g. \end{aligned}\right\} \tag{8}$$

We may, however, also compute f from g by the formula[1]

$$f = 1 - \frac{g^2}{2\,r_0^2\cos^2\frac{1}{2}(v - v_0)}\,. \tag{9}$$

[1] OPPOLZER, *Lehrbuch zur Bahnbestimmung*, Vol. I, p. 97.

The computation of the true anomalies may be avoided by a few simple transformations, with the aid of the formula

$$\sqrt{r\,r_0}\sin\tfrac{1}{2}(v-v_0)=a\sqrt{1-e^2}\sin\bar{g}\,. \tag{10}$$

Let
$$\gamma=\sqrt{r_0\,(1-f)}\ . \tag{11}$$

Then the first of (8) may be written
$$\gamma=\sqrt{2\,a}\,\sin g\ . \tag{12}$$

Eliminating $\sin g$ from (10) by means of (12), we have
$$\sqrt{r\,r_0}\sin\tfrac{1}{2}(v-v_0)=a\sqrt{1-e^2}\frac{\gamma}{\sqrt{2a}}=\sqrt{\tfrac{p}{2}}\,\gamma\ . \tag{13}$$

From (9) we obtain by (11)
$$r\,r_0\cos^2\tfrac{1}{2}(v-v_0)=\frac{g^2}{2\,\gamma^2}\,, \tag{14}$$

which, added to the square of (13) gives
$$r\,r_0=\frac{g^2}{2\,\gamma^2}+\frac{p\,\gamma^2}{2}\,, \tag{15}$$

or
$$g^2=2r\,r_0\,\gamma^2-p\,\gamma^4\,. \tag{16}$$

g is negative for dates before and positive for dates after the epoch t_0 of E_0 or v_0.

After γ has been computed by (12), f and g are readily obtained from (11) and (16). The eccentric anomaly E involved in γ may be obtained in the usual way by the solution of KEPLER'S equation; p is given by (7), page 252, and r, which also appears in (16), by $r=a\,(1-e\cos E)$.

Numerous other general relations involving g and f might be deduced. Thus, for instance, by squaring and adding
$$x=fx_0+gx_0'\,;\qquad y=fy_0+gy_0'\,;\qquad z=fz_0+gz_0'\,,$$
we easily derive
$$r^2=fr_0^2+2\,fg\,(r_0\,r_0')+g^2G^2\,,$$
which may serve as a check equation. Also, since $g=\frac{rr_0\sin(v-v_0)}{\sqrt{p}}$, we obtain from (9), after some simple reductions,
$$p\gamma^2=rr_0-rr_0\cos(v-v_0)\ .$$

But KUEHNERT gives, *loco citato*,
$$rr_0\cos(v-v_0)=fr_0^2+g\,(r_0\,r_0').$$

So that
$$p\gamma^2=r_0\,(r-r_0)+r_0\,\gamma^2-g\,(r_0\,r_0').$$

For the parabola we may derive γ as follows. Since
$$g=\frac{rr_0\sin(v-v_0)}{\sqrt{p}}=\frac{rr_0\,2\sin\tfrac{1}{2}(v-v_0)\cos\tfrac{1}{2}(v-v_0)}{\sqrt{2q}}\,, \tag{17}$$

and since
$$r=\frac{q}{\sec^2\tfrac{1}{2}v}\,;\qquad r_0=\frac{q}{\sec^2\tfrac{1}{2}v_0}\,, \tag{18}$$

we obtain from (9) and (11)
$$\gamma=\sqrt{q}\,(\tan\tfrac{1}{2}v-\tan\tfrac{1}{2}v_0)\,, \tag{19}$$

or since
$$\sqrt{q}\tan\tfrac{1}{2}v=\sqrt{r-q}\,;\qquad \sqrt{q}\tan\tfrac{1}{2}v_0=\sqrt{r_0-q}\,, \tag{20}$$

therefore, also,
$$\gamma=\sqrt{r-q}-\sqrt{r_0-q}\,, \tag{21}$$

where the algebraical signs of the square roots are the same as those of the corre-

sponding $\tan \frac{1}{2} v$. The true anomaly v may be obtained in the usual way with the aid of BARKER'S table. v_0 may be derived from

$$\sin v_0 = 2 \sin \tfrac{1}{2} v_0 \cos \tfrac{1}{2} v_0 = \sqrt{2q}\, r'_0 \,, \tag{22}$$

or, after division of this expression by $\cos^2 \frac{1}{2} v_0 = \frac{q}{r}$,
from

$$2 \tan \tfrac{1}{2} v_0 = \frac{\sqrt{2q}\, r_0 r'_0}{q} \,, \tag{23}$$

or

$$\sqrt{2q} \tan \tfrac{1}{2} v_0 = r_0 r'_0 \,. \tag{24}$$

For the circle the series and the closed expressions for f and g assume very simple forms. Thus, if in (4) we let $r = a$ and $r' = 0$, we obtain for the series

$$f = 1 - \frac{\theta^2}{2\, a^3} + \frac{\theta^4}{24\, a^6} \cdots \quad ; \quad g = \theta - \frac{\theta^3}{6\, a^3} + \frac{\theta^5}{120\, a^6} \cdots \tag{25}$$

The corresponding closed expressions are derived from (8) by letting $a = r_0$, and $r' = 0$, as above. Since for the circle the mean, eccentric, and true anomalies are identical, we have

$$2\, \bar{g} = \frac{\theta}{a^{3/2}} \; ; \quad f = \cos 2\, \bar{g} \; ; \quad g = a^{3/2} \sin 2\, \bar{g} \,, \tag{26}$$

and by (12), $\gamma = \sqrt{2a}\, \sin \bar{g}$.

Since the series (25) represent the cosine and sine series, the relations (26) might also have been written out at once.

THE CONSTANTS FOR THE EQUATOR EXPRESSED IN TERMS OF THE HELIOCENTRIC RECTANGULAR COÖRDINATES AND THEIR VELOCITIES.

It is customary in orbit computations to derive the constants for the equator from the elements which define the position of the orbit plane in space. In VIII, page 18, the elements are, therefore, derived first, and as these elements refer to the equator, the computation of the constants becomes quite simple. But to conform to the general usage in stating the final results, it is necessary to transform the elements from the equator to the ecliptic (*cf.* page 14).

In many cases it will be found advantageous to compute the constants for the equator as soon as the heliocentric coördinates and velocities have become known, and then to derive the elements referred to the eliptic directly from the constants. This obviously renders the computation of the equatorial elements altogether unnecessary and admits of perfecting the agreement between theory and observation, before proceeding to the computation of the final elements even if the residuals are computed by the aid of the constants for the equator instead of the series or closed expressions for f and g.

The constants a and A' may be expressed in terms of the heliocentric coördinates and velocities by means of the equation

$$r \sin a \sin (\sin A' + v) = x \tag{1}$$

and an equation defining $\sin a \cos (A' + v)$, which latter we shall derive on the basis of the following well known relations:

$$x = r \{\cos u \cos \Omega - \sin u \sin \Omega \cos i\}; \qquad y = r \{\cos u \sin \Omega + \sin u \cos \Omega \cos i\}, \tag{2}$$
$$z = r \sin u \sin i$$

and

$$\left.\begin{array}{lll} \sin a \sin A = \cos \Omega & \sin b \sin B = \sin \Omega; & \sin c \sin C = 0; \\ \sin a \cos A = -\sin \Omega \cos i; & \sin b \cos B = \cos \Omega \cos i; & \sin c \cos C = \sin i, \end{array}\right\} \tag{3}$$

where x, y, z, Ω, and i are referred to the equator, and where $u = \omega + v$ is the argument of latitude. Since $A' + v = A + \omega + A = u$, we have

$$\sin a \cos (A' + v) = \sin a \cos A \cos u - \sin a \sin A \sin u, \tag{4}$$

or by (3)

$$\sin a \cos (A' + v) = -\sin \Omega \cos i \cos u - \cos \Omega \sin u.$$

Eliminating from this equation $\sin \Omega \cos u$ by means of the second, and subsequently $\sin u \sin i$ by means of the third of (2), we obtain

$$\sin a \cos (A' + v) = \sin u \cos \Omega \cos^2 i - \frac{y}{r} \cos i - \cos \Omega \sin u,$$

$$= - \sin u \cos \Omega \sin^2 i - \frac{y}{r} \cos i,$$

$$= - \frac{z}{r} \sin i \cos \Omega - \frac{y}{r} \cos i.$$

or, replacing by VIII, page 18, $\sin i \cos \Omega$ and $\cos i$ in terms of the coördinates and velocities,

$$\sin a \cos (A' + v) = \frac{z}{r} \frac{xz' - zx'}{\sqrt{p}} - \frac{y}{r} \frac{yx' - xy'}{\sqrt{p}}$$
$$= \frac{1}{r\sqrt{p}} \Big[x' (y^2 + z^2) - x (yy' + zz') \Big],$$

or, since

$$x^2 + y^2 + z^2 = r^2 , \qquad xx' + yy' + zz' = rr' , \tag{5}$$

$$\sin a \cos (A' + v) = \frac{1}{r\sqrt{p}} \Big[x' (r^2 - x^2) - x (rr' - xx') \Big]$$
$$= \frac{1}{r\sqrt{p}} \Big[x' r^2 - x rr' \Big],$$

or, finally

$$\sin a \cos (A' + v) = \frac{rx' - xr'}{\sqrt{p}} ,$$

and, similarly,

$$\sin b \cos (B' + v) = \frac{ry' - yr'}{\sqrt{p}} , \qquad \sin c \cos (C' + v) = \frac{rz' - zr'}{\sqrt{p}} . \tag{6}$$

Equations (6), together with the three equations of the form (1), determine a, b, c, A', B', and C' in terms of the heliocentric rectangular coordinates and their velocities and of r, r', p, and v. The radius vector r and its velocity r' are given by (5), and p is determined by the equation

$$p = r^2 G^2 - (rr')^2 = r^2 (G^2 - (r')^2) , \tag{7}$$

where G^2 is the velocity in the orbit, viz.:

$$G^2 = x'^2 + y'^2 + z'^2 = \frac{2}{r} - \frac{1}{a} . \tag{8}$$

For the parabola (8) becomes

$$G^2 = \frac{2}{r} , \tag{8a}$$

and

$$p = 2q = 2r - (rr')^2 . \tag{7a}$$

Equation (7) is readily verified by squaring and adding side for side the expressions for $e \sin v$ and $e \cos v$ in VIII, page 18.

The mean anomaly v is determined as in VIII, page 18. For the parabola, since $e = 1$, we may also write

$$e \sin v = 2 \sin \tfrac{1}{2} v \cos \tfrac{1}{2} v = r' \sqrt{p} = r' \sqrt{2q} .$$

Eliminating $\cos \frac{1}{2} v$ by means of $r = \dfrac{q}{\cos^2 \frac{1}{2} v}$,

$$\sin \tfrac{1}{2} v = r' \sqrt{\frac{r}{2}} , \tag{9a}$$

or, by (8a),

$$\sin \tfrac{1}{2} v = \frac{r'}{G} . \tag{9b}$$

It remains to arrange the relations existing between the elements referred to the ecliptic and the constants for the equator in a convenient form for the computation of the former from the latter.

When the elements are referred to the ecliptic, the constants for the equator are given in terms of the elements by

$$\left.\begin{aligned}
&\sin a \sin A = \cos \Omega ; \qquad && \sin b \sin B = -\sin \Omega \cos \varepsilon \\
&\sin a \cos A = -\sin \Omega \cos i \qquad && \sin b \cos B = \cos \Omega \cos i \cos \varepsilon - \sin i \sin \varepsilon \\
&\qquad \sin c \sin C = -\sin \Omega \sin \varepsilon && \\
&\qquad \sin c \cos C = \cos \Omega \cos i \sin \varepsilon + \sin i \cos \varepsilon . &&
\end{aligned}\right\} \tag{10}$$

By analogy with (4) we may write

$$\left.\begin{aligned}\sin b \cos(B' + v) &= \sin b \cos B \cos u - \sin b \sin B \sin u \\ \sin c \cos(C' + v) &= \sin c \cos C \cos u - \sin c \sin C \sin u \\ \sin b \sin(B' + v) &= \sin b \sin B \cos u + \sin b \cos B \sin u \\ \sin c \sin(C' + v) &= \sin c \sin C \cos u + \sin c \cos C \sin u\,.\end{aligned}\right\} \qquad (11)$$

If on the right-hand side of equations (11) the constants for the equator be replaced by (10) in terms of the ecliptical elements, then equations (11) will serve for the determination of these elements in terms of the left-hand members of (11), the numerical values of which have already become known by (1) and (6).

Multiplying the first and third equations of (11) by $\tan\varepsilon$ and subtracting the products from the second and fourth, respectively, we have by (10)

$$\begin{aligned}&\sin c \cos(C' + v) - \sin b \cos(B' + v)\tan\varepsilon = \\ &\quad(\cos\Omega \cos i \sin\varepsilon + \sin i \cos\varepsilon)\cos u - \sin\Omega \sin\varepsilon \sin u \\ &\quad-(\cos\Omega \cos i \cos\varepsilon - \sin i \sin\varepsilon)\cos u \tan\varepsilon + \sin\Omega \cos\varepsilon \sin u \tan\varepsilon \\ &\sin c \sin(C' + v) - \sin b \sin(B' + v)\tan\varepsilon = \\ &\quad\sin\Omega \sin\varepsilon \cos u + (\cos\Omega \cos i \sin\varepsilon + \sin i \cos\varepsilon)\sin u \\ &\quad-\sin\Omega \cos\varepsilon \cos u \tan\varepsilon - (\cos\Omega \cos i \cos\varepsilon - \sin i \sin\varepsilon)\sin u \tan\varepsilon\,,\end{aligned}$$

or, transposing the right- and left-hand members,

$$\left.\begin{aligned}\frac{\sin i}{\cos\varepsilon}\cos u &= \sin c \cos(C' + v) - \sin b \cos(B' + v)\tan\varepsilon \\ \frac{\sin i}{\cos\varepsilon}\sin u &= \sin c \sin(C' + v) - \sin b \sin(B' + v)\tan\varepsilon\,,\end{aligned}\right\} \qquad (12)$$

which determine $\sin i$ and u. We have further

$$\omega = u - v\,; \quad A = A' - \omega\,; \quad B = B' - \omega\,; \quad C = C' - \omega\,. \qquad (13)$$

Finally Ω may be derived from the first and the third or fifth equation of (10), and $\cos i$ from the second. As a partial check we may use the well known relation

$$\tan i = \frac{\sin b \sin c \sin(C - B)}{\sin a \cos A}\,. \qquad (14)$$

It will be observed that the semi-major axis a is defined by (8) and the parameter p by (7). The eccentricity is given by

$$e^2 = \frac{a - p}{a}\,. \qquad (15)$$

The residuals in VI, page 16, may be computed either by means of the f and g series, or by means of the closed expressions for f and g, or with the aid of the constants for the equator. In any case, it is most convenient to compute the constants first and from them the elements. If the constants for the equator are to be used in the comparison between theory and observation, then they should be computed as soon as the heliocentric coördinates and their velocities and r and r' have been derived by VI; otherwise their computation may be deferred until the observations have been properly represented. It is best to defer the computation of the elements to the last in any case.

Collecting results for the computation of the constants for the equator, and incidentally of a and e, which are not needed but reveal the general character of the orbit, we arrive at the following order of computation:

From the heliocentric coördinates and velocities, obtained in VI, page 16, compute G^2 and a by (8) or G^2 by (8a), and p or q by (7) or (7a); then v as in VIII, page 18, or for the parabola by (9a) or (9b); finally the constants for the equator by (1) and (6) and e from (15).

The elements which determine the position of the orbit plane in space, with reference to the ecliptic, are given in terms of the constants by formulæ (12) to (14).

THE DIFFERENTIAL CORRECTION OF THE FUNDAMENTAL DATA.

The arrangement of the differential formulæ developed on pages 11 to 13 and collected in [VII], page 17, is intended for the correction of the direct solution without regard to the eccentricity. A slight change in the arrangement of these formulæ will facilitate their application to parabolic and circular orbits and will permit also of ready transition from parabolic and circular to general orbits and *vice versa.* Thus, if in the original solution an assumption has been made regarding the eccentricity, and if subsequently it becomes apparent that the observations can not be represented under such assumption, then practically no extra labor will be required for the determination of the general orbit.

If we solve equations (25), page 12, for $\partial x_0'$ and $\partial y_0'$, respectively, and let

$$P_x = \frac{C_{\prime\prime\prime} \cos \alpha_{\prime\prime\prime}\, \partial_\prime \alpha_\prime - C_\prime \cos \alpha_\prime\, \partial_\prime \alpha_{\prime\prime\prime}}{C_\prime C_{\prime\prime\prime} \sin(\alpha_{\prime\prime\prime} - \alpha_\prime)} \quad ; \quad Q_x = \frac{A_\prime C_{\prime\prime\prime} \cos \alpha_{\prime\prime\prime} - A_{\prime\prime\prime} C_\prime \cos \alpha_\prime}{C_\prime C_{\prime\prime\prime} \sin(\alpha_{\prime\prime\prime} - \alpha_\prime)}$$

$$P_y = \frac{C_{\prime\prime\prime} \sin \alpha_{\prime\prime\prime}\, \partial_\prime \alpha_\prime - C_\prime \sin \alpha_\prime\, \partial_\prime \alpha_{\prime\prime\prime}}{C_\prime C_{\prime\prime\prime} \sin(\alpha_{\prime\prime\prime} - \alpha_\prime)} \quad ; \quad Q_y = \frac{A_\prime C_{\prime\prime\prime} \sin \alpha_{\prime\prime\prime} - A_{\prime\prime\prime} C_\prime \sin \alpha_\prime}{C_\prime C_{\prime\prime\prime} \sin(\alpha_{\prime\prime\prime} - \alpha_\prime)} \,, \tag{1}$$

then

$$\partial x_0' = P_x - Q_x\, \partial \rho_0 \quad ; \quad \partial y_0' = P_y - Q_y\, \partial \rho_0 \,. \tag{2}$$

Substituting these values for $\partial x_0'$ and $\partial y_0'$ in the equations (25), page 12, and letting

$$P_{z_\prime} = \frac{\partial \delta_\prime + C_\prime \sin \delta_\prime (\cos \alpha_\prime P_x + \sin \alpha_\prime P_y)}{C_\prime \cos \delta_\prime} \quad ; \quad Q_{z_\prime} = \frac{B_\prime + C_\prime \sin \delta_\prime (\cos \alpha_\prime Q_x + \sin \alpha_\prime Q_y)}{C_\prime \cos \delta_\prime}$$

$$P_{z_{\prime\prime\prime}} = \frac{\partial \delta_{\prime\prime\prime} + C_{\prime\prime\prime} \sin \delta_{\prime\prime\prime} (\cos \alpha_{\prime\prime\prime} P_x + \sin \alpha_{\prime\prime\prime} P_y)}{C_{\prime\prime\prime} \cos \delta_{\prime\prime\prime}} \; ; \; Q_{z_{\prime\prime\prime}} = \frac{B_{\prime\prime\prime} + C_{\prime\prime\prime} \sin \delta_{\prime\prime\prime} (\cos \alpha_{\prime\prime\prime} Q_x + \sin \alpha_{\prime\prime\prime} Q_y)}{C_{\prime\prime\prime} \cos \delta_{\prime\prime\prime}} \,, \tag{3}$$

we obtain

$$\partial z_0' = P_{z_\prime} - Q_{z_\prime}\, \partial \rho_0 \quad ; \quad \partial z_0' = P_{z_{\prime\prime\prime}} - Q_{z_{\prime\prime\prime}}\, \partial \rho_0 \,. \tag{4}$$

Formulæ (1) to (4) may be applied in all classes of orbits. If no assumption be made regarding the eccentricity in determining the corrections to the fundamental data ρ_0, x_0', y_0', and z_0', then equations (4) give

$$\partial \rho_0 = \frac{P_{z_{\prime\prime\prime}} - P_{z_\prime}}{Q_{z_{\prime\prime\prime}} - Q_{z_\prime}} \,, \tag{5}$$

and $\partial z_0'$, $\partial x_0'$, $\partial y_0'$ are obtained from (4) and (2). It will be observed that this process remains the same no matter whether an assumption regarding the eccentricity was made in the previous approximation or not.

If the corrections to the fundamental data are to yield a parabolic orbit, then, irrespective of the character of the previous approximation, the corrected fundamental data must satisfy the equation

$$x_0'^2 + y_0'^2 + z_0'^2 = \frac{2}{r_0} \,. \tag{6}$$

Five conditions therefore are available for the determination of the unknowns, namely, equations (2), (4), and (6). If these equations are inconsistent, then a parabolic solution is not possible. The five equations referred to may be applied in various ways for the computation of a parabola and for testing its sufficiency.

In the case of a circular orbit the corrected fundamental data must satisfy the equation

$$x_0'^2 + y_0'^2 + z_0'^2 = \frac{1}{r_0}, \tag{7}$$

and similar considerations prevail as for the parabola. Convenient formulæ for determining parabolic and circular orbits and testing their sufficiency are developed for the parabola on page 279, and for the circle on page 305.

As it may become necessary to pass from one class of orbit to another, it is advisable always to use the foregoing formulæ in determining the differential corrections to the fundamental data in place of the expression for $\partial\rho_0$ and the formulæ following it in [VII], (page 17). In this connection it will be observed that the last of formulæ [VII] afford a check only on the computation of the differential corrections from the adopted linear relations. The sufficiency of the linear relations and of the final results may be rigidly tested by computing the residuals as in VI, (page 16).

In the derivation of the formulæ for the differential correction of our first hypothesis or of any preliminary orbit, (formulæ [VII], page 17), the squares of the variations are neglected (*cf.* page 11), and the linear variations ∂f and ∂g are developed in series progressing in ascending powers of the intervals ((16), page 11). These series are obtained from the series for f and g ((13), page 10). The applicability of the f and g series to longer arcs is discussed on page 19. A comparison of the ∂f and ∂g series with the f and g series shows that the applicability of the former to longer arcs rests on the same conditions as that of the latter.

If an inspection of n, r, and r' indicates that the number of terms of the f and g series given in (13), page 10, is not sufficient, the simple remedy exists of computing as many additional terms as are found to be appreciable from formulæ (1) to (7), pages 247–248, or we may use the closed expressions for f and g. But in the ∂f and ∂g series some of the neglected terms depend upon $\partial r'$, which must remain unknown until the differential corrections of the fundamental quantities ρ_0, x_0', y_0', z_0' have been found, and therefore these terms can not be computed in advance, except in circular orbits where $r' = 0$. In general, it will then become necessary to have recourse to closed expressions for ∂f and ∂g, and these will be derived below. But in some cases it may be found more convenient to make allowance for the neglected terms in the course of the differential correction, particularly so if the number of additional terms to be considered is very limited.

When the slow convergence of the ∂f and ∂g series arises from a small value of r rather than from long intervals, then it is advisable to combine with the additional terms in the ∂f and ∂g series a correction so as to express ∂r in terms of $\partial\rho$ more accurately than is accomplished by the linear relation $\partial r = \cos\beta\,\partial\rho$

given in (24) and (25), page 12, except that the following additional terms will appear:

$$
\left.
\begin{array}{ll}
\text{in} \quad \partial_{,}\alpha_{,} \,, & \dfrac{\cos\alpha_{,}}{\rho_{,}} y_0 \dfrac{\theta_{,,,}^2 \partial r'}{2 r_0^4} - \dfrac{\cos\alpha_{,}}{\rho_{,}} \dfrac{3\,\theta_{,,,}^2 \cos\beta}{r_0^4} \left[\dfrac{y_0 \cos\beta}{r_0} - \tfrac{1}{2} \dfrac{\eta_0}{\rho_0} \right] (\partial\rho_0)^2 \\
& + \dfrac{\sin\alpha_{,}}{\rho_{,}} x_0 \dfrac{\theta_{,,,}^2 \partial r'}{2 r_0^4} + \dfrac{\sin\alpha_{,}}{\rho_{,}} \dfrac{3\,\theta_{,,,}^2 \cos\beta}{r_0^4} \left[\dfrac{x_0 \cos\beta}{r_0} - \tfrac{1}{2} \dfrac{\xi_0}{\rho_0} \right] (\partial\rho_0)^2 \;; \\
\text{in} \quad \partial_{,}\alpha_{,,,} \,, & + \dfrac{\cos\alpha_{,,,}}{\rho_{,,,}} y_0 \dfrac{\theta_{,}^2 \partial r'}{2 r_0^4} - \dfrac{\cos\alpha_{,,,}}{\rho_{,,,}} \dfrac{3\,\theta_{,}^2 \cos\beta}{r_0^4} \left[\dfrac{y_0 \cos\beta}{r_0} - \tfrac{1}{2} \dfrac{\eta_0}{\rho_0} \right] (\partial\rho_0)^2 \\
& \dfrac{\sin\alpha_{,,,}}{\rho_{,,,}} x_0 \dfrac{\theta_{,}^2 \partial r'}{2 r_0^4} + \dfrac{\sin\alpha_{,,,}}{\rho_{,,,}} \dfrac{3\,\theta_{,}^2 \cos\beta}{r_0^4} \left[\dfrac{x_0 \cos\beta}{r_0} - \tfrac{1}{2} \dfrac{\xi_0}{\rho_0} \right] (\partial\rho_0)^2 \;; \\
\text{in} \quad \partial\delta_{,} \,, & - \dfrac{\sin\delta_{,}}{\rho_{,}} \Big\{ - \sin\alpha_{,} y_0 \dfrac{\theta_{,,,}^2 \partial r'}{2 r_0^4} \quad \sin\alpha_{,} \dfrac{3\,\theta_{,,,}^2 \cos\beta}{r_0^4} \left[\dfrac{y_0 \cos\beta}{r_0} - \tfrac{1}{2} \dfrac{\eta_0}{\rho_0} \right] (\partial\rho_0)^2 \\
& \qquad \cos\alpha_{,} x_0 \dfrac{\theta_{,,,}^2 \partial r'}{2 r_0^4} \quad \cos\alpha_{,} \dfrac{3\,\theta_{,,,}^2 \cos\beta}{r_0^4} \left[\dfrac{x_0 \cos\beta}{r_0} - \tfrac{1}{2} \dfrac{\xi_0}{\rho_0} \right] (\partial\rho_0)^2 \Big\} \\
& + \dfrac{\cos\delta_{,}}{\rho_{,}} \Big\{ - z_0 \dfrac{\theta_{,,,}^2 \partial r'}{2 r_0^4} - \dfrac{3\,\theta_{,,,}^2 \cos\beta}{r_0^4} \left[\dfrac{z_0 \cos\beta}{r_0} \quad \tfrac{1}{2} \dfrac{\zeta_0}{\rho_0} \right] (\partial\rho_0)^2 \Big\} \;; \\
\text{in} \quad \partial\delta_{,,,} \,, & - \dfrac{\sin\delta_{,,,}}{\rho_{,,,}} \Big\{ + \sin\alpha_{,,,} y_0 \dfrac{\theta_{,}^2 \partial r'}{2 r_0^4} - \sin\alpha_{,,,} \dfrac{3\,\theta_{,}^2 \cos\beta}{r_0^4} \left[\dfrac{y_0 \cos\beta}{r_0} - \tfrac{1}{2} \dfrac{\eta_0}{\rho_0} \right] (\partial\rho_0)^2 \\
& \qquad + \cos\alpha_{,,,} x_0 \dfrac{\theta_{,}^2 \partial r'}{2 r_0^4} - \cos\alpha_{,,,} \dfrac{3\,\theta_{,}^2 \cos\beta}{r_0^4} \left[\dfrac{x_0 \cos\beta}{r_0} - \tfrac{1}{2} \dfrac{\xi_0}{\rho_0} \right] (\partial\rho_0)^2 \Big\} \\
& + \dfrac{\cos\delta_{,,,}}{\rho_{,,,}} \Big\{ + z_0 \dfrac{\theta_{,}^2 \partial r'}{2 r_0^4} - \dfrac{3\,\theta_{,}^2 \cos\beta}{r_0^4} \left[\dfrac{z_0 \cos\beta}{r_0} - \tfrac{1}{2} \dfrac{\zeta_0}{\rho_0} \right] (\partial\rho_0)^2 \Big\} \,.
\end{array}
\right\} \quad (9)
$$

In these expressions ∂r has been eliminated by the relation $\partial r = \cos\beta\,\partial\rho$. But when r is small, it is advisable to introduce for $\dfrac{\partial r}{\partial\rho}$ a more accurate expression than $\cos\beta$. Since $r^2 = x^2 + y^2 + z^2$, omitting the subscript zero for the present, we have rigidly

$$(r + \partial r)^2 = (x + \partial x)^2 + (y + \partial y)^2 + (z + \partial z)^2 ,$$

or by (17), (18), and (19), page 11,

$$2r\,\partial r + (\partial r)^2 = 2r \cos\beta\,\partial\rho + (\partial\rho)^2 . \tag{10}$$

Let

$$\frac{\partial r}{\partial\rho} = \frac{2r\cos\beta + \partial\rho}{2r + \partial r} = \Gamma , \tag{11}$$

then, by replacing ∂r in the denominator of (11) by $\Gamma\partial\rho$,

$$2r\cos\beta + \partial\rho = \Gamma(2r + \Gamma\partial\rho) .$$

After multiplying both sides of this equation by $\partial\rho$, we obtain the following quadratic equation in $\Gamma\partial\rho$,

$$(\Gamma\partial\rho)^2 + 2r\,\Gamma\partial\rho = (2r\cos\beta + \partial\rho)\,\partial\rho ,$$

the solution of which gives

$$\Gamma\partial\rho = \partial r = - r + \sqrt{r^2 + (2r\cos\beta + \partial\rho)\,\partial\rho} , \tag{12}$$

or

$$\partial r = - r + \sqrt{(r + \cos\beta\,\partial\rho)^2 + \sin^2\beta\,(\partial\rho)^2} .$$

Since by differentiation $\partial r = \cos\beta\,\partial\rho$ is the first term in the expansion of ∂r in terms

of the heliocentric coordinates, as in (15), page 11. The variations of the latter are expressed rigidly in terms of the variations of the fundamental quantities ρ_0, x_0', y_0', z_0' and of f and g, except for the term $\partial g\,\partial\omega_0'$, which is of the order θ^4 multiplied by a variation, that is, of a higher order than the terms retained in the variations ∂f and ∂g. In expressing ∂f and ∂g in terms of the variations of the fundamental quantities, linear relations, correct inclusively to terms of the order θ^3 multiplied by a variation, are adopted, except that ∂r_0 is expressed rigidly in terms of $\partial\rho_0$. By this procedure all terms are included which numerically may become comparable with the smallest terms retained in ∂f and ∂g, as far as this is possible without complicating the resulting expressions.

On the basis of the foregoing considerations, we may now determine the most advantageous course to pursue if after the direct solution or first hypothesis it becomes necessary to improve the orbit. The correction of the fundamental quantities may be performed by means of differential relations in which the variations ∂f and ∂g are expressed either, as above, in series, or in closed form. Closed expressions for ∂f and ∂g are to be adopted when the ∂f and ∂g series do not converge rapidly. The convergency of these series depends in the main on θ and r. But as all the quantities necessary for estimating the magnitude of the various coefficients in ∂f and ∂g are available from the direct solution, a mere inspection of the θ^3 terms in (16) page 11, or of the terms in (9), will reveal at once whether the largest of these theoretically small coefficients are greatly in excess of numbers that may be neglected in a five- or six-place computation, as the case may be. If the magnitude of any of these terms does not exceed 50 units of the last decimal, the series in ∂f and ∂g may be considered satisfactory, although it is obviously not possible to fix the limit definitely, as it also depends upon the magnitude of the residuals which are to be removed. If these are small, the limit may be taken considerably smaller, etc.

The relation $\partial r_0 = \Gamma\,\partial\rho_0$ enters into the differential formulæ whether closed expressions are adopted or not, but for closed expressions, which are generally employed only for longer arcs, when fairly accurate elements are available, we may let $\Gamma = \cos\beta$.

Assuming now that the series for ∂f and ∂g are to be adopted in the differential correction of the fundamental quantities, we estimate first of all the two terms in θ^3 in the formulæ for f_1, f_2, f_3, page 12, in the manner indicated above. They should be included if they count within the last decimal adopted for the computation. For Γ we use the approximate value $\cos\beta$. The differential correction is then performed as in [VII], page 17, or more conveniently as in (1) to (5). As soon as $\partial\rho_0$ has been computed, we may test the convergency of the series for ∂r by (13). If the series be divergent, the method of differential correction fails, and we must resort to the method of arbitrary variation. But as then $\partial\rho > \frac{r}{1\;2}$, it is easily recognized that this case scarcely will occur in practice. If the series converges

very slowly, we compute Γ accurately to the required number of decimals by (14). If Γ differ considerably from $\cos \beta$, we may conclude that the differential correction will not remove the residuals entirely without taking into account the additional terms (9) or (16) in the expressions for the residuals, and it is advisable to compute these terms by (15) and (16) before proceeding any further in the computation of the differential corrections than is necessary for the determination of $\partial r'$, which is required in (16). By variation of the formula for rr' in VI, page 16, we easily derive, omitting the subscript zero,

$$r\,\partial r' = x\,\partial x' + y\,\partial y' + z\,\partial z' + (x'\xi + y'\eta + z'\zeta)\frac{\partial\rho}{\rho} - r'\Gamma\,\partial\rho\,, \tag{17}$$

from which $\partial r'$ may be derived, after $\partial x'$, $\partial y'$, $\partial z'$ have been computed by (2) and (4).

If now the terms given in (16) be transposed to the left-hand sides of (24) and (25), page 12, and be applied as corrections to the residuals, then it will only be necessary to recompute P_x, P_y, P_z with the corrected residuals, all other auxiliary quantities remaining the same.

The computation of the corrections to the residuals by (14), (15), (16), and (17), presents no difficulties and requires but little computation as the terms are small, and all the quantities involved are available from the previous computations.

With the improved values of P_x, P_y, P_z the computation of the differential corrections to the fundamental quantities ρ_0, x'_0, y'_0, z'_0 is carried through in the usual manner.

It will be observed that in the foregoing directions for taking account of higher terms in the differential corrections, no allowance for the difference between $\cos \beta$ and Γ has been made in the expressions for f_x, f_y, and f_z. This difference may, in general, be neglected in these expressions, but should it involve a considerable change in the three terms multiplied by Γ, then it will be advisable to recompute also the f_x, f_y, f_z and all auxiliary quantities which depend on them.

From the preceding considerations we conclude that the computation of the additional terms in the expressions for the residuals will be required only for large values of $\partial\rho$, combined with slow convergence of the ∂r series. But as such cases chiefly arise from excessively small values of r, the ∂f and ∂g series will then also converge slowly, and, as stated above, these cases are met by the use of closed expressions for ∂f and ∂g.

But it may happen that after the differential correction of the fundamental data has been carried through in its simplest form, the resulting residuals point toward a possible slight improvement of the orbit. This is then generally accomplished as outlined above, and in a far more convenient manner than by a second application of the differential formulæ.

Closed expressions for ∂f and ∂g are given by KUEHNERT in *A. N.* 2266 together with his closed expressions for f and g. These expressions apply to elliptic

orbits and become indeterminate for the parabola. Special formulæ for the circle and the parabola are deduced below (for parabola *cf.* also page 300). These may be derived from KUEHNERT'S expressions, for the circle at once, and for both, after γ has been introduced in place of the product $\sqrt{2a}\sin g$ by (12), page 249. This product is indeterminate for the parabola, but its equivalent γ is definitely given by (19) or (21), page 249.

The expression for γ also gives $\sin 2g = \left(\frac{2}{a}\right)^{\frac{1}{2}} \gamma \sqrt{1 - \frac{\gamma^2}{2a}}$. In the notation adopted in formulæ (8) to (12), pages 248-249, KUEHNERT's expressions for ∂f and ∂g now become, after some simple reductions,

$$\left.\begin{aligned} M &= -\frac{a\gamma}{r r_0}\left[r\gamma + \sqrt{1 - \frac{\gamma^2}{2a}}\left(2r_0\gamma\sqrt{1 - \frac{\gamma^2}{2a}} + \frac{r_0 r_0' \gamma^2}{\sqrt{2}} - \sqrt{2}\,\frac{3}{2}\theta\right)\right] \\ N &= -a\left[\frac{3}{2}\left(\theta\left(1 - \frac{\gamma^2}{r}\right) - g\right) + \frac{\gamma^3}{r}\left(\sqrt{2}\, r_0\sqrt{1 - \frac{\gamma^2}{2a}} + \frac{r_0 r_0' \gamma}{2}\right)\right], \end{aligned}\right\} \quad (18)$$

where g is introduced by means of (8), page 248, to eliminate $2g - \sin 2g$ from N;

$$\left.\begin{aligned} \partial f &= \left\{\frac{\gamma^2}{r r_0}\left[\frac{r}{r_0} + 2\left(1 - \frac{\gamma^2}{2a}\right)\right] + \frac{2M}{r_0^2}\right\}\partial r_0 + M\,\partial(G_0^2) + \frac{\sqrt{2}\,\gamma^3}{r r_0}\sqrt{1 - \frac{\gamma^2}{2a}}\,\partial(r_0 r_0') \\ \partial g &= \left\{\frac{\sqrt{2}\,\gamma^3}{r}\sqrt{1 - \frac{\gamma^2}{2a}} + \frac{2N}{r_0^2}\right\}\partial r_0 + N\,\partial(G_0^2) + \frac{\gamma^4}{r}\,\partial(r_0 r_0'), \end{aligned}\right\} \quad (19)$$

where $G_0^2 = \frac{2}{r_0} - \frac{1}{a}$ represents the velocity in the orbit.

It will be observed that the expression $\sqrt{1 - \frac{\gamma^2}{2a}}$ may be computed directly from g, since $\sqrt{1 - \frac{\gamma^2}{2a}} = \cos g = \cos\frac{1}{2}(E - E_0)$.

For the parabola $G_0^2 = \frac{2}{r_0}$, and $\partial(G_0^2) = -\frac{2}{r_0^2}\partial r_0$. Hence the terms containing M and N destroy one another in ∂f and ∂g respectively. If, then, we also let $a = \infty$, we obtain for the parabola

$$\left.\begin{aligned} \partial f &= \frac{\gamma^2}{r r_0}\left[\frac{r}{r_0} + 2\right]\partial r_0 + \frac{\sqrt{2}\,\gamma^3}{r r_0}\,\partial(r_0 r_0') \\ \partial g &= \frac{\sqrt{2}\,\gamma^3}{r}\,\partial r_0 + \frac{\gamma^4}{r}\,\partial(r_0 r_0'), \end{aligned}\right\} \quad (20)$$

in accordance with the expressions derived below directly and given in (34) and (35), page 300.

For the circle $G_0^2 = \frac{1}{r_0} = \frac{1}{a}$, and $\partial G_0^2 = -\frac{1}{r_0^2}\partial r_0$. Hence the terms containing M and N in ∂f and ∂g may be combined into $\frac{M}{r_0^2}\partial r_0$ and $\frac{N}{r_0^2}\partial r_0$, respectively. If then we let $r = a$ in (18) and (19) and introduce (18) in (19), the latter reduce to

$$\partial f = -\frac{\gamma}{a^3}\sqrt{1 - \frac{\gamma^2}{2a}}\,\sqrt{2}\,\frac{3}{2}\theta\,\partial r_0; \qquad \partial g = -\frac{3}{2a}\left(\theta\left(1 - \frac{\gamma^2}{a}\right) - g\right)\partial r_0. \quad (21)$$

or, if we eliminate γ by (12), page 249, and introduce g by (26), page 250, and also write ∂a for ∂r_0,

$$\partial f = \frac{3}{2}\frac{\theta g}{a^4}\partial a\ ; \qquad \partial g = -\frac{3}{2a}(\theta f - g)\,\partial a\ . \tag{22}$$

These expressions may also be derived directly by differentation of equations (26), page 250.

In order to derive, on the basis of closed expressions for ∂f and ∂g in an elliptic orbit, such corrections to the fundamental data ρ_0, x_0', y_0', and z_0', as will remove the residuals the foregoing equations (19) must be introduced into the developments on page 11 *et seq.* in place of the last two equations in (16), page 11. For the sake of brevity let in (18) and (19)

$$\gamma_c = \sqrt{1 - \frac{\gamma^2}{2a}} = \cos\bar{g} = \cos\tfrac{1}{2}(E - E_0)\ , \tag{23}$$

where the algebraical sign of γ_c is the same as that of $\cos\bar{g}$. Introducing, then, the expressions (19) for ∂f and ∂g into the first of equations (16), page 11, where $\omega = x, y, z$, and noting that by (17), page 11, $\partial\omega_0 = \frac{w_0}{\rho_0}\partial\rho_0$, where $w_0 = \xi_0\ ,\ \eta_0\ ,\ \zeta_0$, we obtain, since $\partial r_0 = \cos\beta\,\partial\rho_0$,

$$\begin{aligned}\partial\omega = &\left\{ f\frac{w_0}{\rho_0} + \frac{2\cos\beta}{r_0^2}(\omega_0 M + \omega_0' N) + \frac{\cos\beta\,\gamma^2}{rr_0}\left(\omega_0\left[\frac{r}{r_0} + 2\gamma_c^2\right] + \sqrt{2}\,\omega_0' r_0\gamma\gamma_c\right)\right\}\partial\rho_0 \\ &+ g\,\partial\omega_0' + (\omega_0 M + \omega_0' N)\,\partial(G_0^2) + \left\{\frac{\sqrt{2}\,\omega_0\gamma^3}{rr_0}\gamma_c + \frac{\omega_0'\gamma^4}{r}\right\}\partial(r_0 r_0')\ .\end{aligned} \tag{24}$$

But since $G_0^2 = x_0'^2 + y_0'^2 + z_0'^2$ and $r_0 r_0' = x_0 x_0' + y_0 y_0' + z_0 z_0'$, therefore, also

$$\partial(G_0^2) = 2\,\Sigma\,\omega_0'\,\partial\omega_0'\quad ;\quad \partial(r_0 r_0') = \Sigma\,\omega_0\,\partial\omega_0' + \Sigma\,\omega_0'\,\partial\omega_0\ , \tag{25}$$

or if $\partial\omega_0$ be expressed as above in terms of $\partial\rho_0$

$$\partial(rr_0') = \Sigma\,\omega_0\,\partial\omega_0' + (\Sigma\,\omega_0'\,w_0)\frac{\partial\rho_0}{\rho_0}\ . \tag{26}$$

Let

$$\varphi_0 = \frac{\Sigma\,\omega_0'\,w_0}{\rho_0} = \frac{x_0'\xi_0 + y_0'\eta_0 + z_0'\zeta_0}{\rho_0}\ , \tag{27}$$

and eliminate $\partial(G_0^2)$ and $\partial(r_0 r_0')$ from (24) by means of the foregoing relations. Then, after some simple reductions, (24) becomes

$$\begin{aligned}\partial\omega = &\left\{ f\frac{w_0}{\rho_0} + \frac{\cos\beta}{r_0^2}[2(\omega_0 M + \omega_0' N) + \omega_0\gamma^2] + \sqrt{2}\cos\beta\,\gamma_c\frac{\gamma^2}{rr_0}[\omega_0' r_0\gamma + \sqrt{2}\,\omega_0\gamma_c] + \right. \\ &\left.\frac{\varphi_0\gamma^3}{rr_0}[\omega_0' r_0\gamma + \sqrt{2}\,\omega_0\gamma_c]\right\}\partial\rho_0 + g\,\partial\omega_0' + \frac{\gamma^3}{rr_0}[\omega_0' r_0\gamma + \sqrt{2}\,\omega_0\gamma_c]\,\Sigma\omega_0\,\partial\omega_0' \\ &+ 2(\omega_0 M + \omega_0' N)\,\Sigma\,\omega_0'\,\partial\omega_0'\ .\end{aligned} \tag{28}$$

Let

$$\left.\begin{aligned} g_\omega &= \frac{\gamma^3}{rr_0}[\omega_0' r_0\gamma + \sqrt{2}\,\omega_0\gamma_c]\quad ;\quad m_\omega = \omega_0 M + \omega_0' N \\ f_\omega &= f\frac{w_0}{\rho_0} + \frac{\cos\beta}{r_0^2}[2m_\omega + \omega_0\gamma^2] + g_\omega\left[\varphi_0 + \frac{\sqrt{2}\cos\beta\,\gamma_c}{\gamma}\right]\ .\end{aligned}\right\} \tag{29}$$

Then

$$\partial\omega = f_\omega\,\partial\rho_0 + g\,\partial\omega_0' + g_\omega \Sigma\,\omega_0\,\partial\omega_0' + 2m_\omega \Sigma\,\omega_0'\,\partial\omega_0'. \tag{30}$$

Equation (30) gives ∂x, ∂y, or ∂z in a closed form for any date, the auxiliary quantities g_x, g_y, g_z; m_x, m_y, m_z; f_x, f_y, f_z being defined by (29). To obtain the values of ∂x, ∂y, or ∂z for a particular date, the values of $f, g, \gamma, \gamma_c, r, M$, and N for that date must be introduced in (29) and (30). γ and γ_c are given by (12), page 249, and by (23); f and g in terms of γ by (11) and (16), page 249; M and N in terms of γ and γ_c by (18), while $r = a(1 - e\cos E)$. Thus we have, for instance, for the first date

$$\left.\begin{aligned}
g_{x_{\prime}} &= \frac{\gamma_{\prime}^2}{r_{\prime} r_0}\left[x_0' r_0 \gamma_{\prime} + \sqrt{2}\, x_0 \gamma_{c_{\prime}}\right]; \qquad m_{x_{\prime}} = x_0 M_{\prime} + x_0' N_{\prime} \\
f_{x_{\prime}} &= f_{\prime}\frac{\xi_0}{\rho_0} + \frac{\cos\beta}{r_0^2}\left[2m_{x_{\prime}} + x_0\gamma_{\prime}^2\right] + g_{x_{\prime}}\left[\varphi_0 + \frac{\sqrt{2}\cos\beta\,\gamma_{c_{\prime}}}{\gamma_{\prime}}\right] \\
\partial x_{\prime} &= f_{x_{\prime}}\partial\rho_0 + \left[g_{\prime} + g_{x_{\prime}} x_0 + 2m_{x_{\prime}} x_0'\right]\partial x_0' + \left[g_{x_{\prime}} y_0 + 2m_{x_{\prime}} y_0'\right]\partial y_0' + \left[g_{x_{\prime}} z_0 + 2m_{x_{\prime}} z_0'\right]\partial z_0',
\end{aligned}\right\} \tag{31}$$

and analogous expressions in y and z for the first date and for all three coördinates for the third date. Thus $f_{y_{\prime\prime\prime}}$ is written out by changing x into y and the subscript one into three in (31), etc.

If, now, equations (15), page 11, be applied to the first and third places, we obtain, by (29) and (30),

$$\begin{aligned}
\partial_{\prime}\alpha_{\prime} = \frac{1}{\rho_{\prime}}&\left[\cos\alpha_{\prime} f_{y_{\prime}} - \sin\alpha_{\prime} f_{x_{\prime}}\right]\partial\rho_0 \\
-\sin\alpha_{\prime}&\left[\frac{g_{\prime}}{\rho_{\prime}} + x_0\frac{g_{x_{\prime}}}{\rho_{\prime}} + x_0' 2\frac{m_{x_{\prime}}}{\rho_{\prime}}\right]\partial x_0' + \cos\alpha_{\prime}\left[x_0\frac{g_{y_{\prime}}}{\rho_{\prime}} + x_0' 2\frac{m_{y_{\prime}}}{\rho_{\prime}}\right]\partial x_0' \\
-\sin\alpha_{\prime}&\left[y_0\frac{g_{x_{\prime}}}{\rho_{\prime}} + y_0' 2\frac{m_{x_{\prime}}}{\rho_{\prime}}\right]\partial y_0' + \cos\alpha_{\prime}\left[\frac{g_{\prime}}{\rho_{\prime}} + y_0\frac{g_{y_{\prime}}}{\rho_{\prime}} + y_0' 2\frac{m_{y_{\prime}}}{\rho_{\prime}}\right]\partial y_0' \\
-\sin\alpha_{\prime}&\left[z_0\frac{g_{x_{\prime}}}{\rho_{\prime}} + z_0' 2\frac{m_{x_{\prime}}}{\rho_{\prime}}\right]\partial z_0' + \cos\alpha_{\prime}\left[z_0\frac{g_{y_{\prime}}}{\rho_{\prime}} + z_0' 2\frac{m_{y_{\prime}}}{\rho_{\prime}}\right]\partial z_0'
\end{aligned}$$

$$\begin{aligned}
\partial\delta_{\prime} = -\frac{1}{\rho_{\prime}}&\left[\sin\delta_{\prime}(\sin\alpha_{\prime} f_{y_{\prime}} + \cos\alpha\, f_{x_{\prime}}) - \cos\delta_{\prime} f_{z_{\prime}}\right]\partial\rho_0 \\
-\sin\delta_{\prime}\cos\alpha_{\prime}&\left[\frac{g_{\prime}}{\rho_{\prime}} + x_0\frac{g_{x_{\prime}}}{\rho_{\prime}} + x_0' 2\frac{m_{x_{\prime}}}{\rho_{\prime}}\right]\partial x_0' - \sin\delta_{\prime}\sin\alpha_{\prime}\left[x_0\frac{g_{y_{\prime}}}{\rho_{\prime}} + x_0' 2\frac{m_{y_{\prime}}}{\rho_{\prime}}\right]\partial x_0' \\
-\sin\delta_{\prime}\cos\alpha_{\prime}&\left[y_0\frac{g_{x_{\prime}}}{\rho_{\prime}} + y_0' 2\frac{m_{x_{\prime}}}{\rho_{\prime}}\right]\partial y_0' - \sin\delta_{\prime}\sin\alpha_{\prime}\left[\frac{g_{\prime}}{\rho_{\prime}} + y_0\frac{g_{y_{\prime}}}{\rho_{\prime}} + y_0' 2\frac{m_{y_{\prime}}}{\rho_{\prime}}\right]\partial y_0' \\
-\sin\delta_{\prime}\cos\alpha_{\prime}&\left[z_0\frac{g_{x_{\prime}}}{\rho_{\prime}} + z_0' 2\frac{m_{x_{\prime}}}{\rho_{\prime}}\right]\partial z_0' - \sin\delta_{\prime}\sin\alpha_{\prime}\left[z_0\frac{g_{y_{\prime}}}{\rho_{\prime}} + z_0' 2\frac{m_{y_{\prime}}}{\rho_{\prime}}\right]\partial z_0' \\
&+ \cos\delta_{\prime}\left[x_0\frac{g_{z_{\prime}}}{\rho_{\prime}} + x_0' 2\frac{m_{z_{\prime}}}{\rho_{\prime}}\right]\partial x_0' \\
&+ \cos\delta_{\prime}\left[y_0\frac{g_{z_{\prime}}}{\rho_{\prime}} + y_0' 2\frac{m_{z_{\prime}}}{\rho_{\prime}}\right]\partial y_0' \\
&+ \cos\delta_{\prime}\left[\frac{g_{\prime}}{\rho_{\prime}} + z_0\frac{g_{z_{\prime}}}{\rho_{\prime}} + z_0' 2\frac{m_{z_{\prime}}}{\rho_{\prime}}\right]\partial z_0',
\end{aligned}$$

need here to be given. The computer will choose from the available methods of solution the one which for his particular problem permits of the most convenient determination of the unknowns on the basis of the numerical values of the coefficients. Nor is it necessary in this case to arrange the solution in a form suitable for the transition from one class of orbit to another as in (2), (4) and (5), since special formulæ will be derived below for the transition from circular and parabolic to elliptic (or hyperbolic) orbits, and since the corrected orbit will prove elliptic (or hyperbolic) if in the preliminary orbit calculations the circle or parabola proves insufficient.

Linear differential relations have been adopted throughout in the foregoing developments on the basis of closed expressions for ∂f and ∂g, as these latter will generally be used only for long arcs for which close approximations to the orbit are almost always available.

Quite frequently the differential correction of ρ_0, x'_0, y'_0, z'_0 may be considerably simplified. For short intervals and not too small a value of r_0, inspection of formulæ (22), page 12, may show that the terms in f_x, f_y, f_z depending on θ^2 and θ^3 are very small in comparison with the first term of each. We may then write

$$f_x = f\cos\delta_{\prime\prime}\cos\alpha_{\prime\prime}\,; \quad f_y = f\cos\delta_{\prime\prime}\sin\alpha_{\prime\prime}\,; \quad f_z = f\sin\delta_{\prime\prime}\,.$$

Introducing these values into formulæ (23), page 12, we obtain for the first and third date,

$$\left.\begin{aligned}
A_{\prime} &= \frac{f_{\prime}}{\rho_{\prime}}\sin\,(\alpha_{\prime\prime}-\alpha_{\prime})\cos\delta_{\prime\prime}\,; \qquad A_{\prime\prime\prime} = \frac{f_{\prime\prime\prime}}{\rho_{\prime\prime\prime}}\sin\,(\alpha_{\prime\prime}-\alpha_{\prime\prime\prime})\cos\delta_{\prime\prime} \\
B_{\prime} &= \frac{f_{\prime}}{\rho_{\prime}}\left[\sin\,(\delta_{\prime\prime}-\delta_{\prime}) + 2\sin\delta_{\prime}\cos\delta_{\prime\prime}\sin^2\frac{1}{2}(\alpha_{\prime\prime}-\alpha_{\prime})\right] \\
B_{\prime\prime\prime} &= \frac{f_{\prime\prime\prime}}{\rho_{\prime\prime\prime}}\left[\sin\,(\delta_{\prime\prime}-\delta_{\prime\prime\prime}) + 2\sin\delta_{\prime\prime\prime}\cos\delta_{\prime\prime}\sin^2\frac{1}{2}(\alpha_{\prime\prime}-\alpha_{\prime\prime\prime})\right],
\end{aligned}\right\} \tag{35}$$

where in $B_{\prime}$ and $B_{\prime\prime\prime}$ the second term frequently may be neglected in comparison with the first.

If the inspection proves these approximate values of $A_{\prime}$, $A_{\prime\prime\prime}$, $B_{\prime}$, $B_{\prime\prime\prime}$ to be sufficient, they may be used in formulæ (1) and (3) on page 255 in place of their more rigid expressions. The subsequent computation remains as indicated on page 255.

To determine the error in σ, we must determine the error in κ in II, page 15. Substituting α' for α'_0, δ for δ'_0, etc., from (2) and (3) into κ, we have

$$\kappa = -R\cos D\,\frac{[\tan\delta_{,,}\cos(A-\alpha_{,,})-\tan D]\,\alpha'_0+\sin(A-\alpha_{,,})(\tan\delta)'_0-\frac{\theta_{,}\theta_{,,,}}{6}\left\{[\tan\delta_{,,}\cos(A-\alpha_{,,})-\tan D]\,\alpha'''+\sin(A-\alpha_{,,})(\tan\delta)'''\right\}}{(\alpha'_0)^2\tan\delta_{,,}-\alpha''_0(\tan\delta)'_0+\alpha'_0(\tan\delta)''_0-\left\{\frac{\theta_{,}\theta_{,,,}}{6}[3(\alpha'_0)^2\tan\delta_{,,}+(\tan\delta)''_0]+\frac{\theta_{,,,}-\theta_{,}}{3}(\tan\delta)'_0\right\}\alpha'''+\left\{\frac{\theta_{,,,}-\theta_{,}}{3}\alpha'_0+\frac{\theta_{,}\theta_{,,,}}{6}\alpha''_0\right\}(\tan\delta)'''}, \tag{4}$$

or, denoting the values of N and κ in III, page 15, where third derivatives are neglected, by N_0 and κ_0

$$\begin{aligned}\kappa = \kappa_0 &+ \frac{\theta_{,}\theta_{,,,}}{6N_0}\left\{[\tan\delta_{,,}\cos(A-\alpha_{,,})-\tan D]\,\alpha'''+\sin(A-\alpha_{,,})(\tan\delta)'''\right\}R\cos D\\ &-\left\{\frac{\theta_{,}\theta_{,,,}}{6N_0^2}[3(\alpha'_0)^2\tan\delta_{,,}+(\tan\delta)''_0]+\frac{\theta_{,,,}-\theta_{,}}{3N_0^2}(\tan\delta)'_0\right\}R\cos D\,\alpha'''\\ &+\left\{\frac{\theta_{,,,}-\theta_{,}}{3N_0^2}\alpha'_0+\frac{\theta_{,}\theta_{,,,}}{6N_0^2}\alpha''_0\right\}R\cos D\,(\tan\delta)'''\ .\quad .\ ,\end{aligned} \tag{4a}$$

where the remaining terms may be omitted for the purpose of estimating the error in κ. This error is of the second order for equal or nearly equal intervals and of the first for unequal intervals. It becomes inappreciable for nearly uniform geocentric motion.

Since

$$\sigma = \kappa\left\{\frac{1}{r^3}-\frac{1}{R^3}\right\}$$

the error in σ is of the same order as the error in κ. By inspection of the formulæ for σ' and σ'' in V, page 16, the same orders of error will be found for these quantities. In a similar manner the errors depending on the fourth derivatives, etc., may be deduced.

In the first hypothesis the derivatives beyond the second are neglected. In order to indicate how a second hypothesis may be based on the first, we shall develop expressions for α''' and $(\tan\delta)'''$. By successive differentiation of

$$\sigma\cos\alpha = x + X\,,\qquad \sigma\sin\alpha = y + Y\,,\qquad \sigma\tan\delta = z + Z$$

and remembering that

$$x''' = \frac{1}{k}\frac{d}{dt}x'' = \frac{1}{k}\frac{d}{dt}\left(-\frac{x}{r^3}\right) = -\frac{x'}{r^3}+3\frac{xr'}{r^4}\,, \qquad y''' = \text{etc.}$$

we obtain the equations

$$\left.\begin{aligned}&\sigma'''\cos\alpha-3\sigma''\sin\alpha\,\alpha'-3\sigma'[\cos\alpha(\alpha')^2+\sin\alpha\,\alpha'']-\sigma\{3\cos\alpha\,\alpha'\alpha''+\sin\alpha[\alpha'''-(\alpha')^3]\}\\&\qquad=-\frac{x'}{r^3}+3\frac{xr'}{r^4}+X'''\\&\sigma'''\sin\alpha+3\sigma''\cos\alpha\,\alpha'-3\sigma'[\sin\alpha(\alpha')^2-\cos\alpha\,\alpha'']-\sigma\{3\sin\alpha\,\alpha'\alpha''-\cos\alpha[\alpha'''-(\alpha')^3]\}\\&\qquad=-\frac{y'}{r^3}+\frac{3yr'}{r^4}+Y'''\\&\sigma'''\tan\delta+3\sigma''(\tan\delta)'+3\sigma'(\tan\delta)''+\sigma(\tan\delta)'''=-\frac{z'}{r^3}+\frac{3zr'}{r^4}+Z'''\,,\end{aligned}\right\} \tag{5}$$

which determine α''' and $(\tan\delta)'''$. Multiply the first of these by $\cos\alpha$, the second by $\sin\alpha$ and add the results. Then

$$\sigma''' = 3\sigma'(\alpha')^2 + 3\sigma\alpha'\alpha'' + \frac{3r'}{r^4}[x\cos\alpha + y\sin\alpha] - \frac{1}{r^3}[x'\cos\alpha + y'\sin\alpha] + X'''\cos\alpha + Y'''\sin\alpha,$$

or since by VI, page 16,

$$x\cos\alpha + y\sin\alpha = \sigma - [X\cos\alpha + Y\sin\alpha]\ ; \quad x'\cos\alpha + y'\sin\alpha = \sigma' - [X'\cos\alpha + Y'\sin\alpha]\ ,$$

therefore, also

$$\sigma''' = \sigma\left(\frac{3r'}{r^4} + \alpha'\alpha''\right) - \sigma'\left(\frac{1}{r^3} - 3(\alpha')^2\right) = -\frac{3r'}{r^4}[X\cos\alpha + Y\sin\alpha] + \frac{1}{r^3}[X'\cos\alpha + Y'\sin\alpha] + X'''\cos\alpha + Y'''\sin\alpha. \tag{6}$$

Substituting this expression for σ''' in the first and second of (5), we solve

$$\alpha''' = (\alpha')^3 - \frac{\alpha'}{r^3} - \frac{3}{\sigma}[\sigma'\alpha'' + \sigma''\alpha'] + \frac{3r'}{\sigma r^4}[X\sin\alpha - Y\cos\alpha] - \frac{1}{\sigma r^3}[X'\sin\alpha - Y'\cos\alpha] - \frac{1}{\sigma}[X'''\sin\alpha - Y'''\cos\alpha]. \tag{7}$$

By similarly eliminating σ''' from the third of (5) by means of (6), we obtain

$$\begin{aligned}(\tan\delta)''' = &- 3\tan\delta\,\alpha'\alpha'' + \frac{(\tan\delta)'}{r^3} - \frac{3}{\sigma}[\sigma''(\tan\delta)' + \sigma'((\tan\delta)'' + \tan\delta\,(\alpha)'_2])\\ &+ \frac{3r'}{\sigma r^4}[(X\cos\alpha + Y\sin\alpha)\tan\delta - Z] - \frac{1}{\sigma r^3}[(X'\cos\alpha + Y'\sin\alpha)\tan\delta - Z']\\ &- \frac{1}{\sigma}[(X'''\cos\alpha + Y'''\sin\alpha)\tan\delta - Z'''].\end{aligned} \tag{8}$$

These expressions give α''' and $(\tan\delta)'''$ in terms of σ, α, $\tan\delta$, their velocities and accelerations, and of r and r'. For a second hypothesis, all these quantities are taken from the first hypothesis. It should be observed, however, that α and δ and the solar coördinates remain constant. After α''' and $(\tan\delta)'''$ have been computed by (7) and (8), we may either compute α', α'', $(\tan\delta)'$, $(\tan\delta)''$ by (2) and (3) and then repeat the solution by the formulæ of the first hypothesis, or we may directly compute the corrected value of $\varkappa$ by (4) or (4a).

As stated in the introduction, however, an expression for σ may be derived which permits of performing a second hypothesis without the necessity of actually determining the corrections to the initial velocities and accelerations, or of computing α''' and $(\tan\delta)'''$ for substitution in (4) or (4a). This may evidently be accomplished by eliminating α''' from (2) and (7) and $(\tan\delta)'''$ from (3) and (8) by successive substitution so as to express α', α'' and $(\tan\delta)'$, $(\tan\delta)''$ in series of ascending powers of the initial velocities and accelerations α'_0, α''_0, etc. By introducing these series for the geocentric velocities and accelerations into the formula given for $\varkappa$ in II, page 15, we shall obtain an expression for the second hypothesis of $\varkappa$ and, therefore, also of σ, etc. This expression will involve the unknown velocity r' in addition to the unknown quantities occurring in the equation for σ in the first hypothesis. The terms distinctive of the second hypothesis will also

contain as factors the first and higher powers of the products and the differences of the intervals.

As soon, therefore, as the velocity r' has been derived in the first hypothesis, the complete expressions of the second hypothesis may be computed. But this procedure would lead to the usual complicated numerical operations, even in minor planet orbits for which r' is generally small. The process would become still more complicated, if extended to higher derivatives than the third of the geocentric coördinates.

As stated above, we have therefore chosen the plan of determining from differential relations such corrections to the fundamental quantities ρ, x', y', z' as will remove whatever residuals may remain after our first hypothesis or direct solution.

With reference to the errors of σ, σ', σ'' in the first hypothesis, POINCARÉ has shown, as stated in the introduction, that even with unequal intervals they are of the second order, if the mean of the three dates be chosen for the epoch.

The method of determining the error of an orbit solution by general considerations based on the order of neglected terms is, at best, unsatisfactory, since account is not taken of the special conditions which may exist. It is far more satisfactory to estimate the error *numerically* and thus to arrive at the accuracy of the result in each particular case. It is even possible at the outset to compare the relative effect of the various sources of errors and to plan the solution accordingly by eliminating some of the errors and neglecting others. This method of procedure will preclude an indiscriminate mechanical application of the formulæ for the solution of an orbit. But it makes it possible to obtain the best orbit derivable from the given data. The accuracy of the orbits computed in the Berkeley Astronomical Department is in a large measure due to such an analysis of the conditions of the problem in each individual case.

The accuracy of the preliminary orbit solution depends in the main on the following four conditions:

(1) The magnitude of the geocentric motion.

(2) The magnitude of the errors of observation.

(3) The magnitude of the parallax, if this be neglected.

(4) The magnitude of the neglected terms containing the third and higher derivatives of α and δ.

As these effects are independent of each other, they may be considered separately; nor is it necessary to study the accuracy of the resulting solution both for solutions with and without assumption regarding the eccentricity. It is sufficient to determine the accuracy of the solution for the general case, that is, without assumption regarding the eccentricity. The feasibility of a special solution, that is, with assumption regarding the eccentricity, then depends on whether the special solution falls within the range of uncertainty of the general solution.

With reference to the *geocentric motion* it is evident at once that the number of figures to which the velocities $\alpha'_{,}$, $\alpha'_{,,,}$, α'_{0}, $\delta'_{,}$, $\delta'_{,,,}$, δ'_{0}, $(\tan\delta)'_{0}$, and the accelerations α''_{0} and $(\tan\delta)''_{0}$ can be determined, depends upon the number of places in the differences of the observed α and δ, expressed in seconds of arc, from which these velocities and accelerations are computed. As the tenths of a second of arc are practically always uncertain, even in the most accurate observations, we may consider these differences to be accurately known to the nearest second of arc, except for other errors of observations the effect of which is to be considered separately. No special calculation therefore, is necessary to determine the number of places accurately known in each velocity and acceleration. This number of places is obtained at once by a mere inspection of the given differences of the observed α and δ.

In a general orbit solution $z=\frac{\rho}{R}$ is to be obtained by interpolation from the tables at the end of this paper with ψ and $\frac{1}{m}$ as arguments. The accuracy to which z can be determined depends, therefore, on the accuracy of $\frac{1}{m}$, but by formulæ III, page 15,

$$\frac{1}{m}=\frac{N}{C_1\,\alpha'+C_2\,(\tan\delta)'}\,\frac{\cos\delta\,R^4}{S}\,, \tag{9}$$

where

$$N=\alpha'^2-\alpha''(\tan\delta)'+\alpha'(\tan\delta)''\ ;\quad C_1=\tan\delta\cos(A-\alpha)-\tan D\ ;\quad C_2=\sin(A-\alpha)\,. \tag{10}$$

In this formula the subscripts referring to the middle place have been suppressed. The accuracy of $\frac{1}{m}$ depends, therefore, upon the accuracy to which the ratio

$$\frac{N}{C_1\,\alpha'+C_2\,(\tan\delta)'} \tag{11}$$

can be computed. The number of known places in $\frac{1}{m}$ may thus be obtained by inspection on the basis of the number of places to which the velocities and accelerations are known. It may, however, occur that the ratio (11) becomes indeterminate. In such cases the intelligent computer will transform his coördinates and velocities from the equator to an arbitrary fundamental plane so chosen as to give the most definite value of $\frac{1}{m}$.

As indicated above, after $\frac{1}{m}$ has been computed and the number of places to which $\frac{1}{m}$ is known has been decided by inspection, the interpolation of z from the tables reveals at once the number of places to which z is known.

The effect of *errors of observation* can not, of course, be determined with absolute precision. Observations of the point images of minor planets, of satellites, or of comets with a well-defined nucleus are always more reliable than observations of diffused comets. For objects observed at low altitudes the observations often are affected more than is necessary by errors arising in the correction for refraction. But, as the nature of the object and the conditions under which the observations were made are always known, it is possible to fix an upper limit for the error of observation. No account can, of course, be taken of gross errors of

observation and reduction. No computer can be held responsible for an inaccuracy in his result due to an uncalled for error on the part of the observer.

Let $u_{,}$, $u_{,,}$, $u_{,,,}$ represent the correct values of the observed α or δ; $e_{,}$, $e_{,,}$, $e_{,,,}$ the corresponding errors of observation; $\partial_1 u'$ and $\partial_1 u''$ the errors arising in the velocity u' and the acceleration u'' from the errors of observation. Then by substituting $u_{,} + e_{,}$ for either $\alpha_{,}$ or $\delta_{,}$; $u_{,,} + e_{,,}$ for either $\alpha_{,,}$ or $\delta_{,,}$; etc., in the formulæ for the velocities and accelerations given in III, page 15, and omitting for the present the factors $\frac{15 \sin 1''}{k}$ and $\frac{\sin 1''}{k}$, so that the velocities are expressed in seconds of arc per mean solar day, we obtain

$$\partial_1 u' = \frac{\frac{t_{,,,}-t_{,,}}{t_{,,}-t_{,}}(e_{,,}-e_{,}) + \frac{t_{,,}-t_{,}}{t_{,,,}-t_{,,}}(e_{,,,}-e_{,,})}{t_{,,,}-t_{,}} = \frac{\left[\frac{t_{,,,}-t_{,,}}{t_{,,}-t_{,}} - \frac{t_{,,}-t_{,}}{t_{,,,}-t_{,,}}\right]e_{,,} - \frac{t_{,,,}-t_{,,}}{t_{,,}-t_{,}}e_{,} + \frac{t_{,,}-t_{,}}{t_{,,,}-t_{,,}}e_{,,,}}{t_{,,,}-t_{,}}\,.$$

To obtain the maximum effect of these errors, let the sign of the term containing $e_{,}$ be changed from $-$ to $+$. Let also $e_{,} = e_{,,} = e_{,,,} = e$, where e represents the maximum error of observation. Then the error of the velocity u' will be less than

$$\partial_1 u' < 2\,\frac{t_{,,,}-t_{,,}}{t_{,,}-t_{,}}\,\frac{e}{t_{,,,}-t_{,}}\,. \tag{12}$$

For equal or nearly equal intervals, the error of u' is, therefore, less than e, divided by the constant interval. For unequal intervals we have by division of $\partial_1 u'$ by u', where, as stated above, u' stands for the velocity in seconds of arc per mean solar day,

$$\frac{\partial_1 u'}{u'} < \frac{t_{,,,}-t_{,,}}{t_{,,}-t_{,}}\,\frac{e}{(t_{,,,}-t_{,,})\,u'_{,,,} + (t_{,,}-t_{,})\,u'_{,}} = \frac{t_{,,,}-t_{,,}}{t_{,,}-t_{,}}\,\frac{e}{d'} = a'_{,}\,, \tag{13}$$

where the divisor $d' = (t_{,,,}-t_{,,})\,u'_{,,,} + (t_{,,}-t_{,})\,u'_{,}$ has been previously computed in the derivation of α' and δ'.

In a similar manner we obtain for the upper limit of the error of the acceleration u'', expressed in seconds of arc per mean solar day per mean solar day,

$$\partial_1 u'' = 2\,\frac{\frac{e_{,,,}-e_{,,}}{t_{,,,}-t_{,,}} - \frac{e_{,,}-e_{,}}{t_{,,}-t_{,}}}{t_{,,,}-t_{,}} = 2\,\frac{\frac{e_{,,,}}{t_{,,,}-t_{,,}} - e_{,,}\left[\frac{1}{t_{,,,}-t_{,,}} + \frac{1}{t_{,,}-t_{,}}\right] + \frac{e_{,}}{t_{,,}-t_{,}}}{t_{,,,}-t_{,}}\,.$$

By changing the sign of $e_{,,}$ and equating the errors to e, as before, we find that the error in the acceleration u'' can not exceed

$$\partial_1 u'' < 4e\,\frac{\frac{1}{t_{,,,}-t_{,,}} + \frac{1}{t_{,,}-t_{,}}}{t_{,,,}-t_{,}}\,.$$

For equal intervals $\partial_1 u''$ is less than $4e$ divided by the square of the constant interval. For unequal intervals

$$\frac{\partial_1 u''}{u''} < \frac{2e\left[\frac{1}{t_{,,,}-t_{,,}} + \frac{1}{t_{,,}-t_{,}}\right]}{u'_{,} - u'_{,,,}} = a''_{,}\,, \tag{14}$$

where all quantities involved are known from previous computation.

As soon as the maximum value of the errors of observation has been fixed it is possible, therefore, to compute the maximum errors of the velocities and accelerations by formulæ (13) and (14).

The parallax may be dealt with in three different ways:

(a) It may be *neglected entirely*. In that case its effect on the velocities and accelerations is the same as that of the errors of observation. The parallax errors may, therefore, be included in the estimate of the maximum error e. It is, of course, not possible to know anything definite about the parallax until after the geocentric distances have become known. By choosing, however, the error of observation sufficiently large to include the probable parallax as judged from the observational data, the effect of the parallax may be determined, at least in a preliminary manner. The estimate of the parallax can be verified and, if necessary, corrected as soon as the geocentric distances have become known.

(b) Or the parallax may be *eliminated partially*, by applying to the solar coördinates at the middle date the corrections $\Delta_2 X$, $\Delta_2 Y$ and $\Delta_2 Z$, given by equations (4), page 234. In the partial elimination of the parallax the coördinates are referred to the projection of the center of the Earth upon the line of sight as origin. It should be remembered that the partial elimination of the parallax does not eliminate the barycentric parallax from the coördinates. Nor does it eliminate the effect of the parallax on the velocities and accelerations. The error in these latter must therefore be derived as though the parallax had been entirely neglected.

(c) Or the parallax may be *eliminated completely* by means of the formulæ derived on pages 236 and following.

The decision as to which of the three methods of dealing with the parallax shall be applied, depends upon the observational data. If the estimated error of observation and the error introduced by neglecting the third and higher derivatives of the coördinates, which will be discussed below, are comparatively large, then little or nothing can be gained by completely eliminating the parallax. For the convenience, however, of subsequent computation it is advisable to apply the partial elimination of the parallax in all cases. This convenience arises from the fact that the partial elimination of the parallax leads to the *geocentric* distance at the middle date and that the parallax need not be considered further in computing the heliocentric coördinates.

Whether, in case of an accurately observed object and of small errors due to the neglect of the higher derivatives, the parallax is to be eliminated partially or completely, will depend on whether the complete elimination will greatly enhance the accuracy of the solution. In general, the observations may be represented with the same degree of accuracy with or without elimination of the parallax, but by completely eliminating the parallax a more accurate orbit will result so that a better representation of the *subsequent* motion is attained.

But since all preliminary orbits must necessarily be approximate on account of the shortness of the observed arc and since practical considerations demand that all extra computation, however small, be avoided, the complete elimination of the parallax should be undertaken only in those rare cases in which a solution otherwise can not be accomplished. Such cases may occur for an accurately observed object, if the motion is very small, if the intervals are quite short, and if the parallax factors (barycentric plus geocentric) are very different for the three observations. In all other cases the partial elimination of the parallax will suffice for practical purposes. If, for instance, the parallax factors for the three observations are nearly constant, as would be the case for observations made at moderate intervals at the same observatory at nearly the same hour angle, then the neglect of the parallax considered as an error of observation would not affect the velocities and accelerations, since these depend on the differences of the observed α and δ.

It would, of course, be entirely useless to eliminate the parallax completely if the parallax factors indicate that it is much smaller than the possible errors of observation and the errors due to the neglect of the higher derivatives of α and δ.

(d) In determining the effect of *neglecting the higher derivatives* of α and δ we shall confine ourselves to the consideration of the third derivatives. According to the theory of numerical differentiation, if the intervals are constant, a third derivative may be expressed in a series, the first term of which is the mean of the two third differences on either side of the date. Let this mean be denoted by f'''_u, where u stands for either α or δ.

By equation (2), omitting the algebraical sign, the error of the velocities u' and accelerations u'' due to neglecting the third derivatives, is therefore

$$\partial_2 u' = \frac{\theta_{,}\theta_{,,,}}{6}\,\frac{f'''_u}{\theta^3}\ ; \quad \partial_2 u'' = \frac{\theta_{,,,} - \theta_{,}}{3}\,\frac{f'''_u}{\theta^3}\,, \tag{15}$$

while

$$u' = \frac{f'_u}{\theta}\ ; \quad u'' = \frac{f''_u}{\theta^2}\,, \tag{16}$$

where f'_u and f'''_u are the means of the first and third differences on either side of the middle date corresponding to the constant interval θ. Dividing (15) by (16) we obtain

$$\frac{\partial_2 u'}{u'} = \frac{\theta_{,}\theta_{,,,}}{\theta^2}\,\frac{\frac{f'''_u}{6}}{f'_u} = a'\,, \quad \frac{\partial_2 u''}{u''} = \frac{\theta_{,,,} - \theta_{,}}{\theta}\,\frac{\frac{f'''_u}{3}}{f''_u} = a''\,. \tag{17}$$

For equal intervals $\theta_{,} = \theta_{,,,} = \theta$ and

$$\partial_2 u' = \frac{\frac{f'''_u}{6}}{f'_u}\,u'\,, \quad \partial_2 u'' = 0\,, \tag{18}$$

except for the effect of fourth and higher differences.

If four observations at equal intervals be available, the f'''_u are given at once by the observations and the velocities derived from three observations may be

mined from the observations on the basis that the observations are correct to the nearest second of arc. Equations (20) and (21) now further determine how many of the places given by the geocentric motion are lost on account of the various errors involved, so that the number of places accurately given in $\frac{1}{m}$ has finally become known in any particular case and, as stated above, the accuracy of $\frac{1}{m}$ determines the accuracy to which z can be interpolated from the tables. In this manner a fairly close estimate of the accuracy of the resulting orbit computation may be obtained.

An orbit becomes indeterminate when $\frac{1}{m}$ becomes indeterminate. As stated above, in some cases inspection will show that the indeterminateness may be removed by referring the coördinates to a new fundamental plane. But if the indeterminateness is due entirely to errors of observation then no solution can be made. In this case, which is not probable in practice, the errors of observation would be comparable to the motion.

Finally the indeterminateness may arise from the neglect of the third and higher derivatives. Too much weight has been attached in the past to this possibility. In fact it has been used as the chief argument against the convergence of the LAPLACEAN methods. Experience, however, as well as the foregoing discussion, are sufficient proof that indeterminateness will rarely arise from this cause. In fact, with short intervals, such as are always available, and particularly for slow motion, the first direct solution has, in most cases, yielded a satisfactory representation of the observations.

If an indeterminateness should arise on account of the neglect of the third and higher differences, this fact can be inferred from the differences of the observed α and δ, at the outset, before the solution is undertaken. The remedy then lies in the use of at least the third differences by basing the computation on more than three observations by the formula derived on page 230 and following, or by having recourse to shorter intervals, if such be available. For equation (19) shows that for one-half the interval the third differences will be reduced to approximately one-eighth of their value, etc., so that, theoretically at least, it is always possible to choose intervals so small that the third and higher differences will not affect the solution at all. But it should be remembered that the geocentric arc will be less and the number of decimals to which the orbit can be computed will be reduced in the ratio in which the number of seconds in the geocentric arc is reduced by the shorter intervals.

It is safe to say that, by carefully judging the conditions of the problem at the outset in every case, the computer will never run the risk of performing a fruitless computation if he apply the LAPLACEAN methods in the form in which they are here presented. This has been the experience in all orbit computations at the Students' Observatory of the University of California in recent years.

The order in which the effect of the various errors referred to above should be considered must be decided by the computer in each individual case. Very

this most unfavorable case, we shall assume, as above, that the errors ϵ are numerically equal, but that their algebraical signs are such that every error counts fully in one or the other error sum. The error sums will then reduce to $\frac{3}{2}\epsilon$ and $\frac{16}{3}\epsilon$. It is clear, of course, from equations (24) that in practice they can not reach their maximum for both the velocities and the accelerations at the same time. In order to judge the relative error made by neglecting the third and higher differences in the case of three observations, we might let $\epsilon_1 = \epsilon_5 = 0$, but if five observations are made the basis of computation, then we shall also have to deal with five errors of observation. On that account it is more accurate to base the comparison on the error sums as given above. For equal intervals, therefore, we must compare

$$\text{in } \theta u' : \frac{1}{6} f''' (a + i w) \text{ with } \frac{3}{2}\epsilon ; \text{ in } \theta^2 u'' : \frac{1}{12} f^{iv} (a + i w) \text{ with } \frac{32}{6}\epsilon .$$

It follows, therefore, that little can be gained by the use of more than three observations if

$$f''' (a + i w) \leq 9\epsilon ; \qquad f^{iv} (a + i w) \leq 64\epsilon .$$

As was found above, the accuracy of an acceleration is diminished in a greater degree by the errors of observation than that of a velocity. A detailed discussion of the velocities may therefore be dispensed with. In order to gain anything then, by basing the computation on more than three observations, f^{iv}_u would have to be considerably greater than 64ϵ; that is, for every second of error of observation, the fourth difference would have to exceed 1′. In the case of comets, the errors of observations are quite frequently greater than 5″. To gain anything then, the fourth differences in α and δ would have to be in excess of 5′.

From these considerations it appears that the criticisms that have been applied to the LAPLACEAN methods, and which were refuted above, are still more unjustifiable when the errors of observations are taken into consideration. In general, one element of uncertainty in any method of orbit solution depends upon the percentage error in the motion. This element is necessarily about the same for all methods. It is brought out here very distinctly, that in attacking the LAPLACEAN methods altogether too much emphasis has been laid in the past on the effect of the third and higher differences. The conclusion drawn above, is, therefore, fully justified.

In the case of unequal intervals, the error in the accelerations was found to be approximately $\frac{f'''}{3}$. Comparing this error with the error sum $\frac{16}{3}\epsilon$, which method of comparison, although not as rigid as the foregoing comparisons, is sufficient for our purposes, it is found that in order to gain anything f''' would have to be in excess of 16ϵ, so that in the case of unequal intervals the inclusion of a fourth observation would be more justifiable. It is, however, more advantageous, even in the case of unequal intervals, to base the computation, whenever possible, merely on three observations and to remove, if necessary, any inaccuracies in the resulting elements by the simple methods of differential correction to be discussed later.

which is the desired equation of the sixth degree, since $\sigma_0 = \rho_0 \cos\delta_{\prime\prime}$. Expanding the squares within the second parenthesis and introducing the auxiliary quantities

$$a^2 = (a_x^2 + a_y^2 + a_z^2)\cos^2\delta_{\prime\prime}\,; \quad b = (a_x X_0' + a_y Y_0' + a_z Z_0')\cos\delta_{\prime\prime}\,; \quad G_s^2 = X_0'^2 + Y_0'^2 + Z_0'^2\,, \tag{11}$$

where, provided the parallax be entirely neglected, the velocity G_s may also be computed from the well-known relation

$$G_s^2 = \frac{2}{R} - 1\,, \tag{11a}$$

we may write equation (10) in the form

$$(\rho_0^2 - 2\rho_0 R\cos\psi + R^2)(a^2\rho_0^2 - 2b\rho_0 + G_s^2)^2 = 4 \tag{12}$$

or

$$F(\rho_0) = \Gamma\gamma^2 - 4 = 0 \tag{13}$$

where

$$\Gamma = \rho_0^2 - 2\rho_0 R\cos\psi + R^2\,; \quad \gamma = a^2\rho_0^2 - b\rho_0 + G_s^2. \tag{14}$$

A first approximation to the value of ρ_0 in equation (13) may be obtained by taking $z = \frac{\rho_0}{R}$ from the tables at the end of this paper with the arguments ψ and $\frac{1}{m}$. The formulæ for ψ and $\frac{1}{m}$ are given on page 15, formulæ III. Let the approximate value of ρ_0 obtained with the aid of the tables be denoted by ρ_1 and let

$$(\rho_1^2 - 2\rho_1 R\cos\psi + R^2)(a^2\rho_1^2 - 2b\rho_1 + G_s^2) - 4 = M_1\,, \tag{15}$$

or, by equations (13) and (14),

$$\Gamma_1\gamma_1^2 - 4 = M_1\,. \tag{16}$$

Then it will be necessary to determine a correction $\Delta\rho_1$ to ρ_1 in such a manner that $\rho_2 = \rho_1 + \Delta\rho_1$ will satisfy (12), or (13) and (14). We may write approximately

$$\Delta F(\rho_1) = \left[\frac{dF(\rho_0)}{d\rho_0}\right]_{\rho_0=\rho_1}\Delta\rho_1 = -M_1\,. \tag{17}$$

But from (13) and (14) we have

$$\left[\frac{dF(\rho_0)}{d\rho_0}\right]_{\rho_0=\rho_1} = 2\gamma_1[2\Gamma_1(a\rho_1 - b) + \gamma_1(\rho_1 - R\cos\psi)]\,. \tag{18}$$

Hence from (17) and (18)

$$\Delta\rho_1 = -\frac{M_1}{2\gamma_1[2\Gamma_1(a\rho_1 - b) + \gamma_1(\rho_1 - R\cos\psi)]} \tag{19}$$

and

$$\rho_2 = \rho_1 + \Delta\rho_1\,. \tag{20}$$

If ρ_2 does not satisfy (12), or (13) and (14), then we let

$$F(\rho_2) = M_2$$

and repeat the approximations until a value of ρ is found which satisfies (12), or (13) and (14). In general, not more than two approximations will be required.

If we derive the equation in ρ_0 in a slightly different form, approximate values of the roots may be found graphically with more convenience than with the aid of the tables at the end of this paper. The modified form also seems to be preferable for numerical computation.

Equation (3) gives

$$r_0 = +[\rho_0^2 - 2\rho_0 R\cos\psi + R^2]^{\frac{1}{2}}\,, \tag{21}$$

Let $f(z_2) = M_1$ and continue the approximations until $f(z) = 0$.

To obtain a first approximation to z from the tables, we enter them as before in case of equations (12) or (13) and (14) with the arguments ψ and $\frac{1}{m}$.

To obtain the roots of equation (24) graphically, we determine the values of z for the intersections of the parabola

$$y = (z - p')^2 \tag{28}$$

and the curve

$$y = \frac{h}{[(z-c)^2 + \varkappa^2]^{1/2}} - q'^2 , \tag{29}$$

where we shall take the z as abscissæ and the y as ordinates.

The parabola is the same in all cases. Its axis coincides with the line $z = p'$, and $y = 0$ is a tangent at the vertex. We may, therefore, once for all, plot the parabola $y = z^2$ on a convenient scale.

The curve (29) is symmetrical with respect to $z = c$ and lies wholly above the line $y = -q'^2$, because the negative sign of the square root is excluded, r always being positive. For $z = \pm\infty$, $y = -q'^2$; hence the line $y = -q'^2$ is an asymptote to the curve. At $z = c$, $y = \frac{h}{\varkappa} - q'^2$ the curve has its maximum. By letting $\frac{d^2 y}{d z^2} = 0$ we readily find for the points of inflection $z - c = \frac{\pm\varkappa}{\sqrt{2}}$, $y = \sqrt{\frac{2}{3}}\frac{h}{\varkappa} - q'^2$. The distance of the points of inflection above the line $y = -q'^2$ is therefore $\sqrt{\frac{2}{3}}$ or .816 . . . times the distance of the vertex of the curve above the same line.

Since the parabola lies wholly above the z axis, we are concerned only with that part of the asymptotic curve which lies above $y = 0$. For $y = 0$ we find from (29)

$$\frac{h}{[(z-c)^2 + \varkappa^2]^{1/2}} = q'^2 ; \quad (z-c)^2 = \frac{h^2}{q'^4} - \varkappa^2 ,$$

or

$$z - c = \pm\sqrt{\frac{h^2}{q'^4} - \varkappa^2} . \tag{30}$$

An approximate value of $z - c$ may be obtained from (30) by formulæ (23), if in the latter we let $G_s^2 = \frac{2}{R} - 1$ and $R = 1$. Then $G_s^2 = 1$ and $q'^2 = \frac{1}{a^2} - \frac{b^2}{a^4} = \frac{1}{a^2}\left(1 - \frac{b^2}{a^2}\right)$, while $h = \frac{2}{a^2}$.

Since q'^2 is positive, we have $b^2 < a^2$.

We may now write (30) in the form

$$z - c = \pm\sqrt{\frac{4}{\left(1 - \frac{b^2}{a^2}\right)^2} - \varkappa^2} = \pm\frac{1}{1 - \frac{b^2}{a^2}}\sqrt{4 - \varkappa^2\left(1 - \frac{b^2}{a^2}\right)^2} , \tag{31}$$

or

$$z - c = \pm\frac{1}{a^2 q'^2}\sqrt{4 - \varkappa^2 a^4 q'^4} . \tag{32}$$

Denote this value of $z - c$ by $(z-c)_m$. Then

$$\text{Mod. } (z-c)_m = \mathfrak{z}_m < \frac{2}{a^2 q'^2} .$$

duce the two halves of the curve lying on either side of $z=0$. But, since only that portion of the curve which lies above $y=0$ is under consideration, we may plot to every $\pm z$ the corresponding value $y=\eta-q'^2$ Owing to the adopted limiting value ϑ_m of ϑ, negative values of y will not occur.

In practice it is convenient to plot the parabola on tracing-paper once for all and the asymptotic curve on a separate sheet for each individual case. It is, however, to be remembered that only the positive portion of this curve needs to be drawn. If, then, in the plot of the curve we also draw the line $z=p'$, the intersections of the parabola and the asymptotic curve may be found by superposing the parabola on the curve in such a manner that the axes $y=0$ of the two plots coincide and that the axis of the parabola coincides with the line $z=p'$. The abscissæ of the points of intersection then represent the required values of z.

As the asymptotic curve may become quite steep for large values of $\frac{h}{s}$, it is advisable to plot the abscissæ on a larger scale than the ordinates. The scale two to one has been adopted in the applications that have so far been made of this graphical method of determining the roots of the equation of the sixth degree.

The number of *mathematical* solutions of the problem will be the same as the number of intersections for which z is positive. It can be shown that the number of positive roots of the equation of the sixth degree will be either one or three.

For practical purposes it is sufficient to study the behavior of the roots by considering the intersections of the parabola and the asymptotic curve, which latter, for the sake of brevity, we shall simply call the "curve."

The number of possible intersections of the parabola and the curve depends on the number of possible contacts. If the two curves can have but one point of contact, then there can be but two intersections. The case of no contact, corresponding to a single real root, is excluded, as the equation in z is of the sixth degree and therefore the real roots must occur in pairs. For the same reason any even number of contacts is excluded. If three contacts are possible, then there may be four intersections and therefore four real roots. To determine the number of contacts we write equations (28) and (29) in the form

$$\eta = z'^2 \; ; \qquad \eta = \frac{h}{\sqrt{(z'-c')^2+s^2}} = q'^2 \, , \tag{39}$$

where

$$z' = z - p' \; ; \qquad c' = c - p' \, . \tag{40}$$

The axis of the parabola and the curve then coincides with $z'=0$ and $z'=c'$ respectively, c' being the distance between these axes. The axis of the parabola lies to left of the axis of the curve if c' is positive, and to the right if c' is negative.

Differentiating equations (39) with respect to z' we have

$$\frac{d\eta}{dz'} = 2z' \; ; \qquad \frac{d\eta}{dz'} = -\frac{h\,(z'-c')}{[(z'-c')^2+s^2]^{3/2}} \, . \tag{41}$$

Since the absolute term of this equation is negative, there always exists one positive and one negative root. The other two real roots must either be both positive or both negative. The equation may, therefore, have either one or three positive roots. For instance, if both p' and c are negative, then every coefficient in equation (47) becomes positive, excepting the absolute term. There being in that case only a single variation of sign, the number of positive roots or of solutions is only *one*. But if both p' and c are positive, then the coefficients are alternately positive and negative, excepting the coefficient of z and the absolute term. This condition of signs admits of at most five positive roots and one negative root. But as at least two roots must be imaginary and one real root must be negative, the number of positive roots can not exceed three. The same results would have been obtained if (46) had been expanded as it stands, that is, without introducing the simplification $R = 1$, but equation (47) would then have become very complicated.

Three positive roots, or what is the same, four real roots, can exist only if the curves have three points of contact, that is, if equation (43) or (44) has three real roots. Whether the number of real roots be one or three, the substitution of each real root in (41a) gives the condition which the constants must satisfy in order that the curves may be in contact for the particular point defined by such root z'. We might thus determine the limits within which the roots must lie. But in practice the number and the approximate value of the positive roots is more conveniently determined graphically from the intersections of the parabola and the curve. Equation (44) has one or three real roots according to whether

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 > \text{or} < 0 . \tag{48}$$

Hence for positive values of p more than one solution can never exist. But if p is negative, the criterion (48) must be applied to decide the number of contacts.

Let us now consider the case that

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 < 0 ,$$

so that we shall have three points of contact. The number of real roots will be four. Of these one or three may be positive. In order that there may be three positive roots z or three solutions, all three points of contact must lie on the positive side of $z = 0$. We may determine the conditions which must be satisfied so that there may be three positive roots with the aid of equation (43). In this equation $z' = z - p$, so that the roots z' are measured from the axis of the parabola. Now let $c' = c - p$ be negative. Then every coefficient in (43) will be positive and, therefore, all three roots will be negative. If, however, c' is positive, then all three roots will be positive. In the first case the axis of the parabola lies to the right of the axis of the curve and the points of contact are

contained between $z = c$ and $z = p'$. In the second case the axis of the parabola lies to the left of $z = c$ and the points of contact lie between $z = p'$ and $z = c$. Both cases apply to positive and negative values of c and p'. Let us designate for both parabola and curve that part which lies to the right and the left of their axes by the positive and the negative half, respectively. Then in the first case, when c' is negative, the points of contact lie on the positive half of the curve and on the negative half of the parabola, while in the second case they lie on the negative half of the curve and on the positive half of the parabola. The following cases may occur:

I $c' = c - p'$ is negative	II $c' = c - p'$ is positive
(a) p' negative, c negative	(a) p' negative, c negative ($p' > c$)
(b) [p' negative, c positive]	(b) p' negative, c positive
(c) p' positive, c negative	(c) [p' positive, c negative]
(d) p' positive, c positive ($p > c$)	(d) p' positive, c positive.

The cases (Ib) and (IIc) must necessarily be excluded in this grouping, as the first is provided for under IIb and the second under Ic. Cases Ia and IIa need not be considered here, it having been shown above (page 286) that only one positive root exists, if both p' and c are negative. It remains to consider the cases Ic, Ib, IIb, and IId. For Ic and Id the points of contact lie between $z = c$ and $z = p'$. The axis $z = 0$ lies to right of $z = c$ for Ic and to the left for Id. In the latter case, therefore, the three points of contact lie on the positive side of $z = 0$ and there are three positive roots. In the former case, Ic, the number of positive roots will be one or three according to whether the point of inflection on the positive half of the curve lies on the negative or the positive side of the axis $z = 0$. For this point of inflection $z = c + \frac{s}{\sqrt{2}}$. Hence, there will be one or three positive roots according to whether $c + \frac{s}{\sqrt{2}} <$ or > 0. Let $\tan\psi' = \sqrt{2}$. Then, since $c = \cos\psi$ is negative, there will be one solution if $180 - \psi' < \psi < 180°$, and three solutions if $90° < \psi < 180° - \psi'$.

Similarly, we observe that for cases IIb and IId the points of contact lie between $z = p'$ and $z = c$, while the axis $z = 0$ lies to the right of $z = p'$ for IIb and to the left for IIb. In the latter case, therefore, the three points of contact lie to the right of $z = 0$ and there are three positive roots, while in the former case, IIb, there will be one or three positive roots according to whether the axis $z = 0$ lies to the right or the left of the point of inflection of the negative branch of the curve. For this point of inflection $z = c - \frac{s}{\sqrt{2}}$. The number of positive roots will, therefore, be one or three according to whether $c - \frac{s}{\sqrt{2}} <$ or > 0. Hence, since $c = \cos\psi$ is positive, there will be one positive root, if $\psi' < \psi < 90°$ and three positive roots if $0 < \psi < \psi'$, where $\tan\psi' = \sqrt{2}$ and $\psi' = 54° 44'$.

Collecting results, we find that three positive roots exist if *first:*

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 < 0,$$

where by (45a)

$$\frac{p}{3} = \frac{1}{81}[9(2s^2 + q'^2) - 7c'^2]; \quad \frac{q}{2} = \frac{5c'}{9}\left[\frac{p}{3} + \frac{1}{9}\left(m^2 - \frac{11}{5}q'^2\right)\right]$$

$$m^2 = c'^2 + s^2; \quad c' = c - p'$$

and if *secondly:*

$$\text{either } p' > 0, \quad c > 0,$$

$$\text{or } p' > 0, \quad c < 0, \quad 90^\circ < \psi < 125^\circ 16';$$

$$\text{or } p' < 0, \quad c > 0, \quad 0^\circ < \psi < 54^\circ 44'. \qquad (49)$$

In all other cases the equation of the sixth degree in ρ_0 or z ((12) or (46)) will have only one positive root.

In applying these criteria it suffices to use approximate values of the quantities involved. If either p' or c are negative, the *first* criterion needs to be applied only if ψ lies between the limits specified for the existence of three solutions. It should also be remembered that only one solution exists, if p is positive, that is, if $9(2s^2 + q'^2) - 7c'^2$ is positive. The sign of p is readily ascertained from a mere inspection of the numerical values of s, q', and c', all of which are available. The value of q needs to be computed only if p is negative and then only with sufficient accuracy to determine the algebraical sign of $\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$. Thus, the number of positive roots is determined practically without additional computation. In most cases a mere inspection of the available auxiliary quantities will suffice.

Having thus disposed of the number of positive roots, we may proceed to a further consideration and a recapitulation of the graphical determination of the numerical values of z', from which $z = \frac{\rho}{R} = z' + p'$. By (39) the roots z' are defined as the intersections of

$$y = z'^2 \quad \text{and} \quad y = \frac{h}{[(z' - c')^2 + s^2]^{3/2}} - q'^2, \qquad (50)$$

where by (36) and (37), (35) and (40) we plot the asymptotic curve for equidistant values of ϑ from

$$y = \eta - q'^2 = \frac{h}{s^3}\cos^3\vartheta - q'^2; \qquad z' = \zeta + c'. \qquad (51)$$

Thus the asymptotic curve may be plotted directly with reference to the axes of the parabola, which are $y = 0$, $z' = 0$.

Formulæ (38) gives the maximum positive value ϑ_m of ϑ required in (51), and on page 284 it was pointed out that only that part of the curve needs to be plotted which lies between $z = 0$ and $z = c + \zeta_m$, where $\zeta_m = s \tan\vartheta_m$. Expressed in terms of z' we are, therefore, concerned only with that part of the curve which lies between $z' = -p'$ and $z' = s\tan\vartheta_m + c'$. Now, as it has been decided that ϑ should always be chosen positive, we must plot the portions of the curve to the right and the

left of $z'=0$ from $z'=\zeta+c'$ and $z'=-\zeta+c'$, respectively, together with the value of η corresponding to ζ. Hence, the largest value of ζ which must be used with the negative sign is obtained from $z'=-\zeta+c'=-p'$ or from $\zeta=c'+p'=c$. This result is also at once evident from the fact that the curve is symmetrical with reference to $\zeta=0$ and that $\zeta=-c$ for $z=0$, c being the distance from $z=0$ to $\zeta=0$.

We may also easily derive the numerical value of ϑ which corresponds to the identical conditions $z=0$ or $z'=-p$ or $\zeta=-c$. Expressing ζ in terms of ϑ we have $s\tan\vartheta=-c$ or $\tan\vartheta=-\frac{c}{s}=-\cot\psi=\tan(\psi-90°)$. Hence, if $\psi>90°$, the values of ϑ required in (51) are contained between $\vartheta=\psi-90°$ and $\vartheta=\vartheta_m$ and η and ζ need not be computed for values of ϑ outside of this range; that is, all roots correspond to values of ϑ within the range. Similarly, if $\psi<90°$, we again have the limits $\vartheta=\psi-90°$ and $\vartheta=\vartheta_m$. The negative value of the lower limit in this case signifies that negative values of ζ may be determined from $\zeta=s\tan(-\vartheta)$. In every case, then, the values of ϑ required in (51) are contained between $\vartheta=\psi-90°$ and $\vartheta=\vartheta_m$.

In determining the intersections of the parabola and the curve it is necessary to plot only a short arc of each on either side of the intersections. The scale may be chosen sufficiently large to suit any desired degree of accuracy.

For the intersections the coördinates of the curve satisfy the condition $(\zeta+c')^2=\eta-q'^2$. Let $y'=(\zeta+c')^2-(\eta-q'^2)$. Then at the intersections $y'=0$. Remembering the definition of η and ζ in terms of ϑ by (36) and (37), we may find the region of an intersection with the aid of a table of natural tangents and cosines and of a multiplication table like CRELLE'S *Rechentafeln* or mechanism like a slide rule, by noting down approximate values of $z'=\zeta+c'$ and $y=\eta+q'^2$ for a few values of ϑ between the limits $\vartheta=\psi-90°$ and $\vartheta=\vartheta_m$ and estimating the value of ϑ for which y' changes sign. Let this value of ϑ be called ϑ_0. Then compute η and ζ, etc., for a few values of ϑ, greater and less than ϑ_0, to the degree of accuracy to which it is desired to find z' at the intersection from the plot. Having determined z' at the intersection from the plot, we finally have $z=z'+p'$.

But the graphical solution may be simplified still further by plotting

$$y'=(\zeta+c')^2-(\eta-q'^2)=f(\vartheta) \qquad (52)$$

for a few values of ϑ greater and less than ϑ_0 with ϑ as abscissa and y' as ordinate. The solution is then given by the intersection of the curve $y'=f(\vartheta)$ with the ϑ axis. In case of three positive roots y' will change sign for three values of ϑ_0 between $\psi-90°$ and ϑ_m, and each of the three roots is obtained in the same manner as in the case of a single root. The intersections of the curve $y'=f(\vartheta)$ with the ϑ axis give the accurate values of ϑ_0 corresponding to the positive roots. For each intersection ϑ_0 the value of $z=\frac{\rho}{R}$ is then given by (35), or

$$z=\zeta_0+c=s\tan\vartheta_0+c\,. \qquad (53)$$

Instead of plotting the values $y' = f(\vartheta)$, the values of y' may be arranged in tabular form with ϑ as argument. From this table the value of ϑ_0 corresponding to $y' = 0$ may then be interpolated. Graphical constructions are thereby entirely avoided.

The investigation of the number of positive roots and the various methods[1] of deriving their numerical values having been disposed of, it remains to consider, (a) whether the parabolic hypothesis is feasible, and (b), if so, whether in case of three positive roots any of these roots may be discarded as fictitious. It is evident that a parabolic solution is feasible only if it is also a general solution. In the general solution the positive roots of the fundamental equation (7), page 8, are obtained by interpolation from the tables at the end of this paper with ψ and $\frac{1}{m}$ as arguments. In a parabolic solution $\frac{1}{m}$ is required only if the first approximation of the geocentric distance is obtained from the tables, but even if the geocentric distance is determined graphically, it is necessary to perform the simple computation of $\frac{1}{m}$ and to interpolate the geocentric distance from the tables for the general solution in order to test the feasibility of the parabolic solution. The maximum number of positive roots in the general solution[2] is two. If none of the positive roots for the parabolic solution agree, within the uncertainty of the solution, with a positive root for the general solution as taken from the tables, then the parabolic hypothesis must be abandoned.

(a) On pages 270–278 it was shown how the uncertainty of $\frac{1}{m}$ and, therefore, by means of the tables also of z, may be expressed numerically in any given case. If we denote by Δz the uncertainty of z, then the parabolic hypothesis is infeasible, if the value of z for the parabolic solution differs from the value of z for the general solution by more than Δz, or in case of multiple roots, if none of the values of z for the parabolic solutions agree with the values of z for the general solutions within the uncertainty Δz for each general solution. The parabolic hypothesis, therefore, may be abandoned in the course of the computation, and it is not necessary to proceed to the representation of the first and third places. But the parabolic hypothesis should not be abandoned at this stage of the computation, if the uncertainty Δz is caused chiefly by the neglected third differences of the coördinates as estimated from the first and second differences. The rejection of the parabolic hypothesis, therefore, should be based only on that part of the uncer-

[1]Since writing this paper Mr. B. A. BERNSTEIN of the University of California and I have derived another very simple graphical construction of the roots, which dispenses with the numerical calculation of the coördinates. Mr. BERNSTEIN has applied the same principle to the determination of the roots in deriving an orbit without hypothesis regarding the eccentricity, but for practical purposes the roots in the general case are more readily obtained by interpolation from the tables at the end of this paper. A. O. LEUSCHNER and B. A. BERNSTEIN, *Note on the Graphical Solutions of the Fundamental Equations in the Short Methods of Determining Orbits. Bulletin of the American Mathematical Society*, Vol. XVIII, Number 6.

[2]For one of the most complete discussions of the number of solutions in the general case compare Mrs. YOUNG (GRACE CHISHOLM), *Monthly Notices of the Royal Astronomical Society*, Vol. LVII, page 379.

tainty Δz which is due to the errors of observation and to the limitation of the number of available places expressed in seconds of arc in the first and second differences of the observed coördinates.

Whether the parabolic hypothesis be abandoned or not, any residuals arising from the neglected third and higher differences may be removed by differential correction, but the accuracy of the resulting orbit still will be determined by the accuracy of z as conditioned by the errors of observation and by the number of places available in the first and second differences of the coördinates. The value of Δz referred to here is the maximum uncertainty of z. Hence $z_1 = z + \Delta z$ and $z_2 = z - \Delta z$ may be considered the maximum and minimum values of z consistent with the observations. To each of these limits corresponds a limiting value of the elements. The difference between these elements represents *the range of numerical solutions*. It is frequently of interest to compute this range for at least the period and eccentricity and thus to obtain a fairly accurate estimate of the accuracy of the adopted orbit.

(b) In case of multiple solutions it is easily shown that only one of the three roots for the parabolic solution can agree with a root for the general solution, so that the other two parabolic roots may be discarded as fictitious. This is of importance because it will obviate the possibility of adopting one of the fictitious roots as the physical solution. As stated above, it is evident that a parabolic solution in order to be feasible must agree with a general solution. Since the general case admits of at most two solutions, the case of three parabolic solutions is at once excluded. The case of two parabolic solutions does not exist, as it was shown above that the number of positive roots for the parabolic solution is *either three or one*. Hence at most one of the parabolic roots can agree with a general root.

But if the tables indicate the existence of two general solutions, whether one of them be parabolic or not, then it is not possible to discriminate between the roots in the course of the computation, for both roots must correspond to values of r which are either both smaller or both greater than R, according to whether in the first of the equations (1) $\varkappa$ is positive or negative, since in the nature of the solution not only the fundamental equation (7), page 8, but also the equations (1) and (3), page 279, from which it is derived, are satisfied.

It may happen that a value of z cannot be found in the tables to correspond to the given arguments $\frac{1}{m}$ and ψ. The value of $\frac{1}{m}$ is then affected by a serious error, which may be due either to an unusual error of observation or computation or to the error produced by neglecting the third and higher differences. Since, however, the magnitude of the third differences can be estimated with a fair degree of accuracy (page 274), the case where a serious error in $\frac{1}{m}$ would arise from the neglect of the third differences can be avoided. It is, therefore, in general always

possible to determine the cause of the inaccuracy of $\frac{1}{m}$ and to discard the observations if necessary. If care be taken in the application of these methods, the neglected third differences will never lead to a value of $\frac{1}{m}$ that can not be found in the tables together with the corresponding argument ψ. Any discrepancy that may exist will be slight, and then the value of z may be taken from the tables for the tabular value of $\frac{1}{m}$ corresponding most nearly to the computed value of $\frac{1}{m}$.

If the parallax has been neglected entirely, then it contributes to the error of observation. In the rare cases in which the possibility of the solution of the orbit depends upon the elimination of the parallax, the values of m and (m) in equation (27), page 240, may differ so considerably that they may not be considered equal for the purpose of interpolating a first approximation of z from the tables. But in such cases the tables no longer apply, because if m and (m) differ considerably, the equation referred to no longer has one root equal to zero and, therefore, does not reduce to the equation of the seventh degree which the tables are intended to solve.[1]

After an initial value of z or ρ, where $z = \frac{\rho}{R}$, has been found in accordance with the foregoing directions, it remains to determine their final value z_f or ρ_f. Let the initial value of z or ρ be denoted by z_1 or ρ_1; then the final value ρ_f or z_f is obtained by successive application of either formulæ (15), (16), (19), and (20) or of formulæ (24), (25), (27a) and (27b).

The elimination of the parallax and aberration is accomplished in accordance with the directions given on pages 233 to 246 in the general solution of an orbit. But it should be observed that m and (m) do not enter into the solution of a parabolic orbit and that m needs to be computed only if a first approximation of $z = \frac{\rho}{R}$ is to be taken from the tables at the end of this paper. But the parallax enters into the ratio $\frac{\sigma'}{\sigma}$ which is required in (6) and which by (7) may be replaced by $\frac{\lambda}{\varkappa}$ in the auxiliary quantities (8), if the parallax be neglected. To allow for the parallax in $\frac{\sigma'}{\sigma}$, we make use of equations (37), page 243, in place of equation (9).

By (8) and if we let

$$\left.\begin{aligned} [X]' &= X' + (\varDelta X)' + \beta\cos\alpha \\ [Y]' &= Y' + (\varDelta Y)' + \beta\sin\alpha \\ [Z]' &= Z' + (\varDelta Z)' + \beta\tan\delta, \end{aligned}\right\} \qquad (54)$$

equations (37), page 243, may be written

$$\xi' = \alpha_x\sigma_0 - [X]', \qquad \eta' = \alpha_y\sigma_0 - [Y]'; \qquad \zeta' = \alpha_z\sigma_0 - [Z]'. \qquad (55)$$

These equations then take the place of equations (9) if it be intended to perform a complete elimination of the parallax, so that the only change is in the numerical

[1] The solution, however, is easily accomplished by the graphical method, referred to in footnote 2, page 290.

values of the solar coördinates, while the nature of the solution remains the same as if the parallax were neglected.

The final value ρ_f or $z_f = \frac{\rho_f}{R_0}$ obtained in the preceding solution is the value of ρ_0 or z_0, that is, the value of z or ρ at the middle or normal date, and

$$\sigma_0 = \rho_0 \cos \delta_{\prime\prime} = \frac{z_0}{R_0} \cos \delta_{\prime\prime} \,. \tag{56}$$

The derivation of the heliocentric velocities x'_0, y'_0, z'_0 and the representation of the first and third places are accomplished practically in the same manner as in the general case, Part I. The necessary formulæ are given under V and VI, page 16. It should be observed, however, that some minor modifications in the use of these formulæ have been proposed in the preceding pages. These modifications are introduced in the Synopsis of Formulæ preceding the tables.

As a check on the parabolic solution we have

$$\frac{1}{a} = \frac{2}{r_0} - x_0'^2 + y_0'^2 + z_0'^2 \,, \tag{57}$$

where $\frac{1}{a}$ should be equal to zero within the uncertainty of the numerical operations.

If the residuals of the first and third places are found to be satisfactory, the elements, constants for the equator, and an ephemeris may be computed by VIII, page 18, and in accordance with the additional directions given on pages 251 to 254. If the residuals are not sufficiently small for the purpose in hand, then it will be necessary to determine such corrections to the fundamental data σ_0, x'_0, y'_0, z'_0, as will remove the residuals. Any small difference $\frac{1}{a}$, resulting from (1), may be removed at the same time and the observations may thus be forced to conform as closely as possible to a strictly parabolic orbit. Differential formulæ for the removal of the residuals and of $\frac{1}{a}$ are developed in the next paragraph. If $\frac{1}{a}$ should exceed the uncertainty of the computation, that is, if it should be larger than a few units of the last decimal employed in the calculation, it follows that a mistake has been made in the previous computation and the same must be revised.

The differential formulæ developed in the next paragraph will always produce a parabola, but the residuals can be reduced to zero only if the orbit is accurately parabolic.

THE DIFFERENTIAL CORRECTION OF PARABOLIC ORBITS.

The first approximation of the orbit of a comet may be made either with or without assumption regarding the eccentricity. If it does not yield a satisfactory representation of the observations, and if it is intended to base the second approximation on the assumption that the orbit is parabolic, then, as stated on page 279, the corrected fundamental data ρ, x', y', and z' must satisfy the equation

$$x'^2 + y'^2 + z'^2 = \frac{2}{r}. \tag{1}$$

Let $\partial\rho$, ∂r, $\partial x'$, $\partial y'$, and $\partial z'$ be the required corrections to the initial quantities ρ_0, r_0, x_0', y_0', and z_0', so that $\rho = \rho_0 + \partial\rho$, $r = r_0 + \partial r$, $x' = x_0' + \partial x'$, etc. Then the corrections $\partial\rho$, $\partial x'$, $\partial y'$, and $\partial z'$ must be determined so as to remove the residuals of the first approximation and satisfy equation (1). Equation (1) may be written rigidly

$$(x_0' + \partial x')^2 + (y_0' + \partial y')^2 + (z_0' + \partial z')^2 = \frac{2}{r_0 + \partial r_0} = \frac{2}{r_0} - \frac{2\,\partial r}{r_0^2} + \frac{2\,(\partial r)^2}{r_0^3} - \frac{2\,(\partial r)^3}{r_0^4} + \cdots,$$

or, since $x_0'^2 + y_0'^2 + z_0'^2 = \frac{2}{r_0} - \frac{1}{a_0}$,

$$2\,x_0'\,\partial x' + 2\,y_0'\,\partial y' + 2\,z_0'\,\partial z' + (\partial x')^2 + (\partial y')^2 + (\partial z')^2 = \frac{1}{a_0} - \frac{2\,\partial r}{r_0^2} + \frac{2\,(\partial r)^2}{r_0^3} + \frac{2\,(\partial r)^3}{r_0^4} + \cdots \tag{2}$$

If the first approximation was made on the assumption that the orbit is parabolic, then $\frac{1}{a_0}$ will be small and of the order of the uncertainty of the numerical operations. The equations which determine the corrections $\partial\rho$, $\partial x'$, $\partial y'$, and $\partial z'$ so as to remove the residuals are given in (2) and (4), page 255, and are

$$\partial x' = P_x - Q_x\,\partial\rho\ ; \qquad \partial y' = P_y - Q_y\,\partial\rho\ ; \qquad \partial z' = P_{z_,} - Q_{z_,}\,\partial\rho = P_{z_{,,,}} - Q_{z_{,,,}}\,\partial\rho\,. \tag{3}$$

Since ∂r may be expressed rigidly in terms of $\partial\rho$ by formulæ (11) or (12a), pages 258–259, equations (2) and (3) furnish five conditions for the determination of the four unknowns $\partial\rho$, $\partial x'$, $\partial y'$, and $\partial z'$. To reduce the conditions to the necessary and sufficient number, any one of conditions (3) may be discarded. But the greatest symmetry of the resulting formulæ will be attained by discarding one of the expressions for $\partial z'$ or by combining the two expressions for $\partial z'$ into one, by forming their mean either with or without the assignment of weights. If the observations can be represented by a parabola, then it is immaterial which of conditions (3) be discarded, but if the orbit is not parabolic, then residuals will appear in the α or δ of the first and third places according to the plan adopted in reducing the number of conditions from five to four. Thus, if either of the expressions for $\partial z'$ be discarded, then a residual will appear in the corresponding declination, since $P_{z_,}$ depends on $\partial\delta_,$ and $P_{z_{,,,}}$ on $\partial\delta_{,,,}$. By combining the two expressions for $\partial z'$

preliminary orbit calculations. In the solution of equation (6), an approximate value of $\partial\rho$ is first obtained from $Q\,\partial\rho = P$. With this the remaining terms on the right-hand side of (6) are computed with the aid of (3). If the sum of these terms be denoted by ΔP, then a more accurate value of $\partial\rho$ results from $Q\,\partial\rho = P + \Delta P$. With the new value of $\partial\rho$, the corrections $\partial x'$, $\partial y'$, and $\partial z'$ are recomputed by (3), etc., and this process is continued until the corrections remain constant.

At this point a test may be applied to ascertain whether a parabolic solution is feasible on the basis of the given geocentric places. For this purpose we substitute the final corrections to the fundamental data into both of the equations (25), page 12, and compute $\partial\delta_{,}$ and $\partial\delta_{,,,}$. If the computed values of these residuals agree within the probable errors of observation with the original values which are to be removed by the process of differential correction, then a parabolic orbit is feasible and the solution of the differential formulæ is checked at the same time. It will be observed that whatever discrepancy may exist will reveal itself in the residual corresponding to the equation which has been discarded in the parabolic solution, or in accordance with the weights, if both expressions for $\partial z'$ in (4), page 255, have been retained. If the corrections defined in (16), page 259, were taken into account in the process of differential correction, then the test is to be applied to the residuals $\partial\delta_{,}$ and $\partial\delta_{,,,}$ as modified by these corrections.

If the test, as applied by equations (25), page 12, indicates that a parabolic solution is infeasible, then we may pass at once from the parabola to a general orbit by solving equation (5), page 255, for $\partial\rho_0$ and then equations (2) and (4), page 255, for $\partial x'_0$, $\partial y'_0$, $\partial z'_0$.

After final corrections to ρ_0, x'_0, y'_0, and z'_0 have been adopted, the corrections to x_0, y_0, and z_0 are obtained in the same manner as in the general case, viz.:

$$\partial x = \frac{\xi_0}{\rho_0}\partial\rho\,; \qquad \partial y = \frac{\eta_0}{\rho_0}\partial\rho\,; \qquad \partial z = \frac{\zeta_0}{\rho_0}\partial\rho$$

With the corrected heliocentric coördinates and velocities, the comparison between theory and observation and the computation of the elements is then performed by formulæ VI and VIII, Part I, whereby, however, the modifications outlined in the preceding pages should be observed.

The auxiliary quantities P and Q arise in the solution of equations (24) and (25), page 12, and are defined in (1) and (3), page 255. On pages 256 and following it has been shown that in extreme cases, namely, for long intervals or small heliocentric distances, or both, these equations may not be sufficiently rigid to give the final results. The accuracy of the differential solution may then be increased and a repetition of the differential correction avoided, the same as in the general case, by subtracting expressions (16), page 259, from the residuals, as soon as approximate values of $\partial\rho$, $\partial r'$, and Γ become known through the solution of equations (6),

pendent relation. But as this equation when differentiated involves the additional variation ∂v, it is necessary to provide one more equation by means of which ∂v may be eliminated. Before deriving this equation we shall differentiate (10) and (11) above and (11), page 249. Then we shall obtain

from (10):

$$\partial r = \partial r_0 + 2\left[\gamma + \frac{r_0 r_0'}{\sqrt{2}}\right]\partial\gamma + \sqrt{2}\,\gamma\,\partial(r_0 r_0'),$$

or by (9a)

$$\partial r = \partial r_0 + 2\sqrt{r-q}\,\partial\gamma + \sqrt{2}\,\gamma\,\partial(r_0 r_0')\,; \tag{12}$$

from (11)

$$\partial q = \left[\sqrt{2}\,r_0 + 2(r_0 r_0')\gamma\right]\partial\gamma + \sqrt{2}\,\gamma\,\partial r_0 + \gamma^2\,\partial(r_0 r_0')\,; \tag{13}$$

from (11), page 249, by logarithmic differentiation after squaring both sides:

$$\frac{2\,\partial\gamma}{\gamma} = \frac{\partial r_0}{r_0} - \frac{\partial f}{1-f} = \frac{\partial r_0}{r_0} - \frac{r_0}{\gamma^2}\,\partial f\,,$$

or

$$2\,\partial\gamma = \frac{\gamma}{r_0}\,\partial r_0 - \frac{r_0}{\gamma}\,\partial f\,. \tag{14}$$

After $\partial\gamma$ has been expressed in terms of ∂r_0 and $\partial(r_0 r_0')$, equations (13) and (14) give ∂q and ∂f in terms of the same variations. $\partial\gamma$ is given by (12) after ∂r has been eliminated. To derive the additional equation needed for the elimination of ∂r we differentiate the well-known equation

$$\frac{\theta}{q^{3/2}} = \sqrt{2}\,\{(\tan\tfrac12 v - \tan\tfrac12 v_0) + \tfrac13(\tan^3\tfrac12 v - \tan^3\tfrac12 v_0)\}\,, \tag{15}$$

which gives

$$-3\,\frac{\theta}{q^{5/2}}\,\partial q = \sqrt{2}\,\{\sec^4\tfrac12 v\,\partial v - \sec^4\tfrac12 v_0\,\partial v_0\}\,, \tag{16}$$

where ∂v and ∂v_0 may be expressed in terms of ∂r and ∂r_0 after differentiation of the equation $r = q\sec^2\frac12 v$, etc. This gives

$$\partial r = \sec^2\tfrac12 v\,\partial q + q\sec^2\tfrac12 v\tan\tfrac12 v\,\partial v\,,$$

or, after both sides have been multiplied by $\frac{1}{q}\sec^2\frac12 v\cot\frac12 v$,

$$\sec^4\tfrac12 v\,\partial v = \frac{\sec^2\tfrac12 v\cot\tfrac12 v}{q}\,\partial r - \frac{\sec^4\tfrac12 v\cot\tfrac12 v}{q}\,\partial q\,,$$

or, since

$$\sec^2\tfrac12 v = \frac{r}{q} \quad\text{and}\quad \cot\tfrac12 v = \frac{\sqrt{q}}{\sqrt{r-q}}\,,$$

$$\sec^4\tfrac12 v\,\partial v = \frac{r}{q^{3/2}\sqrt{r-q}}\,\partial r - \frac{r^2}{q^{5/2}\sqrt{r-q}}\,\partial q\,, \tag{17}$$

and an analogous expression for $\sec^4\frac12 v_0\,\partial v_0$. Equation (16) then reduces to

$$\frac{r\,\partial r}{\sqrt{r-q}} = \frac{r_0\,\partial r_0}{\sqrt{r_0-q}} + \left[\frac{r^2}{\sqrt{r-q}} - \frac{r_0^2}{\sqrt{r_0-q}} - \frac{3}{\sqrt{2}}\,\theta\right]\frac{\partial q}{q}\,, \tag{18}$$

which is the equation required for the elimination of ∂r from (12). Equation (18) contains ∂q, but the differention of the square of (9) gives, by (9) and by (20), page 249,

$$\partial r_0 - \partial q = (r_0 r_0')\,\partial(r_0 r_0') = \sqrt{2}\sqrt{r_0-q}\,\partial(r_0 r_0') = \sqrt{2q}\tan\tfrac12 v_0\,\partial(r_0 r_0')\,, \tag{19}$$

by means of which ∂q may be expressed in terms of ∂r_0 and $\partial(r_0 r_0')$. We shall, however, for the present retain ∂q in (18) and introduce ∂q for $\partial(r_0 r_0')$ in (12) by (19). After γ has been replaced by (21), page 249, and both sides of (12) have been divided by $\sqrt{r-q}$, (12) may be written

$$\frac{\partial r}{\sqrt{r-q}} = \frac{\partial r_0}{\sqrt{r-q}} + 2\,\partial\gamma + \frac{\sqrt{r-q}-\sqrt{r_0-q}}{\sqrt{r-q}}\left[\frac{\partial r_0}{\sqrt{r_0-q}} - \frac{\partial q}{\sqrt{r_0-q}}\right],$$

or

$$\frac{\partial r}{\sqrt{r-q}} = 2\,\partial\gamma + \frac{\partial r_0}{\sqrt{r_0-q}} - \left[\frac{1}{\sqrt{r_0-q}} - \frac{1}{\sqrt{r-q}}\right]\partial q. \tag{20}$$

Equating now (18) and (20), after both sides of the former equation have been divided by r, we obtain for the required equation in $\partial\gamma$

$$2\,\partial\gamma = \frac{r_0 - r}{r\sqrt{r_0-q}}\,\partial r_0 + \left[\frac{\sqrt{r-q}}{q} + \frac{rq - r_0^2}{rq\sqrt{r_0-q}} - \frac{3\theta}{\sqrt{2}\,rq}\right]\partial q. \tag{21}$$

In order to obtain the desired expressions for ∂f and ∂g it remains to eliminate $\partial\gamma$ from (14) and (13) by (21) and to replace ∂q by $\partial(r_0 r_0')$ by (19). Thus equating (14) and (21), after γ has been replaced in the former by (21), page 249, we have

$$\begin{aligned}\frac{r_0}{\sqrt{r-q}-\sqrt{r_0-q}}\,\partial f = &\left[\frac{\sqrt{r-q}-\sqrt{r_0-q}}{r_0} - \frac{r_0 - r}{r\sqrt{r_0-q}}\right]\partial r_0 \\ &-\left[\sqrt{r-q} + \frac{rq - r_0^2}{r\sqrt{r_0-q}} - \frac{3\theta}{\sqrt{2}\,r}\right]\frac{\partial q}{q}.\end{aligned} \tag{22}$$

If in this equation we eliminate θ by (15) and ∂q by (19) and introduce true anomalies and q throughout by (20), page 249, and by $r = q\sec^2\frac{1}{2}v$, etc., we obtain without further reduction

$$\begin{aligned}\frac{q\sec^2\frac{1}{2}v_0}{\sqrt{q}\tan\frac{1}{2}v - \sqrt{q}\tan\frac{1}{2}v_0}\,\partial f = &\left[\frac{\sqrt{q}\tan\frac{1}{2}v - \sqrt{q}\tan\frac{1}{2}v_0}{q\sec^2\frac{1}{2}v_0} - \frac{q\sec^2\frac{1}{2}v_0 - q\sec^2\frac{1}{2}v}{q\sec^2\frac{1}{2}v\,\sqrt{q}\tan\frac{1}{2}v_0} - \frac{\sqrt{q}\tan\frac{1}{2}v}{q}\right.\\ &\left. - \frac{q^2\sec^2\frac{1}{2}v - q^2\sec^4\frac{1}{2}v_0}{q^2\sec^2\frac{1}{2}v\sqrt{q}\tan\frac{1}{2}v_0} + 3\frac{\sqrt{q}\,(\tan\frac{1}{2}v - \tan\frac{1}{2}v_0)}{q\sec^2\frac{1}{2}v} + \frac{\sqrt{q}\,(\tan^3\frac{1}{2}v - \tan^3\frac{1}{2}v_0)}{q\sec^2\frac{1}{2}v}\right]\partial r_0\\ &+\frac{\sqrt{2}}{\sqrt{q}}\tan\tfrac{1}{2}v_0\left[\sqrt{q}\tan\tfrac{1}{2}v + \frac{q^2\sec^2\frac{1}{2}v - q^2\sec^4\frac{1}{2}v_0}{q\sec^2\frac{1}{2}v\,\sqrt{q}\tan\frac{1}{2}v_0} - q^{3/2}\left\{\frac{3\,(\tan\frac{1}{2}v - \tan\frac{1}{2}v_0)}{q\sec^2\frac{1}{2}v}\right.\right.\\ &\left.\left.+\frac{\tan^3\frac{1}{2}v - \tan^3\frac{1}{2}v_0}{q\sec^2\frac{1}{2}v}\right\}\right]\partial(r_0 r_0').\end{aligned} \tag{23}$$

After multiplying both sides of this equation by $\sqrt{q}\sec^2\frac{1}{2}v$, we note that q disappears from the coefficient of ∂r_0, while the coefficient of $\partial(r_0 r_0')$ retains the factor $\sqrt{q}$. Of the six terms of the coefficient of ∂r_0 the sum of the second and fourth becomes

$$\frac{\sec^2\frac{1}{2}v_0\,(\sec^2\frac{1}{2}v_0 - 1)}{\tan\frac{1}{2}v_0} = \sec^2\tfrac{1}{2}v_0\tan\tfrac{1}{2}v_0\,.$$

Then the sum of the second and fourth and the third terms reduces to

$$\sec^2\tfrac{1}{2}v_0\tan\tfrac{1}{2}v_0 - \sec^2\tfrac{1}{2}v\tan\tfrac{1}{2}v = -(\tan\tfrac{1}{2}v - \tan\tfrac{1}{2}v_0) - (\tan^3\tfrac{1}{2}v - \tan^3\tfrac{1}{2}v_0)\,.$$

The latter of these two terms cancels against the last term in the coefficient of ∂r_0, while the former may be combined with the fifth term. The coefficient of ∂r_0 then becomes

$$(\tan\tfrac{1}{2}v - \tan\tfrac{1}{2}v_0)\left[\frac{\sec^2\frac{1}{2}v}{\sec^2\frac{1}{2}v_0} + 2\right]. \tag{24}$$

The coefficient of $\sqrt{2q}\,\partial(r_0 r_0')$ in (23) is

$$\begin{aligned}\tan\tfrac{1}{2}v_0\tan\tfrac{1}{2}v\sec^2\tfrac{1}{2}v + \sec^2\tfrac{1}{2}v - \sec^4\tfrac{1}{2}v_0 - 3\tan\tfrac{1}{2}v_0(\tan\tfrac{1}{2}v - \tan\tfrac{1}{2}v_0)\\ - \tan\tfrac{1}{2}v_0(\tan^2\tfrac{1}{2}v - \tan^2\tfrac{1}{2}v_0).\end{aligned} \tag{25}$$

The sum of the first three terms of this expression may be written

$$\tan\tfrac{1}{2}v\tan\tfrac{1}{2}v_0 + \tan^3\tfrac{1}{2}v\tan\tfrac{1}{2}v_0 + 1 + \tan^2\tfrac{1}{2}v - 1 - 2\tan^2\tfrac{1}{2}v_0 - \tan^4\tfrac{1}{2}v_0. \tag{26}$$

Of the five trigonometrical terms of this expression the sum of the second and fifth is

$$\tan\tfrac{1}{2}v_0(\tan^3\tfrac{1}{2}v - \tan^3\tfrac{1}{2}v_0),$$

and this cancels against the last term of (25). The sum of the first term and one-half the fourth of (26) gives

$$\tan\tfrac{1}{2}v_0(\tan\tfrac{1}{2}v - \tan\tfrac{1}{2}v_0), \tag{27}$$

while the remainder of (26) becomes

$$(\tan\tfrac{1}{2}v + \tan\tfrac{1}{2}v_0)(\tan\tfrac{1}{2}v - \tan\tfrac{1}{2}v_0). \tag{28}$$

Finally the sum of (27), (28), and the fourth term of (25) gives for the coefficient of $\sqrt{2q}\,\partial(r_0 r_0')$:

$$(\tan\tfrac{1}{2}v - \tan\tfrac{1}{2}v_0)^2. \tag{29}$$

After introducing (24) and (29) in (23) we obtain, therefore,

$$\frac{q\sec^2\frac{1}{2}v_0\sec^2\frac{1}{2}v}{\tan\frac{1}{2}v - \tan\frac{1}{2}v_0}\partial f = (\tan\tfrac{1}{2}v - \tan\tfrac{1}{2}v_0)\left[\frac{\sec^2\frac{1}{2}v}{\sec^2\frac{1}{2}v_0} + 2\right]\partial r_0 + \sqrt{2q}\,(\tan\tfrac{1}{2}v - \tan\tfrac{1}{2}v_0)^2\,\partial(r_0 r_0'). \tag{30}$$

or by (19), page 249, and if we introduce $r = q\sec^2\frac{1}{2}v$, etc.,

$$\partial f = \frac{\gamma^2}{r_0 r}\left[\frac{r}{r_0} + 2\right]\partial r_0 + \frac{\sqrt{2}\,\gamma^3}{r_0 r}\,\partial(r_0 r_0'). \tag{31}$$

The substitution of this value of ∂f in (14) gives

$$\partial\gamma = -\frac{\gamma}{r}\partial r_0 - \frac{\gamma^2}{\sqrt{2}\,r}\,\partial(r_0 r_0'). \tag{32}$$

Eliminating $\partial\gamma$ from (13) by means of (32) we have

$$\partial g = \frac{\sqrt{2}\,\gamma}{r}\left[r - r_0 - \sqrt{2}(r_0 r_0')\gamma\right]\partial r_0 + \frac{\gamma^2}{r}\left[r - r_0 - \sqrt{2}(r_0 r_0')\gamma\right]\partial(r_0 r_0'), \tag{33}$$

and by (10), and collecting results from (31) and (33), we finally obtain in accordance with (20), page 262,

$$\partial f = \frac{\cos\beta\,\gamma^2}{r_0 r}\left[\frac{r}{r_0} + 2\right]\partial\rho_0 + \frac{\sqrt{2}\,\gamma^3}{r_0 r}\,\partial(r_0 r_0') \tag{34}$$

$$\partial g = \frac{\cos\beta\sqrt{2}\,\gamma^3}{r}\partial\rho_0 + \frac{\gamma^4}{r}\partial(r_0 r_0'), \tag{35}$$

where $\cos\beta\,\partial\rho_0 = \partial r_0$.

Introducing these values of ∂f and ∂g into the first of equations (16), page 11,

where $\omega = x, y, z$, and noting that by (17), page 11, $\partial\omega_0 = \frac{w_0}{\rho_0}\,\partial\rho_0$, where $w_0 = \xi_0, \eta_0, \zeta_0$, we obtain

$$\partial\omega = \left\{ f\frac{w_0}{\rho_0} + \omega_0 \frac{\cos\beta\,\gamma^2}{r_0 r}\left[\frac{r}{r_0} + 2\right] + \omega_0' \frac{\cos\beta}{r}\sqrt{2}\,\gamma^3 \right\}\partial\rho_0 + g\,\partial\omega_0'$$
$$+ \left\{ \frac{\sqrt{2}\,\omega_0\gamma^3}{r_0 r} + \frac{\omega_0'\gamma^4}{r} \right\}\partial(r_0 r_0'),$$

or

$$\left.\begin{aligned}\partial\omega = \left\{ f\frac{w_0}{\rho_0} + \frac{\cos\beta}{r_0 r}\gamma^2\left[\omega_0\left(\frac{r}{r_0}+2\right) + \sqrt{2}\,\omega_0' r_0\gamma\right]\right\}\partial\rho_0 + g\,\partial\omega_0' \\ + \frac{\gamma^3}{r_0 r}\left\{\omega_0' r_0\gamma + \sqrt{2}\,\omega_0\right\}\partial(r_0 r_0').\end{aligned}\right\} \tag{36}$$

But, since

$$r_0 r_0' = x_0 x_0' + y_0 y_0' + z_0 z_0' = \Sigma\,\omega_0\,\omega_0',$$

therefore also

$$\partial(r_0 r_0') = \Sigma\,\omega_0\,\partial\omega_0' + \Sigma\,\omega_0'\,\partial\omega_0$$

or, if $\partial\omega_0$ be expressed as above in terms of $\partial\rho_0$

$$\partial(r_0 r_0') = \Sigma\,\omega_0\,\partial\omega_0' + (\Sigma\,\omega_0'\,w_0)\frac{\partial\rho_0}{\rho_0}. \tag{37}$$

Let

$$\varphi_0 = \frac{\Sigma\,\omega_0'\,w_0}{\rho_0} = \frac{x_0'\xi_0 + y_0'\eta_0 + z_0'\zeta_0}{\rho_0}, \tag{38}$$

and eliminate $\partial(r_0 r_0')$ from (36) by means of (37). Then

$$\partial\omega = \left\{ f\frac{w_0}{\rho_0} + \frac{\cos\beta}{r_0 r}\gamma^2\left[\omega_0\left(\frac{r}{r_0}+2\right) + \sqrt{2}\,\omega_0' r_0\gamma\right] + \frac{\varphi_0\gamma^3}{r_0 r}\left[\omega_0' r_0\gamma + \sqrt{2}\,\omega_0\right]\right\}\partial\rho_0$$
$$+ g\,\partial\omega_0' + \frac{\gamma^3}{r_0 r}\left[\omega_0' r_0\gamma + \sqrt{2}\,\omega_0\right]\Sigma\,\omega_0\,\partial\omega_0',$$

or if

$$g_\omega = \frac{\gamma^3}{r_0 r}\left[\omega_0' r_0\gamma + \sqrt{2}\,\omega_0\right] \tag{39}$$

and

$$f_\omega = f\frac{w_0}{\rho_0} + \frac{\cos\beta}{r_0 r}\gamma^2\left[\omega_0\left(\frac{r}{r_0}+2\right) + \sqrt{2}\,\omega_0' r_0\gamma\right] + \varphi_0 g_\omega, \tag{40}$$

then

$$\partial\omega = f_\omega\,\partial\rho_0 + g\,\partial\omega_0' + g_\omega\,\Sigma\,\omega_0\,\partial\omega_0'. \tag{41}$$

The expression for f_ω may be simplified by means of (39). For

$$f_\omega = f\frac{w_0}{\rho_0} + \frac{\cos\beta}{r_0 r}\gamma^2\left[\frac{\omega_0 r}{r_0} + \sqrt{2}\left(\omega_0' r_0\gamma + \sqrt{2}\,\omega_0\right)\right] + \varphi_0 g_\omega,$$

or

$$f_\omega = f\frac{w_0}{\rho_0} + \frac{\cos\beta}{r_0^2}\omega_0\gamma^2 + g_\omega\left[\varphi_0 + \sqrt{2}\frac{\cos\beta}{\gamma}\right]. \tag{42}$$

Equation (41) gives ∂x, ∂y, or ∂z in a closed form for any date, the auxiliary quantities $g_x, g_y, g_z, f_x, f_y, f_z$ being defined by (39) and (42). To obtain the values of ∂x, ∂y, or ∂z for a particular date we must introduce in (39), (41), and (42)

the values of f, γ, and r for that date. Thus we have, for instance, for the first date

$$g_{x_{,}} = \frac{\gamma_{,}^{2}}{r_0 r_{,}}\left[x_0' r_0 \gamma_{,} + \sqrt{2}\, x_0\right] ; \; f_{x_{,}} = f_{,} \frac{\xi_0}{\rho_0} + \frac{\cos\beta}{r_0^2} x_0 \gamma_{,}^2 + g_{x_{,}}\left[\varphi_0 + \frac{\sqrt{2}\cos\beta}{\gamma_{,}}\right] \tag{43}$$

$$\partial x_{,} = f_{x_{,}} \partial\rho_0 + [g_{,} + g_{x_{,}} x_0]\, \partial x_0' + g_{x_{,}} y_0\, \partial y_0' + g_{x_{,}} z_0\, \partial z_0'. \tag{44}$$

and analogous expressions for

$$g_{y_{,}},\; g_{x_{,}},\; f_{y_{,}},\; f_{z_{,}},\; g_{x_{,,,}},\; g_{y_{,,,}},\; g_{z_{,,,}},\; f_{x_{,,,}},\; f_{y_{,,,}},\; f_{z_{,,,}}, \text{ and } \partial y_{,},\; \partial z_{,},\; \partial x_{,,,},\; \partial y_{,,,},\; \partial z_{,,,}.$$

If now equations (15), page 11, be applied to the first and third places, we obtain by (39), (41), and (42), for the first place

$$\left.\begin{aligned}
\partial_{,}\alpha_{,} = {} & A_{f_{,}} \partial\rho_0 - \sin\alpha_{,} \left[\frac{g_{,}}{\rho_{,}} + x_0 \frac{g_{x_{,}}}{\rho_{,}}\right] \partial x_0' - \sin\alpha_{,}\, y_0 \frac{g_{x_{,}}}{\rho_{,}}\, \partial y_0' - \sin\alpha_{,}\, z_0 \frac{g_{x_{,}}}{\rho_{,}}\, \partial z_0' \\
& + \cos\alpha_{,}\, x_0 \frac{g_{y_{,}}}{\rho_{,}} \partial x_0' + \cos\alpha_{,} \left[\frac{g_{,}}{\rho_{,}} + y_0 \frac{g_{y_{,}}}{\rho_{,}}\right] \partial y_0' + \cos\alpha_{,}\, z_0 \frac{g_{y_{,}}}{\rho_{,}}\, \partial z_0' \\
\partial\delta_{,} = {} & B_{,}\, \partial\rho_0 - \sin\delta_{,} \cos\alpha_{,} \left[\frac{g_{,}}{\rho_{,}} + x_0 \frac{g_{x_{,}}}{\rho_{,}}\right] \partial x_0' - \sin\delta_{,} \cos\alpha_{,}\, y_0 \frac{g_{x_{,}}}{\rho_{,}}\, \partial y_0' - \sin\delta_{,} \cos\alpha_{,}\, z_0 \frac{g_{x_{,}}}{\rho_{,}}\, \partial z_0' \\
& - \sin\delta_{,} \sin\alpha_{,}\, x_0 \frac{g_{y_{,}}}{\rho_{,}} \partial x_0' - \sin\delta_{,} \sin\alpha_{,} \left[\frac{g_{,}}{\rho_{,}} + y_0 \frac{g_{y_{,}}}{\rho_{,}}\right] \partial y_0' - \sin\delta_{,} \sin\alpha_{,}\, z_0 \frac{g_{y_{,}}}{\rho_{,}} \partial z_0' \\
& + \cos\delta_{,}\, x_0 \frac{g_{z_{,}}}{\rho_{,}} \partial x_0' + \cos\delta_{,}\, y_0 \frac{g_{z_{,}}}{\rho_{,}}\, \partial y_0' + \cos\delta_{,} \left[\frac{g_{,}}{\rho_{,}} + z_0 \frac{g_{z_{,}}}{\rho_{,}}\right] \partial z'.
\end{aligned}\right\} \tag{45}$$

Similarly, analogous expressions are obtained for $\partial_{,}\alpha_{,,,}$ and $\partial\delta_{,,,}$. The auxiliary quantities $A_{f_{,}}$, $A_{f_{,,,}}$, $B_{f_{,}}$, $B_{f_{,,,}}$ have the same definition as $A_{,}$, $A_{,,,}$, $B_{,}$, $B_{,,,}$ in formulæ (23), page 12.

Equations (45) may be simplified by the introduction of further auxiliary quantities. Let $A_{g_{,}}$, $A_{g_{,,,}}$, $B_{g_{,}}$, $B_{g_{,,,}}$ be defined as functions of $g_{\omega_{,}}$, $g_{\omega_{,,,}}$ in the same manner as $A_{,}$, $A_{,,,}$, $B_{,}$, $B_{,,,}$ or here $A_{f_{,}}$, $A_{f_{,,,}}$, $B_{f_{,}}$, $B_{f_{,,,}}$ are defined by (23), page 12, as functions of $f_{\omega_{,}}$ and $f_{\omega_{,,,}}$, so that

$$\left.\begin{aligned}
A_{g_{,}} &= \frac{1}{\rho_{,}} [\cos\alpha_{,}\, g_{y_{,}} - \sin\alpha_{,}\, g_{x_{,}}] \\
A_{g_{,,,}} &= \frac{1}{\rho_{,,,}} [\cos\alpha_{,,,}\, g_{y_{,,,}} - \sin\alpha_{,,,}\, g_{x_{,,,}}] \\
B_{g_{,}} &= -\frac{1}{\rho_{,}} [\sin\delta_{,}\, (\sin\alpha_{,}\, g_{y_{,}} + \cos\alpha_{,}\, g_{x_{,}}) - \cos\delta_{,}\, g_{z_{,}}] \\
B_{g_{,,,}} &= -\frac{1}{\rho_{,,,}} [\sin\delta_{,,,}\, (\sin\alpha_{,,,}\, g_{y_{,,,}} + \cos\alpha_{,,,}\, g_{x_{,,,}}) - \cos\delta_{,,,}\, g_{z_{,,,}}],
\end{aligned}\right\} \tag{46}$$

and let $C_{,}$ and $C_{,,,}$ be defined as in (23), page 12, then

$$\left.\begin{aligned}
\partial_{,}\alpha_{,} &= A_{f_{,}}\, \partial\rho_0 - [\sin\alpha_{,}\, C_{,} - x_0 A_{g_{,}}]\, \partial x_0' + [\cos\alpha_{,}\, C_{,} + y_0 A_{g_{,}}]\, \partial y_0' + z_0 A_{g_{,}}\, \partial z_0' \\
\partial\, \delta_{,} &= B_{f_{,}}\, \partial\rho_0 - [\sin\delta_{,} \cos\alpha_{,}\, C_{,} - x_0 B_{g_{,}}]\, \partial x_0' - [\sin\delta_{,} \sin\alpha_{,}\, C_{,} - y_0 B_{g_{,}}]\, \partial y_0' \\
&\qquad + [\cos\delta_{,}\, C_{,} + z_0 B_{g_{,}}]\, \partial z_0' \\
\partial_{,}\alpha_{,,,} &= A_{f_{,,,}}\, \partial\rho_0 - [\sin\alpha_{,,,}\, C_{,,,} - x_0 A_{g_{,,,}}]\, \partial x_0' + [\cos\alpha_{,,,}\, C_{,,,} + y_0 A_{g_{,,,}}]\, \partial y_0' + z_0 A_{g_{,,,}}\, \partial z_0' \\
\partial\, \delta_{,,,} &= B_{f_{,,,}}\, \partial\rho_0 - [\sin\delta_{,,,} \cos\alpha_{,,,}\, C_{,,,} - x_0 B_{g_{,,,}}]\, \partial x_0' - [\sin\delta_{,,,} \sin\alpha_{,,,}\, C_{,,,} - y_0 B_{g_{,,,}}]\, \partial y_0' \\
&\qquad + [\cos\delta_{,,,}\, C_{,,,} + z_0 B_{g_{,,,}}]\, \partial z_0'.
\end{aligned}\right\} \tag{47}$$

Equations (54) and (55) have the same form as equations (8), page 294, and together with (2), page 294, lead to the same equation (6), page 295, in the determination of a parabolic orbit. The same considerations as to the arrangement and the test of a parabolic solution apply here as in the case where the ∂f and ∂g were expressed in series (pages 294 to 296). In the foregoing arrangement, any remaining discrepancy will appear as a residual in the first declination.

It will be observed, however, that the form of ∂f and ∂g is the same for all classes of orbits only if they are expressed in series, but that the closed expressions (34) and (35) of ∂f and ∂g introduced here[1] for the parabola differ from KUEHNERT'S general expressions, (18) and (19), page 262, in that they are based on the condition that the orbit shall remain parabolic. Hence, if the application of the test referred to on page 296 reveals the infeasibility of a parabolic solution, and a general solution becomes necessary by solving equation (5), page 255, then this general solution still is in part conditioned by the parabolic relation $\partial G_0^2 = -\frac{2\,\partial r_0}{r_0^2}$ (*cf.* page 262), so that the resulting orbit will be as closely parabolic as the observations permit.

The effect of this process of passing from a parabolic to a general orbit is that the adoption of too short a period or of too small an eccentricity is practically excluded, which is of particular importance in the case of comet observations admitting of a large range of practical solutions.

[1] The closed expressions of ∂f and ∂g for the parabola had been deduced independently before it was discovered that general expressions for ∂f and ∂g had been published by KUEHNERT.

positive and those of z^3 and z remain positive, then there will be no variation of sign in (47), and consequently no solution. Let

$$\left.\begin{array}{l} k = 2p'^2 + \frac{1}{a^2}, \text{ then these conditions are: } p' > 0,\ c < 0 \text{ and} \\ c < -2p',\quad c < -\frac{\left(1+\frac{1}{a^2}\right)p'}{k},\quad c < -2p'a^2;\qquad c > -\frac{1+2k}{8p'},\quad c > -\frac{\frac{1}{a^2}+2k}{8p'}a^2, \end{array}\right\} \quad (5)$$

that is, there is no solution if c is smaller than the smallest of the expressions of the first group and larger than the largest of the expressions of the second group, which is readily ascertained by inspection of the auxiliary quantities.

Suppressing now the subscript zero for quantities referring to the epoch, we note that the condition $G^2 = \frac{1}{r}$ modifies the *general* solution only to the extent that the semi-major axis a shall be equal to the radius vector r at the middle date or epoch. Since three complete observations are used, the solution, therefore, will generally produce a *conditioned* ellipse with the planet at one of the extremities of the minor axis at the epoch. To test the possibility of a circle it is sufficient to ascertain whether r'^2, which may be computed as soon as the heliocentric coördinates and velocities have been derived, is less than the numerical uncertainty of $G^2 = \frac{1}{r} = \frac{1}{a}$. For in $p = r^2 (G^2 - r'^2)$ we may let $r' = 0$, if r'^2 does not affect the definitely known portion of G^2, so that $p = r = a$. The accuracy of r and, therefore, of G^2 is determined as on pages 270–276. But if $p = a(1-e^2) = a$ within the accuracy attainable from the observations, then a *circular* orbit will represent the observations with the same degree of accuracy as the conditioned ellipse. If r' cannot be put equal to zero, the computer should proceed to the solution of a *general* orbit.

This conditioned solution prevents the computer from adopting a large eccentricity within the range of possible solutions. The elements Ω, i and u of the circle are the same as those of the conditioned ellipse while the radius is equal to $r\,(=a)$ of the latter. The residuals are computed most conveniently by means of the constants for the equator.

Similar considerations will sometimes enable the computer to change the *general* into a *circular* solution by letting $a = r$, if their difference is less than the uncertainty of r, and then ascertaining as above whether r' can be put equal to zero. The elements i, Ω, and u of the circle are then the same as those of the general solution, while the radius of the circle is equal to the computed radius vector at the middle date of the general solution.

Attention has already been called in the Introduction (pages 223 and 224) to the relation of the method outlined above to the usual methods of deriving circular orbits, so that this comparison may here be omitted.

general orbit. In the latter case $\partial\rho_0$ is computed by (5), page 255, and then $\partial x_0'$, $\partial y_0'$, and $\partial z_0'$ by (2) and (4), page 255. The remaining computation proceeds as usual by VI and VIII, Part 1, except for such slight modifications as have been proposed above in various places. This solution is conditioned *in part* by $\partial r = \partial a$ and $\partial r' = 0$, so that it precludes the adoption of a large eccentricity, within the range of possible solutions, when a small one is possible.

If it be proposed to attempt a representation of the observations on the basis of an improved circular orbit, then the further procedure will be analogous to that adopted in the differential correction of the fundamental data in a parabolic orbit. The required formulæ are readily derived from those obtained for the parabola.

The corrected fundamental data must satisfy the conditions

$$(x_0' + \partial x_0')^2 + (y_0' + \partial y_0')^2 + (z_0' + \partial z_0')^2 = \frac{1}{r_0 + \partial r_0} \text{ and } r' = 0 .$$

Allowing for the possibility that the initial orbit may not be strictly circular, we have for the same

$$x_0'^2 + y_0'^2 + z_0'^2 = \frac{2}{r_0} - \frac{1}{a_0} = \frac{1}{r_0} + \left(\frac{1}{r_0} - \frac{1}{a_0}\right) = \frac{1}{r_0} - \varDelta .$$

Proceeding now exactly as in the case of the parabola on page 294 *et seq.*, we observe first of all that in equation (2), page 294, $\varDelta$ takes the place of $\frac{1}{a_0}$ and that the coefficient 2 of ∂r and of its second and third powers drops out. Hence, when later ∂r is replaced by $\partial\rho$ and both sides of the equation are divided by 2, this number will appear in the denominators of all the terms containing $\partial\rho$, $(\partial\rho)^2$, and $(\partial\rho)^3$. In place of equations (5) and (6), we shall therefore obtain for circular orbits

$$\left.\begin{aligned} &Q = \frac{\cos\beta}{2\,r_0^3} - [x_0'\,Q_x + y_0'\,Q_y + z_0'\,Q_z] \; ; \qquad P = \frac{\varDelta}{2} - [x_0'\,P_x + y_0'\,P_y + z_0'\,P_z] \\ &Q\,\partial\rho = P - \frac{1}{2}[(\partial x')^2 + (\partial y')^2 + (\partial z')^2] - \frac{1 - 3\cos^2\beta}{4\,r_0^3}(\partial\rho)^2 + \frac{3\cos\beta\,(1 - \frac{5}{3}\cos^2\beta)}{4\,r_0^4}(\partial\rho)^3 , \end{aligned}\right\} \quad (8)$$

in which, on account of the usually large values of r_0, the terms in $(\partial\rho)^2$ and $(\partial\rho)^3$ may practically always be omitted.

The further procedure is analogous to that in parabolic orbits.

As in the direct solution it remains to ascertain whether r' may be put equal to zero within the accuracy of the solution. After the differential correction the number of places to which G^2 is known with accuracy in general is not less than the number of places contained in the mean of the differences of α or δ, expressed in seconds of arc. The same criterion, however, together with the test whether $r = a$ within the accuracy of the solution, may be applied more conveniently to the corrected *general* orbit. In fact, the solution of equations (8) probably never will be of advantage, as the corrected orbit rarely will prove circular.

A collection of formulæ for the direct solution and for the correction of circular orbits is included in the *Synopsis of Formulæ*, given at the end of this paper.

with similar expressions for $\frac{dy}{dt}$ and $\frac{dz}{dt}$. But

$$\frac{dv}{dt}=\frac{k\sqrt{p}}{r^2}=\frac{k(1+e\cos v)^2}{p^{3/2}}\,;\quad \frac{dr}{dt}=\frac{d}{dt}\left[\frac{p}{1+e\cos v}\right]=\frac{p\,e\sin v}{(1+e\cos v)^2}\frac{dv}{dt}=\frac{k\,e\sin v}{\sqrt{p}}\,. \tag{3}$$

Hence

$$\frac{dx}{dt}=-\frac{k(1+e\cos v)}{\sqrt{p}}\sin a\cos(A'+v)+\frac{ke\sin v}{\sqrt{p}}\sin a\sin(A'+v)\,,$$

where the unit of time is a mean solar day. Since $x'=\frac{1}{k}\frac{dx}{dt}$, etc., we finally have after a simple trigonometrical reduction

$$\left.\begin{aligned} x'_0&=-\frac{\sin a}{\sqrt{p}}[\cos(A'+v_0)+e\cos A']\\ y'_0&=-\frac{\sin b}{\sqrt{p}}[\cos(B'+v_0)+e\cos B']\\ z'_0&=-\frac{\sin c}{\sqrt{p}}[\cos(C'+v_0)+e\cos C']\,.\end{aligned}\right\} \tag{4}$$

For the parabola $p=2\,q$ and $e=1$. Hence equations (4) reduce to

$$\left.\begin{aligned} x'_0&=-\sqrt{\frac{2}{q}}\sin a\cos(A'+\tfrac{1}{2}v_0)\cos\tfrac{1}{2}v_0\\ y'_0&=-\sqrt{\frac{2}{q}}\sin b\cos(B'+\tfrac{1}{2}v_0)\cos\tfrac{1}{2}v_0\\ z'_0&=-\sqrt{\frac{2}{q}}\sin c\cos(C'+\tfrac{1}{2}v_0)\cos\tfrac{1}{2}v_0\,.\end{aligned}\right\} \tag{5}$$

For the circle $\frac{dv}{dt}=\mu=\frac{k}{a^{3/2}}$, $r=a$, and $\frac{dr}{dt}=0$. Hence equations (2) become

$$x'_0=-\frac{1}{a^{1/2}}\sin a\cos(A'+v_0)\;;\quad y'_0=-\frac{1}{a^{1/2}}\sin b\cos(B'+v_0)\;;\quad z'_0=-\frac{1}{a^{1/2}}\sin c\cos(C'+v_0)\,, \tag{6}$$

where $A'+v_0=A+u_0$, etc.

Before the approximate values of the fundamental data, derived by the foregoing directions, may serve as the basis of the differential correction, they must be written out to the number of decimals to which the orbit is to be computed, if necessary by adding ciphers. Initial values of the heliocentric coördinates are then accurately computed from $x_0=\rho_0\cos\alpha_{\prime\prime}\cos\delta_{\prime\prime}-X_0$; etc., where $\alpha_{\prime\prime}$, $\delta_{\prime\prime}$ are the coördinates of the adopted zero or middle date.

From the heliocentric coördinates and velocities at the middle date, the residuals of the first and third places are next deduced with the aid of the expressions for f and g as in VI, page 16. The differential correction of the fundamental data may then be performed on the basis of these residuals in the usual manner. It is evident that closed expressions for f and g, ∂f and ∂g must be used in the computation, if the criteria given on pages 260 and 261 indicate the insufficiency of the series.

The particular formulæ to be used for the most convenient solution of the problem according to whether the initial orbit is circular, parabolic, or elliptic are outlined in the *Synopsis of Formulæ*, at the end of this paper.

The initial orbit which serves as the basis of the differential correction is an artificial orbit defined by the following six constants: The observed right ascension and declination at the middle date; the geocentric distance and the heliocentric velocities in rectangular coördinates corresponding to the preliminary orbit.

The preliminary orbit has, therefore, only the four elements ρ_0, x'_0, y'_0, z'_0 in common with the initial orbit, while the remaining two elements $\alpha_{,,}$ and $\delta_{,,}$ of the initial orbit are, in general, not satisfied by the preliminary orbit.

It will be observed that great accuracy is not required in computing ρ_0, x'_0, y'_0, z'_0 from the preliminary orbit and that these values may even receive slight arbitrary changes to suit the convenience of the computer. But after certain values have been adopted for these quantities they must be strictly retained to the required number of decimals in performing the comparison between theory and observation and the differential correction.

Concerning the elimination of the parallax and aberration the usual rules are to be observed. These corrections may therefore be applied in advance, if the artificial initial orbit may be considered a close approximation. It is evident that the initial orbit is more accurate than the preliminary orbit, for while the geocentric distance ρ_0 and the heliocentric velocities x'_0, y'_0, z'_0 are the same for both, the former also represents the observed $\alpha_{,,}$, $\delta_{,,}$, at the middle date. The geocentric distances $\rho_{,}$ and $\rho_{,,,}$ may be computed by (1) in the same manner as ρ_0.

If the initial orbit can not be considered sufficiently accurate for deriving the parallax and aberration corrections in advance, then the parallax should be eliminated from the middle place alone by (4), page 234. For the first and third places the parallax is applied in the comparison between theory and observation with the values of $\rho_{,}$ and $\rho_{,,,}$ resulting in the course of this comparison as in VI, page 16. To allow for a possible change in the parallax in the course of the differential correction, the auxiliary quantities A and B should be corrected as in (41), page 244.

If the accuracy of the initial orbit be doubtful, the aberration may be treated as in the direct solution. The aberration of the fixed stars is included in the reduction of the observed places to the beginning of the year and the solar coördinates X, Y, Z are interpolated for the observed (uncorrected) dates $t_{,}$, $t_{,,}$, $t_{,,,}$. But in order to free the intervals at least in part from the planetary aberration in computing f and g either by series or closed expressions, $\rho_{,}$, $\rho_{,,}$, $\rho_{,,,}$ are also computed by (1) since true dates must be used in deriving the f and g. The true epoch is obtained by ultimately correcting the observed date $t_{,,}$ by means of the final value of ρ_0. Theoretically, the value of ρ_0 for the uncorrected date should be included in the initial fundamental data together with the values of x'_0, y'_0, z'_0 for the true date $t_{,,}$. But this latter refinement is superfluous, inasmuch as slight arbitrary changes of the fundamental data are permissible.

SYNOPSIS OF FORMULÆ.*

A. Direct Solution of Orbits With or Without Assumption Regarding the Eccentricity.

a. For parabolic orbits use Ia, IIa, IIIa, etc.
b. For circular orbits use Ib, IIb, IIIb, etc.
c. For general orbits use Ic, IIc, IIIc, etc.

GENERAL DIRECTIONS.

Ia; Ib; Ic.

It is assumed that the orbit is to be based on three places of short intervals.

Reduce the observed α and δ to the beginning of the year by including the aberration terms. Let the mean places be

$$\alpha_{,}\,,\ \delta_{,}\,;\quad \alpha_{,,}\,,\ \delta_{,,}\,;\quad \alpha_{,,,}\,,\ \delta_{,,,}\,.$$

From one of the Astronomical Ephemerides interpolate the solar coördinates for the observed dates $t_{,}\,,\ t_{,,}=t_0\,,\ t_{,,,}\,.$

$$X_{,}\,,\ Y_{,}\,,\ Z_{,}\,;\qquad X_{,,}\,,\ Y_{,,}\,,\ Z_{,,}\,;\qquad X_{,,,}\,,\ Y_{,,,}\,,\ Z_{,,,}\,.$$

At the same time obtain $X'_0 = X'_{,,}\,,\ Y'_0 = Y'_{,,}\,,\ Z'_0 = Z'_{,,}\,,$ by either of the following formulæ of numerical differentiation:

$$kw\,\frac{df(l)}{dl} = f^{\text{I}}(a+iw) + nf^{\text{II}}(a+iw) + N_1^3(n)\,f^{\text{III}}(a+iw) + \cdots\,;$$

$$l = a + [i+n]\,w = t_0\,;\qquad N_1^3(n) = \frac{3\,n^2-1}{6}\,;\qquad N_1^4(n) = \frac{n\,(2\,n^2-1)}{12}\,;\ \text{etc.}$$

$$kw\,\frac{df(l)}{dl} = f^{\text{I}}(a+[i+\tfrac{1}{2}]\,w) + mf^{\text{II}}(a+[i+\tfrac{1}{2}]\,w) + M_1^3(m)\,f^{\text{III}}(a+[i+\tfrac{1}{2}]\,w) + \cdots\,;$$

$$l = a + [i+\tfrac{1}{2}+m]\,w = t_0\,;\qquad M_1^3(m) = \frac{3\,m^2-\frac{1}{4}}{6}\,;\qquad M_1^4(m) = \frac{m\,(2\,m^2-\frac{5}{2})}{12}\,;\ \text{etc.,}$$

according as to whether t_0 lies nearer to a tabulated argument or to the **mean** of two such arguments. m and n are numerically < 0.25. $N_1^3(n)$ or $M_1^3(m)$ may be taken from Table I or II, Oppolzer, *Bahnbestimmung*, etc., Vol. II, page 515 or 523, but, in general, the terms in which they occur may be neglected. When the solar coördinates are tabulated for every 12 hours, $w = \frac{1}{2}$, $\log \frac{1}{kw} = 2.0654486$.

*All supplementary formulæ and terms printed in small type, especially those contained in the footnotes, may generally be omitted.

It is advisable to *partially*[1] eliminate the *geocentric* (ordinary) parallax from the middle place by applying the following corrections to the solar coördinates at the epoch $t_0 = t_{,,}$:

Reduce the *geocentric* parallax factors to circular measure by multiplying the $(p_\alpha\rho)^s$ by $15 \sin 1''$ and the $(p_\delta\rho)''$ by $\sin 1''$. $\log (15 \sin 1'') = 5.8617 - 10$; $\log \sin 1'' = 4.6856 - 10$.

$$\Delta X_{,,} = \quad (p_\alpha\rho)_{,,} \sin \alpha_{,,} \cos \delta_{,,} + (p_\delta\rho)_{,,} \cos \alpha_{,,} \sin \delta_{,,}$$
$$\Delta Y_{,,} = -(p_\alpha\rho)_{,,} \cos \alpha_{,,} \cos \delta_{,,} + (p_\delta\rho)_{,,} \sin \alpha_{,,} \sin \delta_{,,}$$
$$\Delta Z_{,,} = -(p_\delta\rho)_{,,} \cos \delta_{,,}\,.$$

$$R \cos D \cos A = X_{,,}\,; \quad R \cos D \sin A = Y_{,,}\,; \quad S = R \cos D, \quad R \sin D = Z_{,,}\,.$$
$$\theta_{,} = k\,(t_{,,,} - t_{,,})\,, \quad \theta_{,,,} = k\,(t_{,,} - t_{,})\,; \qquad \log k = 8.2355814 - 10.$$

[1]But a *complete* elimination of the parallax should be performed, if the accuracy of the resulting orbit essentially depends on such elimination, (*cf.* pages 273 and 274). In brief, the parallax should be completely eliminated if the parallax factors for the three dates differ considerably, and if it be estimated that the percentage error of the observed motion due to neglecting the parallax corrections is large and in excess of the errors arising from the errors of observation and from the neglected third and higher derivatives, (*cf.* pages 270 to 278). For the *complete elimination* of the parallax compute for all three dates:

$$\Delta_1 X = -d_1 \cos \delta_1 \cos \alpha_1\,; \quad \Delta_1 Y = -d_1 \cos \delta_1 \sin \alpha_1\,; \quad \Delta_1 Z = -d_1 \sin \delta_1\,.$$

$$p_\alpha^m \rho_m = \frac{d_1}{\sin 1''} \frac{\cos \delta_1}{\cos \delta} \sin (\alpha - \alpha_1)\,; \quad p_\delta^m \rho_m = \frac{d_1}{\sin 1''} \left\{ -\sin \delta_1 \cos \delta + \cos \delta_1 \sin \delta \cos (\alpha - \alpha_1) \right\},$$

where α_1, δ_1 represent the coördinates of the Moon and where

$$\log d_1 = 5.4943 - 10\,; \quad \frac{d_1}{\sin 1''} = 6''.4372\,.$$

Further, for all three dates:

$$p_\alpha \rho = p_\alpha^g \rho_g + p_\alpha^m \rho_m\,; \qquad p_\delta \rho = p_\delta^g \rho_g + p_\delta^m \rho_m\,,$$

where $p_\alpha^g \rho_g$, $p_\delta^g \rho_g$ represent the geocentric parallax factors, usually designated by $p_\alpha \rho$ or $p_\alpha \Delta$, etc., and where all parallax factors are supposed to be expressed in seconds of arc. Further, for all three dates:

$$\Delta_2 X = [(p_\alpha\rho) \sin \alpha \cos \delta + (p_\delta\rho) \cos \alpha \sin \delta] \sin 1''\,; \quad \Delta_2 Y = [-(p_\alpha\rho) \cos \alpha \cos \delta + (p_\delta\rho) \sin \alpha \sin \delta] \sin 1'';$$
$$\Delta_2 Z = -(p_\delta\rho) \cos \delta \sin 1''\,.$$

If the $\Delta_2 X$, $\Delta_2 Y$, $\Delta_2 Z$, be roughly proportional to the time, then the following auxiliary quantities d_x, d_y, d_z are equal to zero and need not to be computed.

$$d_x = \frac{\Delta_2 X_{,,,} (t_{,,} - t_{,}) + \Delta_2 X_{,} (t_{,,,} - t_{,,})}{\Delta_2 X_{,,} (t_{,,,} - t_{,})} - 1\,; \qquad d_y = \frac{\Delta_2 Y_{,,,} (t_{,,} - t_{,}) + \Delta_2 Y_{,} (t_{,,,} - t_{,,})}{\Delta_2 Y_{,,} (t_{,,,} - t_{,})} - 1$$
$$d_z = \frac{\Delta_2 Z_{,,,} (t_{,,} - t_{,}) + \Delta_2 Z_{,} (t_{,,,} - t_{,,})}{\Delta_2 Z_{,,} (t_{,,,} - t_{,})} - 1\,.$$

$$\Delta X = \Delta_1 X + \Delta_2 X, \;\; \Delta Y = \Delta_1 Y + \Delta_2 Y, \;\; \Delta Z = \Delta_1 Z + \Delta_2 Z; \;\; (X) = X + \Delta X, \;\; (Y) = Y + \Delta Y, \;\; (Z) = Z + \Delta Z.$$

Compute further, but only for the middle date (epoch):

$$R \cos D \cos A = (X)_{,,} = X_{,,} + \Delta X_{,,}\,, \quad R \cos D \sin A = (Y)_{,,} = Y_{,,} + \Delta Y_{,,}\,, \quad R \sin D = (Z)_{,,} = Z_{,,} + \Delta Z_{,,}\,.$$

Also for general orbits, Ic:

$$\Delta_2 R = \frac{X_{,,} \Delta_2 X_{,,} + Y_{,,} \Delta_2 Y_{,,} + Z_{,,} \Delta_2 Z_{,,}}{R_{,,}}\,,$$

where we may use the corrected or uncorrected values for X, Y, Z.

$$j \cos a = \left[\frac{1}{R^3} + \frac{2 d_x}{\theta_{,} \theta_{,,,}} \right] \Delta_2 X_{,,}\,; \; j \sin a = \left[\frac{1}{R^3} + \frac{2 d_y}{\theta_{,} \theta_{,,,}} \right] \Delta_2 Y_{,,}\,; \; j \tan d = \left[\frac{1}{R^3} + \frac{2 d_z}{\theta_{,} \theta_{,,,}} \right] \Delta_2 Z_{,,}\,.$$

Compute $(\Delta X)_0'$; $(\Delta Y)_0'$; $(\Delta Z)_0'$ from $\Delta X_{,}$, $\Delta X_{,,}$, $\Delta X_{,,,}$; $\Delta Y_{,}$, $\Delta Y_{,,}$, $\Delta Y_{,,,}$; $\Delta Z_{,}$, $\Delta Z_{,,}$, $\Delta Z_{,,,}$ in the same manner as α_0'; δ_0' are computed from $\alpha_{,}$, $\alpha_{,,}$, $\alpha_{,,,}$; $\delta_{,}$, $\delta_{,,}$, $\delta_{,,,}$; but omitting the factors $\sin 1''$ in formulæ II which here has already been taken account of. Or, if the ΔX, ΔY, ΔZ be roughly proportional to the time, then simply

$$\log \frac{1}{k} = 1.7644\,; \quad (\Delta X)_0' = \frac{1}{k} \frac{\Delta X_{,,,} - \Delta X_{,}}{t_{,,,} - t_{,}}\,; \quad (\Delta Y)_0' = \frac{1}{k} \frac{\Delta Y_{,,,} - \Delta Y_{,}}{t_{,,,} - t_{,}}\,; \quad (\Delta Z)_0' = \frac{1}{k} \frac{\Delta Z_{,,,} - \Delta Z_{,}}{t_{,,,} - t_{,}}\,.$$

IIa; IIb; IIc.[1]

It is most convenient to express the differences in the coördinates in seconds of arc. Then, with the mean solar day as unit of time:

$$\alpha'_{,,,} = \frac{\alpha_{,,} - \alpha_{,}}{t_{,,} - t_{,}}; \quad \alpha'_{,} = \frac{\alpha_{,,,} - \alpha_{,,}}{t_{,,,} - t_{,,}}; \quad \delta'_{,,,} = \frac{\delta_{,,} - \delta_{,}}{t_{,,} - t_{,}}; \quad \delta'_{,} = \frac{\delta_{,,,} - \delta_{,,}}{t_{,,,} - t_{,,}}.$$

Or expressed in circular measure and in $\frac{1}{k}$ mean solar days as unit of time:

$$\alpha'_0 = \frac{\sin 1''}{k}\,\frac{(t_{,,,} - t_{,,})\,\alpha'_{,,,} + (t_{,,} - t_{,})\,\alpha'_{,}}{t_{,,,} - t_{,}}; \quad \delta'_0 = \frac{\sin 1''}{k}\,\frac{(t_{,,,} - t_{,,})\,\delta'_{,,,} + (t_{,,} - t_{,})\,\delta'_{,}}{t_{,,,} - t_{,}}.$$

$$\alpha''_0 = \frac{2}{k^2}\sin 1''\,\frac{\alpha'_{,} - \alpha'_{,,,}}{t_{,,,} - t_{,}}; \quad \delta''_0 = \frac{2}{k^2}\sin 1''\,\frac{\delta'_{,} - \delta'_{,,,}}{t_{,,,} - t_{,}}.$$

$$\log \frac{\sin 1''}{k} = 6.4499934 - 10, \quad \log \frac{2}{k^2}\sin 1'' = 8.5154420 - 10.$$

$$(\tan\delta)'_0 = \sec^2\delta_{,,}\,\delta'_0; \quad (\tan\delta)''_0 = \sec^2\delta_{,,}\,[2\tan\delta_{,,}\,(\delta'_0)^2 + \delta''_0].$$

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[1]In *extremely rare* cases the geocentric motion may be so irregular as to necessitate laborious subsequent computation by requiring successive differential corrections to the fundamental data $\rho_{,}$, x'_0, y'_0, z'_0 based on the velocities and accelerations in α and δ determined from these formulæ. Complications may then be avoided by using shorter intervals. Difficulties may also arise, if the computer is obliged to base the direct solution on long intervals. In such cases more accurate values of α'_0, α''_0, δ'_0, δ''_0 may be derived by plotting the available coördinates with the dates as abscissæ, then drawing a smooth curve for the right ascensions as well as for the declinations; then forming a table of right ascensions and declinations by taking these coördinates from their respective curves for equidistant dates; and finally determining α'_0, α''_0, δ'_0, δ''_0 from the tabulated values by numerical differentiation in a manner similar to that used in Ia, Ib, Ic in deriving the solar velocities X'_0, etc. Three observations may suffice for this purpose, but more should be used if available.

Or more accurate values α', α'', δ', δ'' for the middle date may be determined by means of TAYLOR's Theorem by the following method of solution, based on five observations. The scheme proposed here is readily extended to more than five observations.

Let the five right ascensions be $\alpha_1, \alpha_2, \alpha_3 = \alpha_0, \alpha_4, \alpha_5$, the dates $t_1, t_2, t_3 = t_0, t_4, t_5$, with a similar notation for the declinations.

$$i = 1, 2, 4, 5, \quad a_i = \frac{(t_i - t_0)^4}{4!}, \quad b_i = \frac{(t_i - t_0)^3}{3!}, \quad c_i = \frac{(t_i - t_0)^2}{2!}.$$

$$p_i = b_i, c_i, 1; \quad i = 2, 4, 5, \quad [p_i \cdot 1] = p_i - \frac{a_i}{a_1} p_1.$$

$$p_i = c_i, 1, \quad i = 2, 4; \quad [p_i \cdot 2] = [p_i \cdot 1] - \frac{[b_i \cdot 1]}{[b_5 \cdot 1]}[p_5 \cdot 1].$$

$$i = 2, 4, \quad n_{i,\alpha} = \frac{\alpha_i - [\alpha_0 + 2c_i\,\alpha'_0 + 3b_i\,\alpha''_0]}{2c_i},$$

where α'_0, α''_0 are derived as usual by the formulæ of this group, except for the factors $\frac{1}{k}$ and $\frac{1}{k^2}$, which are omitted for the present, the provisional unit of time being a mean solar day.

Compute $n_{2,\delta}$, $n_{4,\delta}$ in a manner analogous to the computation of $n_{2,\alpha}$, $n_{4,\alpha}$.

$$y_\alpha = \frac{n_{2,\alpha}[1_4 \cdot 2] - n_{4,\alpha}[1_2 \cdot 2]}{[c_2 \cdot 2][1_4 \cdot 2] - [c_4 \cdot 2][1_2 \cdot 2]}; \quad z_\alpha = \frac{n_{4,\alpha}[c_2 \cdot 2] - n_{2,\alpha}[c_4 \cdot 2]}{[c_2 \cdot 2][1_4 \cdot 2] - [c_4 \cdot 2][1_2 \cdot 2]},$$

Compute y_δ and z_δ from the same formulæ after replacing $n_{2,\alpha}$ by $n_{2,\delta}$ and $n_{4,\alpha}$ by $n_{4,\delta}$. Then

$$\alpha' = \alpha'_0 + z_\alpha, \quad \alpha'' = \alpha''_0 + y_\alpha, \quad \delta' = \delta'_0 + z_\delta, \quad \delta'' = \delta''_0 + y_\delta.$$

To express these velocities in units of $\frac{1}{k}$ mean solar days, multiply α', δ' by $\frac{1}{k}$ and α'', δ'' by $\left(\frac{1}{k}\right)^2$, $\log\frac{1}{k} = 1.7644186$, $\log\left(\frac{1}{k}\right)^2 = 3.5288372$.

Let ϑ_0 be a value of ϑ corresponding to $y' = 0$. There will be either one such value ϑ_0 or three, according to the number of positive roots. Then for each root

$$z = s \tan \vartheta_0 + c' .$$

First find the region of each possible ϑ_0 with the aid of a table of natural tangents and cosines and of a multiplication table, such as CRELLE'S *Rechentafeln*, or with the aid of a mechanism, such as a slide rule, by noting approximate values of $z' = \zeta + c'$ and $y = \eta - q'^2$ for a few values of ϑ from ϑ_m down to $\psi = 90$ until the value or values of ϑ are passed for which $y' = z'^2 - y$ changes sign. This is conveniently accomplished, for instance, with a table like CRELLE'S, by copying from the page headed s, the products $\zeta = s \tan \vartheta$ increased by c', c' being written on the edge of a card and held successively over the products in the table. In this manner a small table of z' is formed with ϑ as argument. As only rough values are needed, most computers will write down the z'^2 as readily as the z'. In the next column write down the values of $y = \frac{h}{s} \cos \vartheta - q'^2$ in a similar manner from the page headed $\frac{h}{s}$ using a card on the edge of which the value of q'^2 is written. Then inspection of the z'^2 and y columns readily reveals the approximate value or values ϑ_0 for which y' changes sign.[1]

Then compute y' for at least one value of ϑ greater and less, respectively, than the approximate ϑ_0 to the degree of accuracy to which it is desired to find ϑ_0 and, therefore, z from the plot.

Finally plot the short arc of the curve $y' = f(\vartheta)$ for each possible root and read off the accurate values of ϑ_0 for which $y' = 0$.

2). *By tabulating* $y' = f(\vartheta)$. Proceed as under (1), but instead of plotting $y' = f(\vartheta)$, construct a small table of y' with argument ϑ and derive ϑ_0 by interpolation.

3). *Graphically*. The roots are given by the values of z' at the intersections of the parabola and curve

$$y = z'^2 \text{ and } y = \eta - q'^2 , \; z' = \zeta + c' ,$$

where η and ζ are defined as functions of ϑ as under (1).

The parabola is the same for all orbits and may be plotted once for all.

To plot the short arc or arcs of the curve, first find an approximate value of ϑ_0 as in (1). Then compute y and z' for at least one value ϑ greater and less, respectively, than the approximate ϑ_0 to the degree of accuracy to which it is desired to find z' from the plot. Finally plot the short arc or arcs of the curve and read off the values of z' at the intersections of the curve and the parabola. Then

$$z = z' + p' .$$

[1] To test the feasibility of a parabolic solution *cf.* page 290, (a). In case of three roots the two fictitious roots may be rejected by the application of (b), page 291.

4). *From the Tables at the end of this Paper*, (*Part 7*).

This method gives the solutions without hypothesis regarding the eccentricity.[1]

$$m = -\frac{\varkappa}{R^4 \cos\delta_{\prime\prime}}; \qquad (m) = m\left(1 + \frac{3\Delta_2 R}{R}\right) + \frac{\Delta\varkappa}{R\cos\delta_{\prime\prime}}.$$

With $\frac{1}{m}$ or $\frac{1}{(m)}$ and ψ as arguments interpolate z from the Tables. As the range of practical solutions is often large for preliminary orbits from short arcs, the value of z obtained from the Tables may differ considerably from the value of z corresponding to a parabola. This method of deriving an initial value z_i of z may, therefore, involve several subsequent corrections of z_i, before a value of z is obtained which satisfies the equation of z for the parabola.

Let z_1 be the value of z obtained by any one of the foregoing methods.

$$\mu_1 = (z_1 - c)^2 + s^2; \; M_1 = (z_1 - p')^2 + q'^2 - \frac{h}{\mu_1^{1/2}}; \quad z_2 = z_1 - \frac{M_1}{2(z_1 - p') + \frac{h}{\mu_1^{1/2}} \frac{z_1 - c}{\mu_1}}.$$

Continue these approximations until $M = 0$. Denote the final value by z, without subscript.

IVb. Circular Orbits.

A circular orbit is impossible, if both c and p' are negative; or if the inequalities given on page 306 are fulfilled. The solution must then be performed without assumption regarding the eccentricity by IVc, etc. No criteria regarding the number of solutions, in general, are required.

$$h = \frac{1}{a^2 R^3}; \qquad \cos\vartheta_m = a^2 q'^2 s.$$

In all other respects formulæ IVb are the same as formulæ IVa. The result of the solution so far is a *conditioned* ellipse so that the criterion $r_0' = 0$ must be applied at the end of Vb. (*cf.* page 306.)

Further*, a circular solution may be derived even more easily directly from the general solution** IVc ***by applying in Vc ***the criteria*** $r_0 = a$ ***and*** $r_0' = 0$ (*cf.* page 306).

IVc. General Orbits.[2]

$$m = -\frac{\varkappa}{R^4\cos\delta_{\prime\prime}}; \; (m) = m\left(1 + \frac{3\Delta_2 R}{R}\right) + \frac{\Delta\varkappa}{R\cos\delta_{\prime\prime}}.$$

With $\frac{1}{m}$ or $\frac{1}{(m)}$ and ψ as arguments take $z = z_1$ from the Tables. The several solutions, which can not exceed two in number, are obtained from the Tables.

$$\mu_1 = (z_1 - c)^2 + s^2; \qquad (\nu)_1 = z_1 - (m); \qquad \mu_1^3 (\nu)_1^2 - m^2 = M_1;$$

For check: $M_1 = 0$; or for correction of z_1, if necessary:

$$z_2 = z_1 - \frac{M_1}{2\mu_1^2(\nu)_1[\mu_1 + 3(\nu)_1(z_1 - c)]}.$$

Let z, without subscript, be the final value.

[1] **Attention is called to the last paragraph on page 291 and the second paragraph on page 292.**

[2] **It should be remembered that** (m) is to be computed only in case of *complete elimination* of the parallax. **Otherwise** $(m) = m$ and $(\nu) = \nu$.

Va, Vb, Vc.

$$\rho_0 = R z\,; \qquad \sigma_0 = \rho_0 \cos \delta_{\prime\prime}$$

$$\sigma_0'' = \sigma_0 \left[\frac{C_3}{\varkappa} - \frac{1}{r_0^3} + (\alpha_0')^2 \right] + c_3 - C_3 \frac{\Delta \varkappa}{\varkappa}\,,$$

where r_0 is given below.

$$\rho_{\prime} = \frac{\sigma_0}{\cos \delta_{\prime}} \left[1 - \frac{\lambda}{\varkappa} \theta_{\prime\prime\prime} \right]^1 + \frac{\theta_{\prime\prime\prime}^2 \sigma_0''}{2 \cos \delta_{\prime}}\,; \quad \rho_{\prime\prime\prime} = \frac{\sigma_0}{\cos \delta_{\prime\prime\prime}} \left[1 + \frac{\lambda}{\varkappa} \theta_{\prime} \right]^1 + \frac{\theta_{\prime}^2 \sigma_0''}{2 \cos \delta_{\prime\prime\prime}}.$$

With $\rho_{\prime}$, ρ_0, $\rho_{\prime\prime\prime}$ correct $t_{\prime}$, t_0, $t_{\prime\prime\prime}$ for planetary aberration; $\log \alpha = 7.76128$; *corrected* $t = t - \alpha \rho$. Recompute from the corrected dates

$$\theta_{\prime} = k\,(t_{\prime\prime\prime} - t_{\prime\prime})\,; \qquad \theta_{\prime\prime\prime} = k\,(t_{\prime\prime} - t_{\prime})\,; \qquad \log k = 8.2355814.$$

The computation of σ_0'' and of the terms in $\rho_{\prime}$ and $\rho_{\prime\prime\prime}$ depending on it are practically always superfluous. It may become essential only if the body should be changing the direction of motion in the line of sight, that is, if $\lambda = 0$, and if σ_0'' is large. These conditions may be estimated by inspection.

$$^2\left\{\begin{array}{llll} x_0 = \sigma_0 \cos \alpha_{\prime\prime} - X_{\prime\prime} & ;\ y_0 = \sigma_0 \sin \alpha_{\prime\prime} - Y_{\prime\prime} & ;\ z_0 = \sigma_0 \tan \delta_{\prime\prime} - Z_{\prime\prime}\,; & r_0^2 = x_0^2 + y_0^2 + z_0^2 \\ \quad = \xi_0 - X_{\prime\prime} & ;\ \quad = \eta_0 - Y_{\prime\prime} & ;\ \quad = \zeta_0 - Z_{\prime\prime} & \\ x_0' = a_x \sigma_0 - X_0' & ;\ y_0' = a_y \sigma_0 - Y_0' & ;\ z_0' = a_z \sigma_0 - Z_0'\,; & r_0 r_0' = x_0 x_0' + y_0 y_0' + z_0 z_0'. \end{array}\right.$$

Although not required at this point, the elements which determine the character of the orbit may now be computed from

$$G_0^2 = x_0'^2 + y_0'^2 + z_0'^2\,; \quad \frac{1}{a} = \frac{2}{r_0} - G_0^2\,; \quad p = r_0^2\,[\,G_0^2 - r_0'^2\,]\,; \quad e^2 = \frac{a - p}{a}.$$

For a parabolic orbit $p = 2q = 2r_0 - (r_0 r_0')^2$, and the computation is partially checked by $G_0^2 = \frac{2}{r_0}$. For the *conditioned* ellipse the check is $a = r_0$, but the criterion $r_0' = 0$ must also be applied. Both criteria, $a = r_0$ and $r_0' = 0$, must be applied if the *circular* is to be based on the *general* solution (*cf.* page 306).

VIa, VIc. PARABOLIC AND GENERAL ORBITS.

$$f_2 = -\frac{1}{2 r_0^3}\,; \qquad f_3 = \frac{r_0'}{2 r_0^4}\,; \qquad f_4 = \frac{1}{6 r_0^6} \left[1 - \left(\frac{15}{4} \frac{(r_0 r_0')^2}{r_0} + \frac{3}{4} \frac{r_0}{a} \right) \right]$$

$$f_5 = -\frac{r_0'}{2 r_0^7} \left[1 - \left(\frac{7}{4} \frac{(r_0 r_0')^2}{r_0} + \frac{3}{4} \frac{r_0}{a} \right) \right] \ \cdots$$

$$g_3 = -\frac{1}{6 r_0^3}\,; \qquad g_4 = \frac{r_0'}{4 r_0^4}\,; \qquad g_5 = \frac{1}{12 r_0^6} \left[1 - \left(\frac{9}{2} \frac{(r_0 r_0')^2}{r_0} + \frac{9}{10} \frac{r_0}{a} \right) \right]$$

$$g_6 = -\frac{7}{24} \frac{r_0'}{r_0^7} \left[1 - \left(2 \frac{(r_0 r_0')^2}{r_0} + \frac{6}{7} \frac{r_0}{a} \right) \right] \ \cdots$$

$$f_{\prime} = 1 + \theta_{\prime\prime\prime}^2 f_2 - \theta_{\prime\prime\prime}^3 f_3 + \theta_{\prime\prime\prime}^4 f_4 - \theta_{\prime\prime\prime}^5 f_5 + \cdots\,; \quad g_{\prime} = -\theta_{\prime\prime\prime} - \theta_{\prime\prime\prime}^3 g_3 + \theta_{\prime\prime\prime}^4 g_4 - \theta_{\prime\prime\prime}^5 g_5 + \theta_{\prime\prime\prime}^6 g_6 \cdots$$

$$f_{\prime\prime\prime} = 1 + \theta_{\prime}^2 f_2 + \theta_{\prime}^3 f_3 + \theta_{\prime}^4 f_4 + \theta_{\prime}^5 f_5 + \cdots\,; \quad g_{\prime\prime\prime} = \theta_{\prime} + \theta_{\prime}^3 g_3 + \theta_{\prime}^4 g_4 + \theta_{\prime}^5 g_5 + \theta_{\prime}^6 g_6 \cdots$$

[1]In case of *complete elimination* of parallax replace $\frac{\lambda}{\varkappa}$ by $\frac{\sigma_0'}{\sigma_0} = \frac{\lambda}{\varkappa} - \frac{\beta}{\sigma_0}$.

[2]In the *complete elimination* of the parallax use $(X)_{\prime\prime}$ for $X_{\prime\prime}$, etc., and $[X]_0'$ for X_0', etc.

For parabolic orbits the terms containing a disappear. In general only f_2, f_3, and g_3 are required. The necessity of including the higher terms $\theta_4 f_4$, $\theta_4 g_4$, etc., is readily determined by estimating their numerical values.

For unusually long intervals or excessively small values of r_0 the f and g series, as given, may prove insufficient. Then the f and g should be computed from the closed expressions given in B II, page 330, etc.

VIb. Circular Orbits.

The criterion $r' = 0$ in the *conditioned* or the criteria $a = r$ and $r' = 0$ in the *general* ellipse having been verified, the computation of the orbit may be concluded by first deriving the elements (Ω), (i), (u) referred to the *equator* from the values of x, y, z; x', y', z' for the *conditioned* or the *general* ellipse, as the case may be, and then computing the constants for the equator and the elements referred to the *ecliptic* in one of the usual ways. The formulæ for the elements, therefore, are given here, instead of later, under VIII.

Equator.

$$\sqrt{p}\cos(i) = x_0 y_0' - y_0 x_0'; \qquad r_0 \sin(u_0) = \frac{z_0}{\sin(i)}$$

$$\sqrt{p}\sin(i)\sin(\Omega) = y_0 z_0' - z_0 y_0'; \qquad r_0\cos(u_0) = x\cos(\Omega) + y\sin(\Omega)$$

$$\sqrt{p}\sin(i)\cos(\Omega) = x_0 z_0' - z_0 x_0'; \qquad r_0^2 = x_0^2 + y_0^2 + z_0^2, \text{ check.}$$

p is to be discarded and, therefore, need not be computed. Let *radius* $a = r_0$.

$$\sin a \sin(A) = \cos(\Omega); \qquad \sin b \sin(B) = \sin(\Omega); \qquad (C) = 0$$

$$\sin a \cos(A) = -\sin(\Omega)\cos(i); \quad \sin b\cos(B) = \cos(\Omega)\cos(i); \quad \sin c = \sin(i).$$

$$A' = (A) + (u_0),\ B' = (B) + (u_0),\ C' = (C) + (u_0); \quad \alpha = r_0 \sin a; \quad \beta = r_0 \sin b; \quad \gamma = r_0 \sin c.$$

$$\mu = k'' a^{-3/2}; \quad \log k'' = 3.550007; \quad \text{Epoch of } (u_0) - \textit{true } t_{,,}. \quad \text{Let } \Delta t = (t - t_{,,}).$$

[1] $$x = \alpha\sin(A' + \mu\Delta t); \quad y = \beta\sin(B' + \mu\Delta t); \quad z = \gamma\sin(C' + \mu\Delta t).$$

Ecliptic.

$$\sin m \sin M = \sin(i)\cos(\Omega); \qquad \sin i \sin\Omega = \sin(i)\sin(\Omega)$$

$$\sin m \cos M = \cos(i); \qquad \sin i\cos\Omega = \sin m \sin(M - \varepsilon)$$

$$\sin n \sin N = \sin(i); \qquad \sin i \sin\sigma = \sin\varepsilon\sin(\Omega)$$

$$\sin n \cos M = \cos(i)\cos(\Omega); \qquad \sin i\cos\sigma = \sin n\sin(N - \varepsilon)$$

$$\cos i = \sin m\cos(M - \varepsilon); \qquad u_0 = (u_0) - \sigma.$$

ε = obliquity of the ecliptic at the beginning of the year.

VIa, VIb, VIc—*Continued.*

$$\rho_{,}\cos\delta_{,}\cos\alpha_{,} = X_{,} + f_{,}x_0 + g_{,}x_0' = \xi_{,}; \qquad \rho_{,,,}\cos\delta_{,,,}\cos\alpha_{,,,} = X_{,,,} + f_{,,,}x_0 + g_{,,,}x_0' = \xi_{,,,}$$

$$\rho_{,}\cos\delta_{,}\sin\alpha_{,} = Y_{,} + f_{,}y_0 + g_{,}y_0' = \eta_{,}; \qquad \rho_{,,,}\cos\delta_{,,,}\sin\alpha_{,,,} = Y_{,,,} + f_{,,,}y_0 + g_{,,,}y_0' = \eta_{,,,}$$

$$\rho_{,}\sin\delta_{,} = Z_{,} + f_{,}z_0 + g_{,}z_0' = \zeta_{,}; \qquad \rho_{,,,}\sin\delta_{,,,} = Z_{,,,} + f_{,,,}z_0 + g_{,,,}z_0' = \zeta_{,,,}.$$

For VIb use $x = \alpha\sin(A' + \mu\Delta t)$, etc., in place of $fx_0 + gx_0'$, etc.

[1] If the orbit is to be made the basis of a differential correction the necessary values of x_0', y_0', z_0' are obtained from $x_0' = a^{-1/2}\sin a\sin A'$; $y_0' = a^{-1/2}\sin b\sin B'$; $z_0' = a^{-1/2}\sin c\sin C'$.

Use here the geocentric solar coördinates and correct the first and third observed places for parallax on the basis of the resulting values of $\rho_{,}$ and $\rho_{,,,}$[1].

Form the residuals, $(O-C)$, $\partial_{,}\alpha_{,} = \partial\alpha_{,}\cos\delta_{,}$, $\partial\delta_{,}$, $\partial_{,}\alpha_{,,,} = \partial\alpha_{,,,}\cos\delta_{,,,}$, $\partial\delta_{,,,}$. If these residuals are sufficiently small for the purpose in hand, the elements and an ephemeris may be computed at once by VIII. If it be intended, however, to represent the observations more accurately the necessary corrections to ρ_0, x'_0, y'_0, z'_0 may be determined by means of [VII].

[VIIa], [VIIb], [VIIc].

For the correction of an initial parabolic or general orbit the differential formulæ of this group are based on the ∂f and ∂g *series*. These are usually sufficient for the removal of any residuals that may remain after the direct solution I—VI. They include terms of the order θ^3 multiplied by a variation. The differential formulæ based on *closed expressions* for ∂f and ∂g are given in *B*. But for the correction of an initial *circular* orbit the *closed* expressions for ∂f and ∂g are adopted in this group as well as in *B*.

For the correction of circular orbits the procedure is, therefore, the same in all cases [VIIb].

For the correction of an initial parabolic or general orbit apply either the formulæ of this group—with or without the supplementary terms—or the closed formulæ *B*, page 327, in accordance with the following criterion:

If in the computation of the f and g series by VIa, or VIc, as the case may be, terms containing powers of θ higher than θ^4 are negligible, apply [VIIa] or [VIIc]. Otherwise apply *B*, page 327.

Before adopting any values of ρ_0, x'_0, y'_0, z'_0 as final, compute and test the residuals by VI. As a further check the observations may later also be reproduced from the constants for the equator.

A second application of the differential formulæ will be required only if linear relations do not prevail, that is, for unusually large initial residuals. This condition will be revealed by the residuals remaining after the first differential correction. In the second differential correction it is generally sufficient to recompute only those auxiliary quantities which depend on the new residuals.

As soon as $\partial\rho_0$ (or in case of improvement of a *conditioned* elliptic or parabolic orbit a first approximation of $\partial\rho_0$) has been found, test the applicability of the differential relations to the correction of the fundamental quantities ρ_0, x'_0, y'_0, z'_0. If

$$\partial\rho_0 \quad \frac{1}{1 \quad 2\sin(\beta \quad 45^\circ)}\,\rho_0 ,$$

then the differential formulæ will not converge and must be abandoned. In such

[1] If the parallax has been eliminated *completely*, so that $(X)_{,}$, $(X)_{,,,}$, etc., are available, these corrected solar coördinates may be used here, but then the observations require no further correction in the derivation of the residuals.

cases, which, however, are not to be expected in practice, if proper precautions have been taken in the derivation of the initial velocities and accelerations, recourse must be had to the method of arbitrary variation.

If $\partial\rho_0$ be less than the foregoing limiting value but comparable in magnitude to the same, then the convergence of the differential formulæ will be slow. In that case compute the auxiliary terms given in footnote (1) on page 323 as soon as $\partial\rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$, $\partial r_0'$ have been found. In *no other case* are these auxiliary terms to be calculated.

[VIIb]. FROM CIRCULAR ORBITS.

$$\cos\beta = \frac{\rho_0 - R_0\cos\psi}{r_0}.$$

$$d = \frac{3\cos\beta}{2a}; \quad c_{,} = -\frac{\theta_{,,,}\, g_{,}}{a^3}; \quad c_{,}' = \theta_{,,,} f_{,} + g_{,}; \quad c_{,,,} = \frac{\theta_{,}\, g_{,,,}}{a^3}; \quad c_{,,,}' = -(\theta_{,} f_{,,,} - g_{,,,}).$$

$$f_{x_{,}} = f_{,}\frac{\xi_0}{\rho_0} + d\,[c_{,}\, x_0 + c_{,}' x_0']; \qquad f_{x_{,,,}} = f_{,,,}\frac{\xi_0}{\rho_0} + d\,[c_{,,,}\, x_0 + c_{,,,}' x_0']$$

$$f_{y_{,}} = f_{,}\frac{\eta_0}{\rho_0} + d\,[c_{,}\, y_0 + c_{,}' y_0']; \qquad f_{y_{,,,}} = f_{,,,}\frac{\eta_0}{\rho_0} + d\,[c_{,,,}\, y_0 + c_{,,,}' y_0']$$

$$f_{z_{,}} = f_{,}\frac{\zeta_0}{\rho_0} + d\,[c_{,}\, z_0 + c_{,}' z_0']; \qquad f_{z_{,,,}} = f_{,,,}\frac{\zeta_0}{\rho_0} + d\,[c_{,,,}\, z_0 + c_{,,,}' z_0'].$$

[VIIa], [VIIc]. FROM PARABOLIC OR ELLIPTIC ORBITS.

$$\cos\beta = \frac{\rho_0 - R_0\cos\psi}{r_0};$$

$$f_{x_{,}} = f_{,}\frac{\xi_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}\left(3\theta_{,,,}^2\, x_0 - \theta_{,,,}^3\left[x_0' - \frac{4r_0' x_0}{r_0}\right]\right); \quad f_{x_{,,,}} = f_{,,,}\frac{\xi_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}\left(3\theta_{,}^2\, x_0 + \theta_{,}^3\left[x_0' - \frac{4r_0' x_0}{r_0}\right]\right)$$

$$f_{y_{,}} = f_{,}\frac{\eta_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}\left(3\theta_{,,,}^2\, y_0 - \theta_{,,,}^3\left[y_0' - \frac{4r_0' y_0}{r_0}\right]\right); \quad f_{y_{,,,}} = f_{,,,}\frac{\eta_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}\left(3\theta_{,}^2\, y_0 + \theta_{,}^3\left[y_0' - \frac{4r_0' y_0}{r_0}\right]\right)$$

$$f_{z_{,}} = f_{,}\frac{\zeta_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}\left(3\theta_{,,,}^2\, z_0 - \theta_{,,,}^3\left[z_0' - \frac{4r_0' z_0}{r_0}\right]\right); \quad f_{z_{,,,}} = f_{,,,}\frac{\zeta_0}{\rho_0} + \frac{\cos\beta}{2r_0^4}\left(3\theta_{,}^2\, z_0 + \theta_{,}^3\left[z_0' - \frac{4r_0' z_0}{r_0}\right]\right).$$

[VIIa], [VIIb], [VIIc]—*Continued.*

$$A_{,} = \frac{1}{\rho_{,}}\left[\cos\alpha_{,}\, f_{y_{,}} - \sin\alpha_{,}\, f_{x_{,}}\right] + \cos\delta_{,}\,\frac{(p_a\rho)_{,}}{\rho_{,}^2}$$

$$B_{,} = -\frac{1}{\rho_{,}}\left[\sin\delta_{,}\,(\sin\alpha_{,}\, f_{y_{,}} + \cos\alpha_{,}\, f_{x_{,}}) - \cos\delta_{,}\, f_{z_{,}}\right] + \frac{(p_\delta\rho)_{,}}{\rho_{,}^2}; \quad C_{,} = \frac{g_{,}}{\rho_{,}}$$

$$A_{,,,} = \frac{1}{\rho_{,,,}}\left[\cos\alpha_{,,,}\, f_{y_{,,,}} - \sin\alpha_{,,,}\, f_{x_{,,,}}\right] + \cos\delta_{,,,}\,\frac{(p_a\rho)_{,,,}}{\rho_{,,,}^2}$$

$$B_{,,,} = -\frac{1}{\rho_{,,,}}\left[\sin\delta_{,,,}\,(\sin\alpha_{,,,}\, f_{y_{,,,}} + \cos\alpha_{,,,}\, f_{x_{,,,}}) - \cos\delta_{,,,}\, f_{z_{,,,}}\right] + \frac{(p_\delta\rho)_{,,,}}{\rho_{,,,}^2}; \quad C_{,,,} = \frac{g_{,,,}}{\rho_{,,,}}.$$

[VIIa], [VIIb], [VIIc]—*Continued.*

$$P_x = \frac{C_{,,,} \cos \alpha_{,,,} \partial_{,} \alpha_{,} - C_{,} \cos \alpha_{,} \partial_{,} \alpha_{,,,}}{C_{,} C_{,,,} \sin (\alpha_{,,,} - \alpha_{,})}; \qquad Q_x = \frac{A_{,} C_{,,,} \cos \alpha_{,,,} - A_{,,,} C_{,} \cos \alpha_{,}}{C_{,} C_{,,,} \sin (\alpha_{,,,} - \alpha_{,})}$$

$$P_y = \frac{C_{,,,} \sin \alpha_{,,,} \partial_{,} \alpha_{,} - C_{,} \sin \alpha_{,} \partial_{,} \alpha_{,,,}}{C_{,} C_{,,,} \sin (\alpha_{,,,} - \alpha_{,})}; \qquad Q_y = \frac{A_{,} C_{,,,} \sin \alpha_{,,,} - A_{,,,} C_{,} \sin \alpha_{,}}{C_{,} C_{,,,} \sin (\alpha_{,,,} - \alpha_{,})}$$

$$P_{z,} = \frac{\partial \delta_{,} + C_{,} \sin \delta_{,} (\cos \alpha_{,} P_x + \sin \alpha_{,} P_y)}{C_{,} \cos \delta_{,}}; Q_{z,} = \frac{B_{,} + C_{,} \sin \delta_{,} (\cos \alpha_{,} Q_x + \sin \alpha_{,} Q_y)}{C_{,} \cos \delta_{,}}$$

$$P_{z,,,} = \frac{\partial \delta_{,,,} + C_{,,,} \sin \delta_{,,,} (\cos \alpha_{,,,} P_x + \sin \alpha_{,,,} P_y)}{C_{,,,} \cos \delta_{,,,}}; Q_{z,,,} = \frac{B_{,,,} + C_{,,,} \sin \delta_{,,,} (\cos \alpha_{,,,} Q_x + \sin \alpha_{,,,} Q_y)}{C_{,,,} \cos \delta_{,,,}}.$$

A parabolic, circular, or general orbit, may now be attempted irrespective of the character of the initial orbit.

For the computation of a parabola or conditioned ellipse a single value must be adopted for the P_z as well as for the Q_z. If any possible subsequent discrepancy in the parabolic or circular orbit is to be concentrated in the residual of the declination of the first place, then let $P_z = P_{z,,,}$; $Q_z = Q_{z,,,}$. If it be decided, however, that the final residuals be distributed between the first and third declinations, then this may be accomplished by forming mean values P_z and Q_z from $P_{z,}$, $P_{z,,,}$ and $Q_{z,}$, $Q_{z,,,}$ by assigning to the latter such arbitrary weights as may seem most expedient. In order to obtain equal residuals for the first and third declinations, the weights must be based on the intervals and the rate of variation of $\tan \delta_{,}$ and $\tan \delta_{,,,}$. In general, the ephemeris will hold better by concentrating any discrepancy in the first declination.

[VIIa]. For the Derivation of Parabolic Orbits.

If not computed in Va, obtain

$$G_0^2 = (x_0'^2 + y_0'^2 + z_0'^2); \qquad \frac{1}{a_0} = \frac{2}{r_0} - G_0^2.$$

If the direct solution or the initial orbit was parabolic, then $\frac{1}{a_0}$ will be a small quantity comparable to the inaccuracy of the computation.

$$Q = \frac{\cos \beta}{r_0^2} - [x_0' Q_x + y_0' Q_y + z_0' Q_z]; \qquad P = \frac{1}{2 a_0} - [x_0' P_x + y_0' P_y + z_0' P_z]$$

$$Q \partial \rho_0 = P - \frac{1}{2} [(\partial x_0')^2 + (\partial y_0')^2 + (\partial z_0')^2] - \frac{1 - 3 \cos^2 \beta}{2 r_0^3} (\partial \rho_0)^2 + \frac{3 \cos \beta (1 - \frac{5}{3} \cos^2 \beta)}{2 r_0^4} (\partial \rho_0)^3 \ldots$$

As a first approximation: $\partial \rho_0 = \frac{P}{Q}$;

$$\partial x_0' = - P_x - Q_x \partial \rho_0; \qquad \partial y_0' = - P_y - Q_y \partial \rho_0; \qquad \partial z_0' = - P_z - Q_z \partial \rho_0.$$

Substitute these approximate values of $\partial \rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$ in the right-hand member

[VIIa]. Test of the Possibility of a Parabolic Orbit.

With the final value of $\partial\rho_0$ compute

$$\partial z_0' = P_{z_{\prime}} - Q_{z_{\prime}}\,\partial\rho_0\,; \qquad \partial z_0' = P_{z_{\prime\prime\prime}} - Q_{z_{\prime\prime\prime}}\,\partial\rho_0\,.$$

If the two values of $\partial z_0'$ agree within the limits of accuracy of the numerical computation, then the observations will be represented by a parabolic orbit.

If they disagree, the resulting residuals may be obtained at once by subtracting the residuals computed from

$$(\partial\delta_{\prime})_c = B_{\prime}\,\partial\rho_0 - C_{\prime}\,\sin\delta_{\prime}\,\cos\alpha_{\prime}\,\partial x_0' - C_{\prime}\,\sin\delta_{\prime}\,\sin\alpha_{\prime}\,\partial y_0' + C_{\prime}\,\cos\delta_{\prime}\,\partial z_0'$$
$$(\partial\delta_{\prime\prime\prime})_c = B_{\prime\prime\prime}\,\partial\rho_0 - C_{\prime\prime\prime}\sin\delta_{\prime\prime\prime}\cos\alpha_{\prime\prime\prime}\,\partial x_0' - C_{\prime\prime\prime}\sin\delta_{\prime\prime\prime}\sin\alpha_{\prime\prime\prime}\,\partial y_0' + C_{\prime\prime\prime}\cos\delta_{\prime\prime\prime}\,\partial z_0'$$

from the initial residuals $\partial\delta_{\prime}$ and $\partial\delta_{\prime\prime\prime}$. The value of $\partial z_0'$ to be used in these formulæ for $\partial\delta_{\prime}$ and $\partial\delta_{\prime\prime\prime}$ is the value derived from $\partial z_0' = P_z - Q_z\,\partial\rho_0$. If P_z, Q_z were assumed equal to $P_{z_{\prime\prime\prime}}$, $Q_{z_{\prime\prime\prime}}$, then $\partial\delta_{\prime\prime\prime} - (\partial\delta_{\prime\prime\prime})_c$ will be equal to zero, provided that no error has been committed in the computation of the differential corrections. If, then, $\partial\delta_{\prime} - (\partial\delta_{\prime})_c$ is sufficiently small the corrections to ρ_0, x_0', y_0', z_0', may be adopted as final.

If P_z and Q_z represent weighted means then neither $\partial\delta_{\prime} - (\partial\delta_{\prime})_c$ nor $\partial\delta_{\prime\prime\prime} - (\partial\delta_{\prime\prime\prime})_c$ will be equal to zero, but if these residuals are sufficiently small, then the corrections to ρ_0, x_0', y_0', z_0' may be adopted as final.

If the representation should be unsatisfactory, then the attempt at a parabolic orbit in general should be abandoned and instead the corrections should be determined without regard to the eccentricity, as in [VIIc], etc.

The parabola, however, should not as yet be abandoned, if the residuals after the differential correction appear to be due to the insufficiency of the *linear* differential relations adopted in the differential formulæ.

[VIIc]. For the Derivation of Orbits Without Assumption Regarding the Eccentricity.[1]

$$\partial\rho_0 = \frac{P_{z_{\prime\prime\prime}} - P_{z_{\prime}}}{Q_{z_{\prime\prime\prime}} - Q_{z_{\prime}}}\,;\ \partial x_0' = P_x - Q_x\,\partial\rho_0\,;\ \partial y_0' = P_y - Q_y\,\partial\rho_0\,;\ \partial z_0' = P_{z_{\prime\prime\prime}} - Q_{z_{\prime\prime\prime}}\partial\rho_0 = P_{z_{\prime}} - Q_{z_{\prime}}\,\partial\rho_0\,.$$

[VIIa], [VIIb], [VIIc].

The corrected heliocentric coördinates and velocities are

$$x_0 + \xi_0\,\frac{\partial\rho_0}{\rho_0}\,;\quad y_0 + \eta_0\,\frac{\partial\rho_0}{\rho_0}\,;\quad z_0 + \zeta_0\,\frac{\partial\rho_0}{\rho_0}\,;\quad x_0' + \partial x_0'\,;\quad y_0' + \partial y_0'\,;\quad z_0' + \partial z_0'\,.$$

With these compute r_0 and $r_0\,r_0'$ as in V and then the new residuals by VI. If these prove satisfactory, then compute the elements by VIII.

[1] See footnote on preceding page.

2). *Or compute first* the elements defining the position of the orbit plane with reference to the *equator* and from them the constants for the *equator.*

Equator.

$$\sqrt{p}\cos(i) = xy' - yx'\ ;\quad r\sin(u) = \frac{z}{\sin(i)}$$
$$\sqrt{p}\sin(i)\sin(\Omega) = yz' - zy'\ ;\quad r\cos(u) = x\cos(\Omega) + y\sin(\Omega)$$
$$\sqrt{p}\sin(i)\cos(\Omega) = xz' - zx'\ ;\quad r^2 = x^2 + y^2 + z^2\text{, check.}$$

p should check with its previously computed value.

$$\sin a\sin(A) = \cos(\Omega)\ ;\qquad \sin b\sin(B) = \sin(\Omega)\ ;\qquad (C) = 0$$
$$\sin a\cos(A) = -\sin(\Omega)\cos(i)\ ;\quad \sin b\cos(B) = \cos(\Omega)\cos(i)\ ;\quad \sin c = \sin(i):$$
$$(\omega) = (u) - v\ ;\quad A' = (A) + (\omega),\quad B' = (B) + (\omega),\quad C' = (\omega)\ .$$

$$\sin\tfrac{1}{2}i\sin\tfrac{1}{2}[\Omega+\sigma] = \sin\tfrac{1}{2}[(i)+\epsilon]\sin\tfrac{1}{2}(\Omega)$$
$$\sin\tfrac{1}{2}i\cos\tfrac{1}{2}[\Omega+\sigma] = \sin\tfrac{1}{2}[(i)-\epsilon]\cos\tfrac{1}{2}(\Omega)$$
$$\cos\tfrac{1}{2}i\sin\tfrac{1}{2}[\Omega-\sigma] = \cos\tfrac{1}{2}[(i)+\epsilon]\sin\tfrac{1}{2}(\Omega)$$
$$\cos\tfrac{1}{2}i\cos\tfrac{1}{2}[\Omega-\sigma] = \cos\tfrac{1}{2}[(i)-\epsilon]\cos\tfrac{1}{2}(\Omega)\ ,$$

where e = obliquity of the ecliptic. Another set of formulæ for the transformation of the elements from the equator to the ecliptic is given under [VII b]: $m\sin M$, etc.

$$\omega = (\omega) - \sigma\ ;\qquad \pi = \Omega + \omega,$$

where i, Ω, ω, π are referred to the *ecliptic.*

Ellipse.

$$\tan\tfrac{1}{2}E = \sqrt{\frac{1-e}{1+e}}\tan\tfrac{1}{2}v$$
$$M = E - \frac{e}{\sin 1''}\sin E$$
$$\mu = k''a^{-\frac{3}{2}}$$
$$\log k'' = 3.550\,006\,6$$

Epoch = *true* $t_{,,}$.

Parabola.

With v as argument, take $M_{,,}$ from OPPOLZER, vol. I, table IV.

$$T = true\ t_{,,} - M_{,,}q^{\frac{3}{2}}.$$

Hyperbola.

$$\tan\tfrac{1}{2}F = \sqrt{\frac{e-1}{e+1}}\tan\tfrac{1}{2}v$$
$$T = true\ t_{,,} - \frac{(-a)^{\frac{3}{2}}}{k}\left[e\tan F - \frac{\log\tan(45^\circ+\tfrac{1}{2}F)}{\text{Mod.}}\right].$$

An ephemeris may be computed in the usual manner from the constants for the equator or by means of the series for f and g as in VI, or by means of the closed expressions for f and g as in B.

B. THE DERIVATION OF AN ORBIT FROM THREE OBSERVATIONS ON THE BASIS OF A PREVIOUS APPROXIMATION.

I.

The derivation of an orbit from three observations on the basis of a previous approximation (preliminary orbit) is accomplished either by formulæ A [VII] or by the formulæ of this group.

Formulæ A [VII] are the more convenient of the two groups. They are based on the series for ∂f and ∂g and are applicable to moderate intervals and if the heliocentric distance r_0 at the middle date is not excessively small, that is, in general if in the f and g series A VI, the terms containing higher powers of θ than θ^4 are negligible.

Formulæ A [VII], therefore, in general have a wider range of applicability for minor planets than for comets.

The formulæ of this group are based on the closed expressions for ∂f and ∂g and are applicable to intervals of any length as well as to small values of r_0.

Both groups of formulæ are applicable to the derivation of an orbit with or without assumption regarding the eccentricity on the basis of a preliminary orbit of any eccentricity and readily permit of the transition from one class of orbit to another for the purpose of improving the representation of the observations.

While the possibility of obtaining a hyperbolic orbit from a preliminary parabolic or nearly parabolic orbit is directly included, no special formulæ are given for basing the computation on an *initial* hyperbolic orbit. Since, however, the eccentricity of a hyperbolic orbit rarely differs considerably from unity, the differential formulæ for the correction of an initial parabolic orbit will prove sufficient for the correction of a possible hyperbolic orbit.

The method to be chosen for applying the corrections for parallax and aberration will depend on the accuracy of the geocentric distances which the preliminary orbit may yield for the dates of the new orbit. This may be judged by a rough comparison of the preliminary orbit with the observations.

If it be deemed possible to derive from the preliminary orbit values of the geocentric distances that are close enough to yield the *final* values of the corrections for parallax and planetary aberration, these corrections should be applied in advance. The observations are then to be reduced to the beginning of the year by the usual reduction formulæ, *exclusive* of the aberration terms, and the solar coördinates X, Y, Z are to be interpolated, also for the beginning of the year, from an astronomical ephemeris for the dates corrected for planetary aberration, that is, for the *true* dates.

In case of doubt concerning the accuracy of the geocentric distances of the

preliminary orbit, the *geocentric* parallax and aberration should be *eliminated* by correcting the solar coördinates for the middle date by

$$\Delta X_{,,} = (p_\alpha\rho)_{,,} \sin\alpha_{,,} \cos\delta_{,,} + (p_\delta\rho)_{,,} \cos\alpha_{,,} \sin\delta_{,,}, \quad \Delta Y_{,,} = -(p_\alpha\rho)_{,,} \cos\alpha_{,,} \cos\delta_{,,} + (p_\delta\rho) \sin\alpha_{,,} \sin\delta_{,,}$$
$$\Delta Z_{,,} = -(p_\delta\rho)_{,,} \cos\delta_{,,},$$

and by using the solar coördinates at the beginning of the year interpolated for the *uncorrected* dates together with the observed places reduced to the beginning of the year by the usual reduction formulæ, *inclusive* of the aberration terms.

Let the data resulting from either of the foregoing methods of reduction be

$$\alpha_{,}, \delta_{,}, \quad \alpha_{,,}, \delta_{,,}; \quad \alpha_{,,,}, \delta_{,,,}; \quad X_{,}, Y_{,}, Z_{,}; \quad X_{,,}, Y_{,,}, Z_{,,}; \quad X_{,,,}, Y_{,,,}, Z_{,,,}.$$
$$R\cos D\cos A = X_{,,}; \ R\cos D\sin A = Y_{,,}; \ R\sin D = Z_{,,}; \cos\psi = \sin D\sin\delta_{,,} + \cos D\cos\delta_{,,}\cos(A - \alpha_{,,}).$$

Derive initial values of the fundamental quantities ρ_0, x'_0, y'_0, z'_0 from the preliminary orbit:

The number of decimals to which these quantities should be computed should correspond to the number of decimals to which they may be considered to agree with their physical values. But as only approximate values are required, slight arbitrary changes may be made in them to suit the convenience of the computer. Hence, although theoretically the initial fundamental data x'_0, y'_0, z'_0 should always correspond to the true date t_0, while ρ_0 should correspond to the true date only if the observations have been corrected in advance for parallax and aberration, this distinction, if inconvenient, may be disregarded.

In deriving the fundamental quantities any data that may be available from the numerical work on the preliminary orbit in addition to the elements and the constants for the equator should be used for the simplification of the computation.

Thus ρ_0 for the apparent or the true middle date, as the case may be, is frequently directly given by an ephemeris.

If the numerical work on the ephemeris be available, the velocities x'_0, y'_0, z'_0 for the true date may be derived from the heliocentric coördinates of the ephemeris computation by means of the formulæ for numerical differentiation given in A I.

If the constants for the equator are not available, but if the middle date is covered by an ephemeris giving α, δ, ρ, then the velocities of α'_0, δ'_0, ρ'_0 may be obtained by numerical differentiation and then

$$(\tan\delta)'_0 = \sec^2\delta_{,,}\delta'_0, \quad \sigma' = \cos\delta_{,,}\rho'_0 - \rho_0\sin\delta_{,,}\delta'_0, \quad \sigma_0 = \rho_0\cos\delta_{,,}.$$
$$x'_0 = \cos\alpha_{,,}\sigma' - \sin\alpha_{,,}\sigma\alpha'_0 - X'_0, \quad y'_0 = \sin\alpha_{,,}\sigma'_0 + \cos\alpha_{,,}\sigma_0\alpha'_0 - Y'_0, \quad z'_0 = \tan\delta_{,,}\sigma'_0 + \sigma_0(\tan\delta)'_0 - Z'_0,$$

where X'_0, Y'_0, Z'_0 are to be obtained by numerical differentiation as in A I.

In the majority of cases, only the elements and the constants for the equator are available.

Then derive $v_{,,}$ from the elements (or $u_{,,}$ from a circular orbit) in the usual

manner according to the eccentricity of the preliminary orbit. (If the preliminary orbit is *very nearly* parabolic, let $e=1$, $q=\frac{p}{2}$ and proceed as in case of a parabolic orbit.)

Let t_0 be the epoch of the preliminary orbit, T the date of perihelion passage.

Ellipse

$$E_{\prime\prime}-\frac{e}{\sin 1''}\sin E_{\prime\prime}=M_0+(t_{\prime\prime}-t_0)\mu$$

$$\tan\tfrac{1}{2}v_{\prime\prime}=\sqrt{\frac{1+e}{1-e}}\tan\tfrac{1}{2}E_{\prime\prime}$$

$$r_{\prime\prime}=a(1-e\cos E_{\prime\prime})$$

$$z_{\prime\prime}=r_{\prime\prime}\sin c\sin(C'+v_{\prime\prime})$$

Parabola

$$M_{\prime\prime}=\frac{t_{\prime\prime}-T}{q^{3/2}}$$

with $M_{\prime\prime}$ as argument take $v_{\prime\prime}$ from OPPOLZER, Vol. I, Table IV.

$$r_{\prime\prime}=q\sec^2\tfrac{1}{2}v_{\prime\prime}$$

$$z_{\prime\prime}=r_{\prime\prime}\sin c\sin(C'+v_{\prime\prime})$$

Circle

$$u_{\prime\prime}=(t_{\prime\prime}-t_0)\mu$$

$$z_{\prime\prime}=a\sin c\sin(C'+u_{\prime\prime}).$$

$$\rho_0=\frac{z_{\prime\prime}+Z_{\prime\prime}}{\sin\delta_{\prime\prime}}\ [1]$$

Ellipse

$$x_0'=\frac{\sin a}{\sqrt{p}}[\cos(A'+v_{\prime\prime})+e\cos A']$$

$$y_0'=\frac{\sin b}{\sqrt{p}}[\cos(B'+v_{\prime\prime})+e\cos B']$$

$$z_0'=\frac{\sin c}{\sqrt{p}}[\cos(C'+v_{\prime\prime})+e\cos C']$$

Parabola

$$x_0'=\sqrt{\frac{2}{q}}\sin a\cos(A'+\tfrac{1}{2}v_{\prime\prime})\cos\tfrac{1}{2}v_{\prime\prime}$$

$$y_0'=\sqrt{\frac{2}{q}}\sin b\cos(B'+\tfrac{1}{2}v_{\prime\prime})\cos\tfrac{1}{2}v_{\prime\prime}$$

$$z_0'=\sqrt{\frac{2}{q}}\sin c\cos(C'+\tfrac{1}{2}v_{\prime\prime})\cos\tfrac{1}{2}v_{\prime\prime}$$

Circle

$$x_0'=\frac{\sin a\cos(A'+u_{\prime\prime})}{a^{1/2}}$$

$$y_0'=\frac{\sin b\cos(B'+u_{\prime\prime})}{a^{1/2}}$$

$$z_0'=\frac{\sin c\cos(C'+u_{\prime\prime})}{a^{1/2}}.$$

Compute $\rho_{\prime}$ and $\rho_{\prime\prime\prime}$ in the same manner as ρ_0 to free the intervals from aberration before computing the f and g. These values of $\rho_{\prime}$ and $\rho_{\prime\prime\prime}$ will also serve for correcting the observations in advance for parallax and aberration, if the preliminary orbit is sufficiently accurate for that purpose.

The adopted values of ρ_0, x_0', y_0', z_0' must now be written out to the number of decimals to which the orbit is to be computed, by adding ciphers if necessary.

$$\sigma_0=\rho_0\cos\delta_{\prime\prime};\quad x_0=\sigma_0\cos\alpha_{\prime\prime}-X_{\prime\prime};\quad y_0=\sigma_0\sin\alpha_{\prime\prime}-Y_{\prime\prime};\quad z_0=\sigma_0\tan\delta_{\prime\prime}-Z_{\prime\prime}$$

$$=\xi_0-X_{\prime\prime};\qquad =\eta_0-Y_{\prime\prime};\qquad =\zeta_0-Z_{\prime\prime}.$$

$$r_0^2=x_0^2+y_0^2+z_0^2;\qquad G_0^2=x_0'^2+y_0'^2+z_0'^2;\qquad \frac{1}{a}=\frac{2}{r_0}-G_0^2.$$

x_0, y_0, z_0; x_0', y_0, z_0' are the constants (or elements) of an artificial *initial* orbit which has ρ_0, x_0', y_0', z_0' in common with the preliminary orbit, but accurately represents $\alpha_{\prime\prime}$ and $\delta_{\prime\prime}$.

The eccentricity of the *initial* orbit, therefore, is not necessarily the same as that of the preliminary orbit. Nevertheless, if the latter was parabolic or circular

[1] With ρ_0 thus obtained from the *observed* time we may correct $t_{\prime\prime}$ for planetary aberration and with the reduced time $t_{\prime\prime}=t_{\prime\prime}-\alpha\rho_0$ ($\log\alpha=7.76128$) we may compute a second approximation of $v_{\prime\prime}$ and ρ_0. Then if the parallax and aberration have been eliminated, we may choose as the fundamental data the first approximation of ρ_0 together with the values of x', y', z' corresponding to the second approximation of $v_{\prime\prime}$, but if the corrections for parallax and aberration have been applied in advance, then all four fundamental data should correspond to the second approximation of $v_{\prime\prime}$. But, as stated above, these distinctions are not important. In any case x', y', z' are to be computed but once.

the *differential formulæ* for the correction of an initial parabolic or circular orbit will serve for the correction of the adopted fundamental data.

But in such cases one or more (for the circle) of the heliocentric velocities may first be changed arbitrarily so as to furnish a value of G_0 which will make $\frac{1}{a} = 0$, or $a = r_0$ and $r'_0 = 0$, so that the initial orbit is changed into a strictly parabolic or a strictly circular orbit, as the case may be. To produce an *initial* parabola it is most convenient to change z'_0 so as to satisfy the equation $x'^2_0 + y'^2_0 + z'^2_0 = \frac{2}{r_0}$. To produce an *initial* circle let $a = r_0$; $r'_0 = 0$ and compute x'_0, y'_0, z'_0 as in footnote 1, page 319. The practical gain arising from this arbitrary variation is that the computation of the f and g by their closed expressions is simplified since the application of the more complicated expressions for very nearly parabolic or for hyperbolic orbits is avoided. Nevertheless, to provide for the case that an arbitrary variation of the heliocentric velocities has not been made or that the required arbitrary change would materially alter the form of the initial orbit, the closed expressions for f and g in nearly parabolic and in hyperbolic orbits are included below.

II.

Compute $f_{,}$, $g_{,}$; $f_{,,,}$, $g_{,,,}$ in strict accordance with the ***adopted initial*** fundamental data or orbit.

If these may be computed from the f and g series, apply the proper expressions given in A VI.

If it be necessary to derive the f and g from closed expressions, compute, except for circular orbits,

$$\text{Epoch} = t_{,,}\,; \qquad r_0 r'_0 = x_0 x'_0 + y_0 y'_0 + z_0 z'_0\,; \qquad e \sin v_0 = r'_0 \sqrt{p}$$

$$\log k'' = 3.550\,006\,6\,; \qquad p = r_0^2\,(G_0^2 - r'^2_0)\,; \qquad e \cos v_0 = \frac{p}{r_0} - 1$$

$$\mu = k''\,a^{-\frac{3}{2}}\,; \qquad e^2 = 1 - \frac{p}{a},\ \textbf{check},$$

and then in the usual manner, except where special formulæ are given, according to the character of the *initial* orbit, where for the sake of uniformity $f_{,}$, $f_{,,,}$; $g_{,}$, $g_{,,,}$ are computed from the corresponding $\gamma_{,}$, $\gamma_{,,,}$ for all classes of orbits,[1] except for the circle:

Circle: $\quad 2\bar{g}_{,} = -\frac{\theta_{,,,}}{a^{3/2}}$, $2\bar{g}_{,,,} = \frac{\theta_{,}}{a^{3/2}}$; $f_{,} = \cos 2\bar{g}_{,}$, $f_{,,,} = \cos 2\bar{g}_{,,,}$; $g_{,} = a^{3/2} \sin 2\bar{g}_{,}$, $g_{,,,} = a^{3/2} \sin 2\bar{g}_{,,,}$.

Parabola: r_0, q, T, $r_{,}$, $r_{,,,}$; $\quad v_{,}$, $v_{,,,}$; $\quad \gamma_{,} = \sqrt{r_{,} - q} - \sqrt{r_0 - q}$; $\quad \gamma_{,,,} = \sqrt{r_{,,,} - q} - \sqrt{r_0 - q}$,

where the algebraical sign of each square root is the same as that of the corresponding $\tan \frac{1}{2} v$.

[1] But $f_{,}$, $f_{,,,}$; $g_{,}$, $g_{,,,}$ may also be computed directly from formulæ (8) and (9), page 248, **except for parabolic or nearly parabolic orbits for which** $g_{,}$, $g_{,,,}$ may be computed directly from formula (17), **page 249. Careful attention should be paid to the quadrants of the angles involved, particularly, if the observations are of different oppositions.**

Ellipse: r_0, E_0, M_0; $\mu, t_0, E_{,}, E_{,,,}$; $r_{,}, r_{,,,}$; $2g_{,} = E_{,} - E_0$, $2g_{,,,} = E_{,,,} - E_0$

$$\gamma_{,} = \sqrt{2a}\sin g_{,}\,, \quad \gamma_{,,,} = \sqrt{2a}\sin g_{,,,}\,.$$

Hyperbola: F_0, T; $F_{,}, F_{,,,}$; $v_{,}, v_{,,,}$; $r_{,}, r_{,,,}$;

$$\gamma_{,} = \sqrt{\frac{2r_{,}r_0}{p}}\sin\tfrac{1}{2}(v_{,} - v_0)\,, \quad \gamma_{,,,} = \sqrt{\frac{2r_{,,,}r_0}{p}}\sin\tfrac{1}{2}(v_{,,,} - v_0)\,.$$

Very nearly parabolic orbits: θ_0, P_1, P_3, T; $\varepsilon, \alpha, \beta$; $M_{,}, x_{,}, u_{,}, v_{,}, \theta_{,}, r_{,}$; $M_{,,,}, x_{,,,}, u_{,,,}, v_{,,,}, \theta_{,,,}, r_{,,,}$.

$$\gamma_{,} = \sqrt{\frac{2r_{,}r_0}{p}}\sin\tfrac{1}{2}(v_{,} - v_0)\,, \quad \gamma_{,,,} = \sqrt{\frac{2r_{,,,}r_0}{p}}\sin\tfrac{1}{2}(v_{,,,} - v_0)\,.$$

The usual formulæ for computing the true anomalies appear in A VIII, except for nearly parabolic orbits for which *cf.* OPPOLZER, Vol. I, pages 73, 75. The *radii vectors* may be computed from the polar equation $r = \frac{p}{1 + e\cos v}$, or for the ellipse from $r = a(1 - e\cos E)$, for the parabola from $r = q\sec^2\tfrac{1}{2}v$, etc.

For all, except circular orbits:

$$f_{,} = 1 - \frac{\gamma_{,}^2}{r_0}\,; \quad f_{,,,} = 1 - \frac{\gamma_{,,,}^2}{r_0}\,; \quad g_{,}^2 = [2r_{,}r_0 - p\gamma_{,}^2]\gamma_{,}^2\,; \quad g_{,,,}^2 = [2r_{,,,}r_0 - p\gamma_{,,,}^2]\gamma_{,,,}^2\,.$$

g_i in general is *positive* for dates *after* and *negative* for dates *before* $t_{,,} = t_0$, but the algebraical sign should be carefully checked by reference to the quadrants of the angles involved in formulæ (8), (9), (17), page 248 and 249.[1]

With x_0, y_0, z_0; x_0', y_0', z_0' and with $f_{,}, f_{,,,}$; $g_{,}, g_{,,,}$ as obtained either by series or closed expressions, compute the residuals for the first and third dates as in A VI, the corrections for parallax being applied on the basis of the values of $\rho_{,}$ and $\rho_{,,,}$ resulting from the computation of the places from the initial orbit.

III.

By the application of the proper differential formulæ determine such corrections to ρ_0, x_0', y_0', z_0' as will remove the residuals.

Choose the formulæ of group A [VII] or those given below in accordance with the criteria contained above, page 327, second paragraph.

For the improvement of an initial circular orbit the differential formulæ based on the closed expressions are given in A [VII b] and are not repeated below.

If the differential formulæ based on the closed expressions for the ∂f and ∂g be chosen, apply the formulæ designated 1) *from circular orbits* to the improvement of *initial circular* or *very nearly circular orbits*; those designated by 2) *from parabolic orbits* to the improvement of *initial parabolic*, *very nearly parabolic*, or *hyperbolic orbits*; those designated by 3) *from elliptic orbits* to the improvement of *initial elliptic orbits* in general.

1). *From initial circular orbits.*

Apply the formulæ of A [VII b].

[1] See footnote 1, page 330.

2). *From initial parabolic orbits.*

$$\cos\beta = \frac{\rho_0 - R_0 \cos\psi}{r_0}\,; \qquad \varphi_0 = \frac{x'_0 \xi_0 + y'_0 \eta_0 + z'_0 \zeta_0}{\rho_0}$$

$$\Phi_{,} = \varphi_0 + \frac{\sqrt{2}\cos\beta}{\gamma_{,}}\,; \qquad \Phi_{,,,} = \varphi_0 + \frac{\sqrt{2}\cos\beta}{\gamma_{,,,}}.$$

$$g_{x_{,}} = \frac{\gamma_{,}^3}{r_0 r_{,}}\left[x'_0 r_0 \gamma_{,} + \sqrt{2}\, x_0\right]; \quad f_{x_{,}} = f_{,}\frac{\xi_0}{\rho_0} + \frac{\cos\beta\,\gamma_{,}^2}{r_0^2} x_0 + g_{x_{,}} \Phi_{,}$$

$$g_{y_{,}} = \frac{\gamma_{,}^3}{r_0 r_{,}}\left[y'_0 r_0 \gamma_{,} + \sqrt{2}\, y_0\right]; \quad f_{y_{,}} = f_{,}\frac{\eta_0}{\rho_0} + \frac{\cos\beta\,\gamma_{,}^2}{r_0^2} y_0 + g_{y_{,}} \Phi_{,}$$

$$g_{z_{,}} = \frac{\gamma_{,}^3}{r_0 r_{,}}\left[z'_0 r_0 \gamma_{,} + \sqrt{2}\, z_0\right]; \quad f_{z_{,}} = f_{,}\frac{\zeta_0}{\rho_0} + \frac{\cos\beta\,\gamma_{,}^2}{r_0^2} z_0 + g_{z_{,}} \Phi_{,}$$

$$g_{x_{,,,}} = \frac{\gamma_{,,,}^3}{r_0 r_{,,,}}\left[x'_0 r_0 \gamma_{,,,} + \sqrt{2}\, x_0\right]; \quad f_{x_{,,,}} = f_{,,,}\frac{\xi_0}{\rho_0} + \frac{\cos\beta\,\gamma_{,,,}^2}{r_0^2} x_0 + g_{x_{,,,}} \Phi_{,,,}$$

$$g_{y_{,,,}} = \frac{\gamma_{,,,}^3}{r_0 r_{,,,}}\left[y'_0 r_0 \gamma_{,,,} + \sqrt{2}\, y_0\right]; \quad f_{y_{,,,}} = f_{,,,}\frac{\eta_0}{\rho_0} + \frac{\cos\beta\,\gamma_{,,,}^2}{r_0^2} y_0 + g_{y_{,,,}} \Phi_{,,,}$$

$$g_{z_{,,,}} = \frac{\gamma_{,,,}^3}{r_0 r_{,,,}}\left[z'_0 r_0 \gamma_{,,,} + \sqrt{2}\, z_0\right]; \quad f_{z_{,,,}} = f_{,,,}\frac{\zeta_0}{\rho^0} + \frac{\cos\beta\,\gamma_{,,,}^2}{r_0^2} z_0 + g_{z_{,,,}} \Phi_{,,,}.$$

$$A_{f_{,}} = \frac{1}{\rho_{,}}\left[\cos\alpha_{,} f_{y_{,}} - \sin\alpha_{,} f_{x_{,}}\right] + \cos\delta_{,}\frac{(p_\alpha \rho)_{,}}{\rho_{,}^2}$$

$$B_{f_{,}} = -\frac{1}{\rho_{,}}\left[\sin\delta_{,}(\sin\alpha_{,} f_{y_{,}} + \cos\alpha_{,} f_{x_{,}}) - \cos\delta_{,} f_{z_{,}}\right] + \frac{(p_\delta \rho)_{,}}{\rho_{,}^2}$$

$$A_{g_{,}} = \frac{1}{\rho_{,}}\left[\cos\alpha_{,} g_{y_{,}} - \sin\alpha_{,} g_{x_{,}}\right]$$

$$B_{g_{,}} = -\frac{1}{\rho_{,}}\left[\sin\delta_{,}(\sin\alpha_{,} g_{y_{,}} + \cos\alpha_{,} g_{x_{,}}) - \cos\delta_{,} g_{z_{,}}\right]; \qquad C_{,} = \frac{g_{,}}{\rho_{,}}$$

$$A_{f_{,,,}} = \frac{1}{\rho_{,,,}}\left[\cos\alpha_{,,,} f_{y_{,,,}} - \sin\alpha_{,,,} f_{x_{,,,}}\right] + \cos\delta_{,,,}\frac{(p_\alpha \rho)_{,,,}}{\rho_{,,,}^2}$$

$$B_{f_{,,,}} = -\frac{1}{\rho_{,,,}}\left[\sin\delta_{,,,}(\sin\alpha_{,,,} f_{y_{,,,}} + \cos\alpha_{,,,} f_{x_{,,,}}) - \cos\delta_{,,,} f_{z_{,,,}}\right] + \frac{(p_\delta \rho)_{,,,}}{\rho_{,,,}^2}$$

$$A_{g_{,,,}} = \frac{1}{\rho_{,,,}}\left[\cos\alpha_{,,,} g_{y_{,,,}} - \sin\alpha_{,,,} g_{x_{,,,}}\right]$$

$$B_{g_{,,,}} = -\frac{1}{\rho_{,,,}}\left[\sin\delta_{,,,}(\sin\alpha_{,,,} g_{y_{,,,}} + \cos\alpha_{,,,} g_{x_{,,,}}) - \cos\delta_{,,,} g_{z_{,,,}}\right]; \qquad C_{,,,} = \frac{g_{,,,}}{\rho_{,,,}}.$$

$$A = A_{f_{,}} A_{g_{,,,}} - A_{f_{,,,}} A_{g_{,}}.$$

$$a_1 = [\sin\alpha_{,} C_{,} A_{g_{,,,}} - \sin\alpha_{,,,} C_{,,,} A_{g_{,}}]\,; \qquad b_1 = -[\cos\alpha_{,} C_{,} A_{g_{,,,}} - \cos\alpha_{,,,} C_{,,,} A_{g_{,}}]$$

$$a_2 = [\sin\alpha_{,} C_{,} A_{f_{,,,}} - \sin\alpha_{,,,} C_{,,,} A_{f_{,}} + A x_0]\,; \qquad b_2 = -[\cos\alpha_{,} C_{,} A_{f_{,,,}} - \cos\alpha_{,,,} C_{,,,} A_{f_{,}} - A y_0]$$

$$a_3 = -[\sin\delta_{,} \cos\alpha_{,} C_{,} - x_0 B_{g_{,}}]\,; \qquad b_3 = [\sin\delta_{,} \sin\alpha_{,} C_{,} - y_0 B_{g_{,}}]$$

$$a_4 = [\sin\delta_{,,,} \cos\alpha_{,,,} C_{,,,} - x_0 B_{g_{,,,}}]\,; \qquad b_4 = [\sin\delta_{,,,} \sin\alpha_{,,,} C_{,,,} - y_0 B_{g_{,,,}}]$$

$$c_2 = A z_0\,; \qquad d_1 = -A\,; \qquad e_1 = A_{g_{,}} \partial\alpha_{,,,} - A_{g_{,,,}} \partial\alpha_{,}$$

$$c_3 = -[\cos\delta_{,} C_{,} + z_0 B_{g_{,}}]\,; \qquad d_3 = B_{f_{,}}\,; \qquad e_2 = A_{f_{,}} \partial\alpha_{,,,} - A_{f_{,,,}} \partial\alpha_{,}$$

$$c_4 = -[\cos\delta_{,,,} C_{,,,} + z_0 B_{g_{,,,}}]\,; \qquad d_4 = B_{f_{,,,}}\,; \qquad e_3 = -\partial\delta_{,}$$

$$e_4 = -\partial\delta_{,,,}.$$

$$A_{f_{\prime}} = \frac{1}{\rho_{\prime}}[\cos\alpha_{\prime} f_{y_{\prime}} - \sin\alpha_{\prime} f_{x_{\prime}}] + \cos\delta_{\prime}\frac{(p_{\alpha}\rho)_{\prime}}{\rho_{\prime}^{2}}\,; \quad A_{f_{\prime\prime\prime}} = \frac{1}{\rho_{\prime\prime\prime}}[\cos\alpha_{\prime\prime\prime} f_{y_{\prime\prime\prime}} - \sin\alpha_{\prime\prime\prime} f_{x_{\prime\prime\prime}}] + \cos\delta_{\prime\prime\prime}\frac{(p_{\alpha}\rho)_{\prime\prime\prime}}{\rho_{\prime\prime\prime}^{2}}$$

$$A_{g_{\prime}} = \frac{1}{\rho_{\prime}}[\cos\alpha_{\prime} g_{y_{\prime}} - \sin\alpha_{\prime} g_{x_{\prime}}]\,; \quad A_{g_{\prime\prime\prime}} = \frac{1}{\rho_{\prime\prime\prime}}[\cos\alpha_{\prime\prime\prime} g_{y_{\prime\prime\prime}} - \sin\alpha_{\prime\prime\prime} g_{x_{\prime\prime\prime}}]$$

$$A_{m_{\prime}} = \frac{1}{\rho_{\prime}}[\cos\alpha_{\prime} m_{y_{\prime}} - \sin\alpha_{\prime} m_{x_{\prime}}]\,; \quad A_{m_{\prime\prime\prime}} = \frac{1}{\rho_{\prime\prime\prime}}[\cos\alpha_{\prime\prime\prime} m_{y_{\prime\prime\prime}} - \sin\alpha_{\prime\prime\prime} m_{x_{\prime\prime\prime}}].$$

$$B_{f_{\prime}} = -\frac{1}{\rho_{\prime}}[\sin\delta_{\prime}(\sin\alpha_{\prime} f_{y_{\prime}} + \cos\alpha_{\prime} f_{x_{\prime}}) - \cos\delta_{\prime} f_{z_{\prime}}] + \frac{(p_{\delta}\rho)_{\prime}}{\rho_{\prime}^{2}}$$

$$B_{g_{\prime}} = -\frac{1}{\rho_{\prime}}[\sin\delta_{\prime}(\sin\alpha_{\prime} g_{y_{\prime}} + \cos\alpha_{\prime} g_{x_{\prime}}) - \cos\delta_{\prime} g_{z_{\prime}}]$$

$$B_{m_{\prime}} = -\frac{1}{\rho_{\prime}}[\sin\delta_{\prime}(\sin\alpha_{\prime} m_{y_{\prime}} + \cos\alpha_{\prime} m_{x_{\prime}}) - \cos\delta_{\prime} m_{z_{\prime}}]$$

$$B_{f_{\prime\prime\prime}} = -\frac{1}{\rho_{\prime\prime\prime}}[\sin\delta_{\prime\prime\prime}(\sin\alpha_{\prime\prime\prime} f_{y_{\prime\prime\prime}} + \cos\alpha_{\prime\prime\prime} f_{x_{\prime\prime\prime}}) - \cos\delta_{\prime\prime\prime} f_{z_{\prime\prime\prime}}] + \frac{(p_{\delta}\rho)_{\prime\prime\prime}}{\rho_{\prime\prime\prime}^{2}}$$

$$B_{g_{\prime\prime\prime}} = -\frac{1}{\rho_{\prime\prime\prime}}[\sin\delta_{\prime\prime\prime}(\sin\alpha_{\prime\prime\prime} g_{y_{\prime\prime\prime}} + \cos\alpha_{\prime\prime\prime} g_{x_{\prime\prime\prime}}) - \cos\delta_{\prime\prime\prime} g_{z_{\prime\prime\prime}}]$$

$$B_{m_{\prime\prime\prime}} = -\frac{1}{\rho_{\prime\prime\prime}}[\sin\delta_{\prime\prime\prime}(\sin\alpha_{\prime\prime\prime} m_{y_{\prime\prime\prime}} + \cos\alpha_{\prime\prime\prime} m_{x_{\prime\prime\prime}}) - \cos\delta_{\prime\prime\prime} m_{z_{\prime\prime\prime}}]$$

$$C_{\prime} = \frac{g_{\prime}}{\rho_{\prime}}\,; \quad C_{\prime\prime\prime} = \frac{g_{\prime\prime\prime}}{\rho_{\prime\prime\prime}}.$$

Then $\partial\rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$ are obtained by solving four equations of the form

$$\alpha_i\,\partial\rho_0 + \beta_i\,\partial x_0' + \gamma_i\,\partial y_0' + \delta_i\,\partial z_0' = \nu_i\,, \quad i = 1, 2, 3, 4,$$

where

$$\begin{array}{lll}
\nu_1 = \partial_{\prime}\alpha_{\prime}\;; & \alpha_1 = A_{f_{\prime}}\;; & \beta_1 = -[\sin\alpha_{\prime}\,C_{\prime} - x_0 A_{g_{\prime}} - 2x_0' A_{m_{\prime}}] \\
\nu_2 = \partial\delta_{\prime}\;; & \alpha_2 = B_{f_{\prime}}\;; & \beta_2 = -[\sin\delta_{\prime}\cos\alpha_{\prime}\,C_{\prime} - x_0 B_{g_{\prime}} - 2x_0' B_{m_{\prime}}] \\
\nu_3 = \partial_{\prime}\alpha_{\prime\prime\prime}\;; & \alpha_3 = A_{f_{\prime\prime\prime}}\;; & \beta_3 = -[\sin\alpha_{\prime\prime\prime}\,C_{\prime\prime\prime} - x_0 A_{g_{\prime\prime\prime}} - 2x_0' A_{m_{\prime\prime\prime}}] \\
\nu_4 = \partial\delta_{\prime\prime\prime}\;; & \alpha_4 = B_{f_{\prime\prime\prime}}\;; & \beta_4 = -[\sin\delta_{\prime\prime\prime}\cos\alpha_{\prime\prime\prime}\,C_{\prime\prime\prime} - x_0 B_{g_{\prime\prime\prime}} - 2x_0' B_{m_{\prime\prime\prime}}]
\end{array}$$

$$\begin{array}{ll}
\delta_1 = [z_0 A_{g_{\prime}} + 2z_0' A_{m_{\prime}}]\;; & \gamma_1 = [\cos\alpha_{\prime}\,C_{\prime} + y_0 A_{g_{\prime}} + 2y_0' A_{m_{\prime}}] \\
\delta_2 = [\cos\delta_{\prime}\,C_{\prime} + z_0 B_{g_{\prime}} + 2z_0' B_{m_{\prime}}]\;; & \gamma_2 = -[\sin\delta_{\prime}\sin\alpha_{\prime}\,C_{\prime} - y_0 B_{g_{\prime}} - 2y_0' B_{m_{\prime}}] \\
\delta_3 = [z_0 A_{g_{\prime\prime\prime}} + 2z_0' A_{m_{\prime\prime\prime}}]\;; & \gamma_3 = [\cos\alpha_{\prime\prime\prime}\,C_{\prime\prime\prime} + y_0 A_{g_{\prime\prime\prime}} + 2y_0' A_{m_{\prime\prime\prime}}] \\
\delta_4 = [\cos\delta_{\prime\prime\prime}\,C_{\prime\prime\prime} + z_0 B_{g_{\prime\prime\prime}} + 2z_0' B_{m_{\prime\prime\prime}}]\;; & \gamma_4 = -[\sin\delta_{\prime\prime\prime}\sin\alpha_{\prime\prime\prime}\,C_{\prime\prime\prime} - y_0 B_{g_{\prime\prime\prime}} - 2y_0' B_{m_{\prime\prime\prime}}].
\end{array}$$

Of the available methods of solution the computer will choose the one which admits of the most convenient determination of the unknowns on the basis of the numerical values of the coefficients. Then

$$x = x_0 + \xi_0\frac{\partial\rho_0}{\rho_0},\; y = y_0 + \eta_0\frac{\partial\rho_0}{\rho_0},\; z = z_0 + \zeta_0\frac{\partial\rho_0}{\rho_0}\,;\; x' = x_0' + \partial x_0',\; y' = y_0' + \partial y_0',\; z' = z_0' + \partial z_0'.$$

IV.

x, y, z; x', y', z' are the new values of the heliocentric coördinates and velocities from which the residuals are computed as a check by A VI with new values of f and g, and then the elements by A VIII.

TABLES FROM WHICH $z = \frac{\rho}{R}$ MAY BE INTERPOLATED WITH THE ARGUMENTS ψ AND $\frac{1}{m}$ IN THE DETERMINATION OF PRELIMINARY ORBITS WITHOUT ASSUMPTION REGARDING THE ECCENTRICITY.

In Part I of this volume I have pointed out that the geocentric distance which is defined by the positive real roots of the equation of the seventh degree, to which the orbit problem reduces if no assumption is made regarding the eccentricity, may be obtained directly from an extension of v. OPPOLZER'S Table XIIIa, *Bahnbestimmung*, Vol. I. Such a preliminary extension has been made in the form of the Tables which follow. They serve for the interpolation of $z = \frac{\rho}{R}$ with the arguments ψ and $\frac{1}{m}$ in general orbits; for determining the feasibility of a solution with hypothesis regarding the eccentricity (*cf.* pages 290, 291); and for determining the accuracy of the adopted solution (*cf.* pages 270–278).

The tabular values were computed under my direction by students of the University, Miss SOPHIA H. LEVY, a graduate student, performing by far the major portion of the work. Miss LEVY has not only completed the computations commenced by a number of students, but has also checked most of the work and read the proofs.

The equation from which the Tables were constructed is (*cf.* equation (27), page 240, where $m = (m)$, if parallax be neglected, and where the absolute term m^2 is the square of $-m$)

$$(z^2 - 2z\cos\psi + 1)^3 (z - m)^2 - m^2 = 0.$$

For the computation of the tabular values of $\frac{1}{m}$ this equation was written in the form

$$a = [z(z - 2\cos\psi) + 1]^{-\frac{3}{2}}; \qquad \frac{1}{m} = \frac{1-a}{z}.$$

The values of $\frac{1}{m}$ were computed directly from $z = 0.01$ to $z = 4.00$ for every 10 degrees of ψ. This computation was performed to seven figures and cut down to six figures in the Tables.

It is evident that for a particular value of ψ the second differences of $z^2 - 2z\cos\psi + 1$ are constant, so that this function may be built up by differences for equidistant values of z and then checked at intervals by direct computation.

For the intermediate values $\psi = 5°$, $15°$, etc., the tabular values of $\frac{1}{m}$ were either computed directly or obtained by interpolation from the values for $\psi = 10°$, $20°$, etc., taken to six figures if the interpolation could be performed with sufficient accuracy. The last figure of the tabulated values of $\frac{1}{m}$, therefore, is not absolutely

reliable throughout the Tables and the computer is advised to check the interpolated value of z in each case by substitution in the equation and to correct it, if necessary, by the differential formula (*cf. Synopsis of Formulæ*, A IVc).

To interpolate z (of which there can be at most two values corresponding to given arguments ψ and $\frac{1}{m}$) it is most convenient to first enter the ψ column with its given value and to determine by inspection (along the horizontal lines) the page and region of the given value of $\frac{1}{m}$. Two such regions will be found in case of a double solution. Then the values of $\frac{1}{m}$ corresponding to the given ψ should be accurately interpolated from two or more of the vertical columns, according to the run of the differences.

The interpolated values of $\frac{1}{m}$ furnish a small table with z as argument, from which the required value of z may be interpolated with the given value of $\frac{1}{m}$. In the greater part of the Tables, interpolation to second differences will prove sufficient to obtain the definitive value of z and, therefore, of the geocentric distance. The laborious trials hitherto necessary are thus avoided.

If it is found by inspection that the required value (or values) of z falls within a part of the Table where the interpolation requires the use of more than the second differences, it is more convenient to perform the interpolation approximately and to correct the approximate value of z by the differential formula. A single correction is practically always sufficient.

The author hopes that ultimately means may be placed at his disposal to enable him to further extend the Table so that interpolation by proportional parts will prove sufficient in all parts of the Tables.

The Tables are applicable to all orbit methods in which the solution of the problem has been reduced to the solution of the equation on which the Tables are based. The importance of the Tables as a direct means of obtaining the geocentric distance, therefore, can not be overestimated.

Examples of the use of the Tables for the determination of the geocentric distance, for testing the accuracy of the solution (*cf.* page 270 *et seq.*), and for testing the feasibility of a solution with hypothesis regarding the eccentricity (*cf.* pages 290, 291) will be found in Part 8 of this volume.

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	z / ψ
0°	− 3.061 01	− 3.124 12	− 3.189 42	− 3.257 02	− 3.327 02	− 3.399 54	− 3.474 70	− 3.552 64	− 3.633 50	− 3.717 42	− 3.804 57	0°
5	− 3.049 00	− 3.111 50	− 3.176 14	− 3.243 02	− 3.312 28	− 3.384 00	− 3.458 30	− 3.535 33	− 3.615 22	− 3.698 11	− 3.784 14	5
10	− 3.013 11	− 3.073 73	− 3.136 41	− 3.201 22	− 3.268 24	− 3.337 60	− 3.409 39	− 3.483 74	− 3.560 76	− 3.640 59	− 3.723 38	10
15	− 2.953 59	− 3.011 24	− 3.070 70	− 3.132 11	− 3.195 52	− 3.261 05	− 3.328 76	− 3.398 78	− 3.471 19	− 3.546 11	− 3.623 64	15
20	− 2.871 06	− 2.924 60	− 2.979 73	− 3.036 58	− 3.095 14	− 3.155 52	− 3.217 77	− 3.282 00	− 3.348 25	− 3.416 63	− 3.487 22	20
25	− 2.766 15	− 2.814 67	− 2.864 53	− 2.915 76	− 2.968 42	− 3.022 54	− 3.078 20	− 3.135 42	− 3.194 28	− 3.254 82	− 3.317 10	25
30	− 2.639 81	− 2.682 56	− 2.726 32	− 2.771 14	− 2.817 06	− 2.864 08	− 2.912 24	− 2.961 57	− 3.012 10	− 3.063 86	− 3.116 88	30
35	− 2.493 11	− 2.529 52	− 2.566 62	− 2.604 45	− 2.643 00	− 2.682 34	− 2.722 43	− 2.763 29	− 2.804 93	− 2.847 36	− 2.890 59	35
40	− 2.327 36	− 2.357 00	− 2.387 07	− 2.417 58	− 2.448 48	− 2.479 81	− 2.511 56	− 2.543 70	− 2.576 25	− 2.609 20	− 2.642 54	40
45	− 2.143 92	− 2.166 66	− 2.189 57	− 2.212 63	− 2.235 84	− 2.259 15	− 2.282 60	− 2.306 12	− 2.329 72	− 2.353 39	− 2.377 12	45
50	− 1.944 34	− 1.960 26	− 1.976 11	− 1.991 88	− 2.007 56	− 2.023 14	− 2.038 59	− 2.053 88	− 2.069 01	− 2.083 95	− 2.098 68	50
55	− 1.730 29	− 1.739 66	− 1.748 78	− 1.757 65	− 1.766 27	− 1.774 61	− 1.782 65	− 1.790 37	− 1.797 76	− 1.804 78	− 1.811 45	55
60	− 1.503 57	− 1.506 86	− 1.509 78	− 1.512 35	− 1.514 56	− 1.516 39	− 1.517 82	− 1.518 84	− 1.519 44	− 1.519 61	− 1.519 33	60
65	− 1.266 06	− 1.263 89	− 1.261 33	− 1.258 37	− 1.255 01	− 1.251 25	− 1.247 04	− 1.242 43	− 1.237 37	− 1.231 87	− 1.225 92	65
70	− 1.019 65	− 1.012 86	− 1.005 68	− 0.998 12	− 0.990 20	− 0.981 87	− 0.973 16	− 0.964 05	− 0.954 56	− 0.944 68	− 0.934 41	70
75	− 0.766 31	− 0.755 86	− 0.745 06	− 0.733 94	− 0.722 54	− 0.710 80	− 0.698 75	− 0.686 41	− 0.673 77	− 0.660 83	− 0.647 62	75
80	− 0.508 06	− 0.494 99	− 0.481 67	− 0.468 12	− 0.454 38	− 0.440 42	− 0.426 26	− 0.411 94	− 0.397 42	− 0.382 75	− 0.367 92	80
85	− 0.246 97	− 0.232 36	− 0.217 64	− 0.202 82	− 0.187 93	− 0.172 95	− 0.157 91	− 0.142 82	− 0.127 68	− 0.112 51	− 0.097 33	85
90	+ 0.015 02	+ 0.029 99	+ 0.044 95	+ 0.059 88	+ 0.074 77	+ 0.089 60	+ 0.104 36	+ 0.119 04	+ 0.133 65	+ 0.148 15	+ 0.162 54	90
95	+ 0.275 85	+ 0.290 05	+ 0.304 14	+ 0.318 05	+ 0.331 81	+ 0.345 39	+ 0.358 78	+ 0.371 99	+ 0.385 00	+ 0.397 80	+ 0.410 39	95
100	+ 0.533 58	+ 0.545 92	+ 0.558 05	+ 0.569 89	+ 0.581 48	+ 0.592 80	+ 0.603 84	+ 0.614 60	+ 0.625 09	+ 0.635 30	+ 0.645 21	100
105	+ 0.786 27	+ 0.795 75	+ 0.804 91	+ 0.813 71	+ 0.822 22	+ 0.830 38	+ 0.838 21	+ 0.845 71	+ 0.852 89	+ 0.859 75	+ 0.866 28	105
110	+ 1.032 09	+ 1.037 78	+ 1.043 09	+ 1.048 00	+ 1.052 62	+ 1.056 87	+ 1.060 77	+ 1.064 34	+ 1.067 57	+ 1.070 49	+ 1.073 08	110
115	+ 1.269 28	+ 1.270 34	+ 1.271 05	+ 1.271 39	+ 1.271 45	+ 1.271 17	+ 1.270 58	+ 1.269 70	+ 1.268 49	+ 1.267 02	+ 1.265 28	115
120	+ 1.496 11	+ 1.491 89	+ 1.487 40	+ 1.482 64	+ 1.477 62	+ 1.472 34	+ 1.466 84	+ 1.461 11	+ 1.455 17	+ 1.449 03	+ 1.442 70	120
125	+ 1.710 95	+ 1.701 00	+ 1.690 87	+ 1.680 60	+ 1.670 16	+ 1.659 60	+ 1.648 92	+ 1.638 14	+ 1.627 25	+ 1.616 29	+ 1.605 26	125
130	+ 1.912 34	+ 1.896 34	+ 1.880 30	+ 1.864 28	+ 1.848 25	+ 1.832 26	+ 1.816 30	+ 1.800 38	+ 1.784 52	+ 1.768 72	+ 1.752 98	130
135	+ 2.098 90	+ 2.076 70	+ 2.054 66	+ 2.032 82	+ 2.011 18	+ 1.989 77	+ 1.968 55	+ 1.947 57	+ 1.926 81	+ 1.906 27	+ 1.885 96	135
140	+ 2.269 34	+ 2.240 97	+ 2.213 01	+ 2.185 47	+ 2.158 36	+ 2.131 66	+ 2.105 37	+ 2.079 51	+ 2.054 05	+ 2.028 99	+ 2.004 33	140
145	+ 2.422 48	+ 2.388 17	+ 2.354 55	+ 2.321 59	+ 2.289 26	+ 2.257 57	+ 2.226 51	+ 2.196 06	+ 2.166 20	+ 2.136 94	+ 2.108 24	145
150	+ 2.557 32	+ 2.517 48	+ 2.478 59	+ 2.440 60	+ 2.403 48	+ 2.367 21	+ 2.331 77	+ 2.297 14	+ 2.263 29	+ 2.230 20	+ 2.197 85	150
155	+ 2.672 96	+ 2.628 16	+ 2.584 54	+ 2.542 05	+ 2.500 65	+ 2.460 31	+ 2.420 98	+ 2.382 68	+ 2.345 32	+ 2.308 89	+ 2.273 35	155
160	+ 2.768 62	+ 2.719 56	+ 2.671 89	+ 2.625 55	+ 2.580 51	+ 2.536 72	+ 2.494 12	+ 2.452 68	+ 2.412 35	+ 2.373 10	+ 2.334 88	160
165	+ 2.843 68	+ 2.791 18	+ 2.740 24	+ 2.690 80	+ 2.642 84	+ 2.596 28	+ 2.551 04	+ 2.507 11	+ 2.464 43	+ 2.422 93	+ 2.382 59	165
170	+ 2.897 61	+ 2.842 58	+ 2.789 26	+ 2.737 58	+ 2.687 48	+ 2.638 88	+ 2.591 74	+ 2.546 00	+ 2.501 58	+ 2.458 47	+ 2.416 58	170
175	+ 2.930 10	+ 2.873 52	+ 2.818 76	+ 2.765 71	+ 2.714 31	+ 2.664 48	+ 2.616 17	+ 2.569 34	+ 2.523 87	+ 2.479 78	+ 2.436 94	175
180	+ 2.940 96	+ 2.883 87	+ 2.828 61	+ 2.775 09	+ 2.723 25	+ 2.673 01	+ 2.624 32	+ 2.577 10	+ 2.531 29	+ 2.486 85	+ 2.443 71	180
ψ / z	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	ψ / z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20	0.21	z / ψ
0°	− 3.804 57	− 3.895 10	− 3.989 20	− 4.087 06	− 4.188 89	− 4.294 89	− 4.405 31	− 4.520 39	− 4.640 40	− 4.765 62	− 4.896 37	0°
5	− 3.784 14	− 3.873 50	− 3.966 33	− 4.062 84	− 4.163 21	− 4.267 66	− 4.376 40	− 4.489 68	− 4.607 75	− 4.730 89	− 4.859 38	5
10	− 3.723 38	− 3.809 24	− 3.898 35	− 3.990 87	− 4.086 97	− 4.186 84	− 4.290 68	− 4.398 70	− 4.511 11	− 4.628 17	− 4.750 12	10
15	− 3.623 64	− 3.703 92	− 3.787 06	− 3.873 21	− 3.962 51	− 4.055 10	− 4.151 15	− 4.250 82	− 4.354 30	− 4.461 77	− 4.573 43	15
20	− 3.487 [illegible]	− 3.560 11	− 3.635 38	− 3.713 16	− 3.793 54	− 3.876 62	− 3.962 52	− 4.051 37	− 4.143 29	− 4.238 41	− 4.336 87	20
25	− 3.317 10	− 3.381 18	− 3.447 13	− 3.515 01	− 3.584 88	− 3.656 81	− 3.730 87	− 3.807 15	− 3.885 70	− 3.966 61	− 4.049 96	25
30	− 3.116 88	− 3.171 19	− 3.226 83	− 3.283 82	− 3.342 19	− 3.401 99	− 3.463 23	− 3.525 95	− 3.590 19	− 3.655 97	− 3.723 32	30
35	− 2.890 50	− 2.934 63	− 2.979 48	− 3.025 15	− 3.071 63	− 3.118 95	− 3.167 09	− 3.216 07	− 3.265 86	− 3.316 48	− 3.367 92	35
40	− 2.641 54	− 2.676 24	− 2.710 32	− 2.744 75	− 2.779 52	− 2.814 61	− 2.850 01	− 2.885 70	− 2.921 66	− 2.957 87	− 2.994 30	40
45	− 2.377 12	− 2.400 86	− 2.424 62	− 2.448 36	− 2.472 07	− 2.495 70	− 2.519 25	− 2.542 70	− 2.565 89	− 2.589 11	− 2.612 03	45
50	− 2.098 68	− 2.113 [illegible]	− 2.127 44	− 2.141 42	− 2.155 10	− 2.168 45	− 2.181 46	− 2.194 09	− 2.206 31	− 2.218 10	− 2.229 44	50
55	− 1.811 45	− 1.817 73	− 1.823 59	− 1.829 03	− 1.834 00	− 1.838 57	− 1.842 54	− 1.846 05	− 1.849 05	− 1.851 48	− 1.853 35	55
60	− 1.519 33	− 1.518 59	− 1.517 37	− 1.515 67	− 1.513 47	− 1.510 76	− 1.507 52	− 1.503 76	− 1.499 44	− 1.494 58	− 1.489 15	60
65	− 1.225 92	− 1.219 52	− 1.212 65	− 1.205 33	− 1.197 53	− 1.189 26	− 1.180 52	− 1.171 31	− 1.161 62	− 1.151 46	− 1.140 82	65
70	− 0.934 41	− 0.923 76	− 0.912 72	− 0.901 30	− 0.889 51	− 0.877 34	− 0.864 81	− 0.851 91	− 0.838 65	− 0.825 05	− 0.811 11	70
75	− 0.647 62	− 0.634 13	− 0.620 37	− 0.606 35	− 0.592 08	− 0.577 57	− 0.562 85	− 0.547 87	− 0.532 6[illegible]	− 0.517 31	− 0.501 74	75
80	− 0.367 [illegible]	− 0.352 95	− 0.337 85	− 0.322 62	− 0.307 28	− 0.291 85	− 0.276 33	− 0.260 73	− 0.245 08	− 0.229 37	− 0.213 62	80
85	− 0.097 33	− 0.082 14	− 0.066 95	− 0.051 78	− 0.036 63	− 0.021 53	− 0.006 48	+ 0.008 52	+ 0.023 44	+ 0.038 29	+ 0.053 04	85
90	+ 0.162 54	+ 0.176 81	+ 0.190 96	+ 0.204 97	+ 0.218 83	+ 0.232 54	+ 0.246 09	+ 0.259 46	+ 0.272 66	+ 0.285 67	+ 0.298 41	90
95	+ 0.410 3[illegible]	+ 0.422 75	+ 0.434 89	+ 0.446 80	+ 0.458 48	+ 0.469 9[illegible]	+ 0.481 08	+ 0.492 04	+ 0.502 73	+ 0.513 16	+ 0.523 34	95
100	+ 0.645 21	+ 0.654 84	+ 0.664 1[illegible]	+ 0.673 22	+ 0.681 97	+ 0.690 44	+ 0.698 61	+ 0.706 49	+ 0.714 08	+ 0.721 3[illegible]	+ 0.728 40	100
105	+ 0.866 [illegible]	+ 0.872 40	+ 0.878 30	+ 0.883 98	+ 0.889 25	+ 0.894 23	+ 0.898 90	+ 0.903 28	+ 0.907 36	+ 0.911 16	+ 0.914 6[illegible]	105
110	+ 1.073 08	+ 1.075 37	+ 1.077 35	+ 1.079 04	+ 1.080 44	+ 1.081 50	+ 1.082 40	+ 1.082 98	+ 1.083 3[illegible]	+ 1.083 38	+ 1.083 21	110
115	+ 1.265 78	+ 1.263 [illegible]	+ 1.261 13	+ 1.258 53	+ 1.255 80	+ 1.252 85	+ 1.249 68	+ 1.246 31	+ 1.242 73	+ 1.238 98	+ 1.235 05	115
120	+ 1.443 70	+ 1.436 19	+ 1.429 41	+ 1.422 68	+ 1.415 70	+ 1.408 58	+ 1.401 34	+ 1.393 97	+ 1.386 50	+ 1.378 93	+ 1.371 2[illegible]	120
125	+ 1.605 36	+ 1.594 16	+ 1.583 00	+ 1.571 80	+ 1.560 56	+ 1.549 30	+ 1.538 02	+ 1.526 72	+ 1.515 42	+ 1.504 12	+ 1.492 84	125
130	+ 1.752 98	+ 1.737 3[illegible]	+ 1.721 74	+ 1.706 25	+ 1.690 85	+ 1.675 56	+ 1.660 36	+ 1.645 27	+ 1.630 30	+ 1.615 44	+ 1.600 70	130
135	+ 1.885 96	+ 1.865 88	+ 1.846 04	+ 1.826 42	+ 1.807 06	+ 1.787 90	+ 1.768 99	+ 1.750 33	+ 1.731 90	+ 1.713 70	+ 1.695 74	135
140	+ 2.004 33	+ 1.980 0[illegible]	+ 1.956 1[illegible]	+ 1.932 71	+ 1.909 60	+ 1.886 87	+ 1.864 52	+ 1.842 53	+ 1.820 90	+ 1.799 63	+ 1.778 71	140
145	+ 2.108 24	+ 2.080 10	+ 2.052 52	+ 2.025 48	+ 1.998 76	+ 1.972 96	+ 1.947 46	+ 1.922 46	+ 1.897 95	+ 1.873 90	+ 1.850 32	145
150	+ 2.197 85	+ 2.166 24	+ 2.135 32	+ 2.105 00	+ 2.075 53	+ 2.046 61	+ 2.018 33	+ 1.990 66	+ 1.963 59	+ 1.937 10	+ 1.911 18	150
155	+ 2.273 35	+ 2.238 60	+ 2.204 88	+ 2.171 88	+ 2.139 67	+ 2.108 23	+ 2.077 55	+ 2.047 58	+ 2.018 32	+ 1.989 73	+ 1.961 81	155
160	+ 2.334 88	+ 2.297 67	+ 2.261 43	+ 2.226 12	+ 2.191 71	+ 2.158 18	+ 2.125 49	+ 2.093 62	+ 2.062 54	+ 2.032 22	+ 2.002 64	160
165	+ 2.382 59	+ 2.343 35	+ 2.305 19	+ 2.268 06	+ 2.231 91	+ 2.196 73	+ 2.162 46	+ 2.129 10	+ 2.096 64	+ 2.065 11	+ 2.034 03	165
170	+ 2.416 58	+ 2.375 87	+ 2.336 32	+ 2.297 86	+ 2.260 47	+ 2.224 09	+ 2.188 70	+ 2.154 25	+ 2.120 72	+ 2.088 07	+ 2.056 27	170
175	+ 2.436 94	+ 2.395 35	+ 2.354 95	+ 2.315 69	+ 2.277 55	+ 2.240 45	+ 2.204 38	+ 2.169 28	+ 2.135 12	+ 2.101 89	+ 2.069 53	175
180	+ 2.443 71	+ 2.401 83	+ 2.361 15	+ 2.321 63	+ 2.283 23	+ 2.245 89	+ 2.209 58	+ 2.174 27	+ 2.139 92	+ 2.106 48	+ 2.073 93	180
ψ / z	0.11	0.12	0.13	0.14	0.15	0.16	0.17	0.18	0.19	0.20	0.21	z \ ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.30	0.31	z / ψ
0°	− 4.896 37	− 5.032 96	− 5.175 75	− 5.325 12	− 5.481 48	− 5.645 27	− 5.816 97	− 5.997 09	− 6.186 17	− 6.384 83	− 6.593 73	0°
5	− 4.859 38	− 4.993 55	− 5.133 73	− 5.280 27	− 5.433 58	− 5.594 06	− 5.762 16	− 5.938 39	− 6.123 25	− 6.317 31	− 6.521 19	5
10	− 4.750 12	− 4.877 24	− 5.009 82	− 5.148 17	− 5.292 62	− 5.443 53	− 5.601 28	− 5.766 28	− 5.938 98	− 6.119 84	− 6.309 38	10
15	− 4.573 43	− 4.689 50	− 4.810 20	− 4.935 78	− 5.066 47	− 5.202 56	− 5.344 33	− 5.492 08	− 5.646 16	− 5.806 87	− 5.974 61	15
20	− 4.336 87	− 4.438 82	− 4.544 41	− 4.653 80	− 4.767 15	− 4.884 65	− 5.006 46	− 5.132 80	− 5.263 85	− 5.399 84	− 5.540 96	20
25	− 4.049 96	− 4.135 82	− 4.224 29	− 4.315 44	− 4.409 38	− 4.506 18	− 4.605 93	− 4.708 73	− 4.814 68	− 4.923 87	− 5.036 39	25
30	− 3.723 32	− 3.792 28	− 3.862 86	− 3.935 10	− 4.009 02	− 4.084 65	− 4.162 01	− 4.241 11	− 4.321 98	− 4.404 63	− 4.489 07	30
35	− 3.367 92	− 3.420 18	− 3.473 23	− 3.527 07	− 3.581 69	− 3.637 07	− 3.693 18	− 3.750 00	− 3.807 52	− 3.865 68	− 3.924 46	35
40	− 2.994 30	− 3.030 92	− 3.067 70	− 3.104 62	− 3.141 63	− 3.178 70	− 3.215 80	− 3.252 88	− 3.289 89	− 3.326 79	− 3.363 52	40
45	− 2.612 03	− 2.634 72	− 2.657 13	− 2.679 24	− 2.701 01	− 2.722 38	− 2.743 34	− 2.763 83	− 2.783 81	− 2.803 24	− 2.822 07	45
50	− 2.229 44	− 2.240 28	− 2.250 61	− 2.260 39	− 2.269 59	− 2.278 19	− 2.286 14	− 2.293 42	− 2.300 01	− 2.305 86	− 2.310 94	50
55	− 1.853 35	− 1.854 63	− 1.855 31	− 1.855 36	− 1.854 76	− 1.853 50	− 1.851 56	− 1.848 94	− 1.845 59	− 1.841 52	− 1.836 71	55
60	− 1.489 15	− 1.483 15	− 1.476 58	− 1.469 42	− 1.461 66	− 1.453 31	− 1.444 36	− 1.434 81	− 1.424 64	− 1.413 87	− 1.402 49	60
65	− 1.140 82	− 1.129 70	− 1.118 11	− 1.106 06	− 1.093 54	− 1.080 56	− 1.067 12	− 1.053 23	− 1.038 90	− 1.024 14	− 1.008 96	65
70	− 0.811 109	− 0.796 834	− 0.782 233	− 0.767 316	− 0.752 096	− 0.736 583	− 0.720 786	− 0.704 720	− 0.688 396	− 0.671 827	− 0.655 026	70
75	− 0.501 741	− 0.486 014	− 0.470 111	− 0.454 059	− 0.437 868	− 0.421 551	− 0.405 121	− 0.388 593	− 0.371 978	− 0.355 291	− 0.338 546	75
80	− 0.213 624	− 0.197 853	− 0.182 070	− 0.166 285	− 0.150 509	− 0.134 761	− 0.119 044	− 0.103 375	− 0.087 764	− 0.072 224	− 0.056 764	80
85	+ 0.053 037	+ 0.067 677	+ 0.082 216	+ 0.096 627	+ 0.110 907	+ 0.125 045	+ 0.139 030	+ 0.152 863	+ 0.166 530	+ 0.180 022	+ 0.193 332	85
90	+ 0.298 488	+ 0.311 104	+ 0.323 515	+ 0.335 716	+ 0.347 699	+ 0.359 460	+ 0.370 994	+ 0.382 299	+ 0.393 369	+ 0.404 201	+ 0.414 792	90
95	+ 0.523 345	+ 0.533 243	+ 0.542 893	+ 0.552 280	+ 0.561 401	+ 0.570 258	+ 0.578 846	+ 0.587 170	+ 0.595 228	+ 0.603 025	+ 0.610 549	95
100	+ 0.728 404	+ 0.735 135	+ 0.741 583	+ 0.747 752	+ 0.753 644	+ 0.759 260	+ 0.764 605	+ 0.769 682	+ 0.774 495	+ 0.779 050	+ 0.783 347	100
105	+ 0.914 690	+ 0.917 923	+ 0.920 899	+ 0.923 610	+ 0.926 062	+ 0.928 261	+ 0.930 214	+ 0.931 929	+ 0.933 404	+ 0.934 641	+ 0.935 681	105
110	+ 1.083 209	+ 1.082 802	+ 1.082 168	+ 1.081 314	+ 1.080 248	+ 1.078 976	+ 1.077 506	+ 1.075 846	+ 1.074 004	+ 1.071 964	+ 1.069 798	110
115	+ 1.235 048	+ 1.230 949	+ 1.226 684	+ 1.222 266	+ 1.217 703	+ 1.213 000	+ 1.208 170	+ 1.203 216	+ 1.198 145	+ 1.192 965	+ 1.187 688	115
120	+ 1.371 259	+ 1.363 507	+ 1.355 679	+ 1.347 780	+ 1.339 820	+ 1.331 803	+ 1.323 738	+ 1.315 629	+ 1.307 482	+ 1.299 307	+ 1.291 104	120
125	+ 1.492 839	+ 1.481 553	+ 1.470 298	+ 1.459 068	+ 1.447 869	+ 1.436 704	+ 1.425 583	+ 1.414 505	+ 1.403 474	+ 1.392 500	+ 1.381 575	125
130	+ 1.600 701	+ 1.586 084	+ 1.571 592	+ 1.557 229	+ 1.542 996	+ 1.528 894	+ 1.514 927	+ 1.501 096	+ 1.487 401	+ 1.473 843	+ 1.460 425	130
135	+ 1.695 736	+ 1.678 006	+ 1.660 512	+ 1.643 240	+ 1.626 219	+ 1.609 418	+ 1.592 844	+ 1.576 498	+ 1.560 374	+ 1.544 473	+ 1.528 793	135
140	+ 1.778 710	+ 1.758 136	+ 1.737 903	+ 1.718 006	+ 1.698 437	+ 1.679 194	+ 1.660 268	+ 1.641 658	+ 1.623 354	+ 1.605 356	+ 1.587 654	140
145	+ 1.850 324	+ 1.827 195	+ 1.804 510	+ 1.782 259	+ 1.760 430	+ 1.739 016	+ 1.718 005	+ 1.697 391	+ 1.677 169	+ 1.657 316	+ 1.637 836	145
150	+ 1.911 182	+ 1.885 812	+ 1.860 978	+ 1.836 670	+ 1.812 869	+ 1.789 564	+ 1.766 742	+ 1.744 390	+ 1.722 496	+ 1.701 050	+ 1.680 036	150
155	+ 1.961 808	+ 1.934 523	+ 1.907 857	+ 1.881 797	+ 1.856 320	+ 1.831 411	+ 1.807 054	+ 1.783 232	+ 1.759 931	+ 1.737 138	+ 1.714 823	155
160	+ 2.002 640	+ 1.973 777	+ 1.945 604	+ 1.918 104	+ 1.891 252	+ 1.865 029	+ 1.839 417	+ 1.814 393	+ 1.789 943	+ 1.766 048	+ 1.742 691	160
165	+ 2.034 032	+ 2.003 935	+ 1.974 587	+ 1.945 963	+ 1.918 039	+ 1.890 795	+ 1.864 206	+ 1.838 249	+ 1.812 906	+ 1.788 160	+ 1.763 988	165
170	+ 2.056 266	+ 2.025 283	+ 1.995 091	+ 1.965 664	+ 1.936 974	+ 1.908 997	+ 1.881 710	+ 1.855 088	+ 1.829 109	+ 1.803 755	+ 1.779 002	170
175	+ 2.069 528	+ 2.038 012	+ 2.007 313	+ 1.977 403	+ 1.948 254	+ 1.919 837	+ 1.892 131	+ 1.865 111	+ 1.838 750	+ 1.813 034	+ 1.787 932	175
180	+ 2.073 933	+ 2.042 242	+ 2.011 373	+ 1.981 303	+ 1.952 000	+ 1.923 438	+ 1.895 593	+ 1.868 438	+ 1.841 952	+ 1.816 112	+ 1.790 896	180
ψ / z	0.21	0.22	0.23	0.24	0.25	0.26	0.27	0.28	0.29	0.30	0.31	z \ ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39	0.40	0.41	z / ψ
0	− 6.593 73	− 6.813 56	− 7.045 09	− 7.289 15	− 7.546 66	− 7.818 61	− 8.106 08	− 8.410 26	− 8.732 45	− 9.074 07	− 9.436 70	0
5	− 6.521 19	− 6.735 55	− 6.961 12	− 7.198 66	− 7.449 04	− 7.713 16	− 7.992 06	− 8.286 81	− 8.598 62	− 8.928 80	− 9.278 79	5
10	− 6.309 38	− 6.508 13	− 6.716 69	− 6.935 71	− 7.165 86	− 7.407 88	− 7.662 58	− 7.930 82	− 8.213 54	− 8.511 76	− 8.826 58	10
15	− 5.974 61	− 6.149 76	− 6.332 74	− 6.523 98	− 6.723 97	− 6.933 19	− 7.152 20	− 7.381 55	− 7.621 86	− 7.873 78	− 8.138 00	15
20	− 5.540 96	− 5.687 47	− 5.839 60	− 5.997 59	− 6.161 72	− 6.332 25	− 6.509 46	− 6.693 66	− 6.885 14	− 7.084 23	− 7.291 25	20
25	− 5.036 39	− 5.152 35	− 5.271 84	− 5.394 76	− 5.521 80	− 5.652 46	− 5.787 01	− 5.925 63	− 6.068 31	− 6.215 16	− 6.366 26	25
30	− 4.489 07	− 4.575 31	− 4.663 35	− 4.753 19	− 4.844 82	− 4.938 22	− 5.033 39	− 5.130 29	− 5.228 88	− 5.329 14	− 5.431 00	30
35	− 3.924 46	− 3.983 82	− 4.043 71	− 4.104 09	− 4.164 88	− 4.226 05	− 4.287 52	− 4.349 22	− 4.411 06	− 4.472 97	− 4.534 86	35
40	− 3.363 52	− 3.400 03	− 3.436 28	− 3.472 [illegible]	− 3.507 68	− 3.542 72	− 3.577 23	− 3.611 13	− 3.644 35	− 3.676 80	− 3.708 41	40
45	− 2.822 0[illegible]	− 2.840 25	− 2.857 74	− 2.874 48	− 2.890 44	− 2.905 55	− 2.919 76	− 2.933 03	− 2.945 30	− 2.956 52	− 2.966 63	45
50	− 2.3[illegible]	− 2.315 24	− 2.318 71	− 2.321 32	− 2.323 05	− 2.323 88	− 2.323 76	− 2.322 68	− 2.320 60	− 2.317 52	− 2.313 39	50
55	− 1.836 71	− 1.831 11	− 1.824 80	− 1.817 69	− 1.809 80	− 1.801 10	− 1.791 61	− 1.781 31	− 1.770 20	− 1.758 28	− 1.745 54	55
60	− 1.402 49	− 1.390 5[illegible]	− 1.377 92	− 1.364 73	− 1.350 94	− 1.336 56	− 1.321 60	− 1.306 05	− 1.289 95	− 1.273 28	− 1.256 08	60
65	− 1.008 [illegible]	− 0.993 388	− 0.977 350	− 0.960 [illegible]	− 0.944 180	− 0.927 050	− 0.909 558	− 0.891 727	− 0.873 574	− 0.855 114	− 0.836 363	65
70	− 0.655 [illegible]	− 0.638 005	− 0.620 78[illegible]	− 0.603 369	− 0.585 78[illegible]	− 0.568 035	− 0.550 142	− 0.532 123	− 0.513 9[illegible]	− 0.495 761	− 0.477 450	70
75	− 0.338 546	− 0.321 755	− 0.304 943	− 0.288 0[illegible]5	− 0.271 250	− 0.254 417	− 0.237 645	− 0.220 830	− 0.204 [illegible]	− 0.187 437	− 0.170 846	75
80	− 0.056 [illegible]	− 0.041 365	− 0.026 131	− 0.010 978	0.004 051	+ 0.018 947	+ 0.033 7[illegible]	+ 0.048 303	+ 0.062 747	+ 0.077 022	+ 0.091 121	80
85	+ 0.193 332	+ 0.206 456	+ 0.219 385	+ 0.232 114	+ 0.244 635	+ 0.256 946	+ 0.269 041	+ 0.280 915	+ 0.292 563	+ 0.303 982	+ 0.315 170	85
90	+ 0.414 7[illegible]2	+ 0.425 14[illegible]	+ 0.435 242	+ 0.445 008	+ 0.454 7[illegible]3	+ 0.464 050	+ 0.473 165	+ 0.482 018	+ 0.490 62[illegible]	+ 0.498 973	+ 0.507 073	90
95	+ 0.610 540	+ 0.617 815	+ 0.624 [illegible]	+ 0.631 566	+ 0.638 053	+ 0.644 285	+ 0.650 266	+ 0.655 997	+ 0.661 48[illegible]	+ 0.666 729	+ 0.671 720	95
100	+ 0.783 347	+ 0.787 392	+ 0.791 1[illegible]	+ 0.794 7[illegible]	+ 0.798 0[illegible]3	+ 0.801 [illegible]46	+ 0.804 0[illegible]1	+ 0.806 632	+ 0.809 044	+ 0.811 243	+ 0.813 233	100
105	+ 0.935 681	+ 0.936 486	+ 0.937 000	+ 0.937 487	+ 0.937 683	+ 0.937 68[illegible]	+ 0.937 5[illegible]	+ 0.937 148	+ 0.936 614	+ 0.935 912	+ 0.935 047	105
110	+ 1.069 798	+ 1.067 444	+ 1.064 938	+ 1.062 282	+ 1.059 483	+ 1.056 548	+ 1.053 483	+ 1.050 293	+ 1.046 986	+ 1.043 566	+ 1.040 139	110
115	+ 1.187 088	+ 1.182 310	+ 1.176 845	+ 1.171 2[illegible]	+ 1.165 670	+ 1.159 973	+ 1.154 20[illegible]	+ 1.148 385	+ 1.142 565	+ 1.136 575	+ 1.130 600	115
120	+ 1.291 104	+ 1.282 882	+ 1.274 644	+ 1.266 395	+ 1.258 140	+ 1.249 885	+ 1.241 630	+ 1.233 383	+ 1.225 145	+ 1.216 [illegible]	+ 1.208 717	120
125	+ 1.381 [illegible]	+ 1.370 714	+ 1.359 914	+ 1.349 177	+ 1.338 510	+ 1.327 911	+ 1.317 38[illegible]	+ 1.306 933	+ 1.296 556	+ 1.286 200	+ 1.276 043	125
130	+ 1.460 425	+ 1.447 147	+ 1.434 009	+ 1.421 009	+ 1.408 152	+ 1.395 433	+ 1.382 857	+ 1.370 420	+ 1.358 123	+ 1.345 900	+ 1.333 253	130
135	+ 1.528 [illegible]	+ 1.513 328	+ 1.498 084	+ 1.483 050	+ 1.468 229	+ 1.453 614	+ 1.439 207	+ 1.425 003	+ 1.411 001	+ 1.397 198	+ 1.383 598	135
140	+ 1.587 654	+ 1.570 240	+ 1.553 123	+ 1.536 284	+ 1.519 722	+ 1.503 432	+ 1.487 410	+ 1.471 650	+ 1.456 14[illegible]	+ 1.440 8[illegible]	+ 1.425 868	140
145	+ 1.637 836	+ 1.618 717	+ 1.599 [illegible]	+ 1.581 536	+ 1.563 455	+ 1.545 70[illegible]	+ 1.528 278	+ 1.511 16[illegible]	+ 1.494 367	+ 1.477 [illegible]2	+ 1.461 655	145
150	+ 1.681 036	+ 1.659 448	+ 1.639 270	+ 1.619 495	+ 1.600 112	+ 1.581 111	+ 1.562 491	+ 1.544 216	+ 1.526 302	+ 1.508 734	+ 1.491 502	150
155	+ 1.714 823	+ 1.692 976	+ 1.671 636	+ 1.650 727	+ 1.630 253	+ 1.610 205	+ 1.590 569	+ 1.571 331	+ 1.552 501	+ 1.534 044	+ 1.515 955	155
160	+ 1.742 6[illegible]	+ 1.719 855	+ 1.697 527	+ 1.675 691	+ 1.654 331	+ 1.633 434	+ 1.612 987	+ 1.592 977	+ 1.573 300	+ 1.554 216	+ 1.535 442	160
165	+ 1.763 988	+ 1.740 366	+ 1.717 207	+ 1.694 748	+ 1.672 704	+ 1.651 153	+ 1.630 079	+ 1.609 468	+ 1.589 308	+ 1.569 581	+ 1.550 27[illegible]	165
170	+ 1.779 002	+ 1.754 833	+ 1.731 228	+ 1.708 169	+ 1.685 640	+ 1.663 625	+ 1.642 106	+ 1.621 068	+ 1.600 498	+ 1.580 381	+ 1.560 702	170
175	+ 1.787 932	+ 1.763 437	+ 1.739 511	+ 1.716 148	+ 1.693 328	+ 1.671 036	+ 1.649 252	+ 1.627 961	+ 1.607 [illegible]45	+ 1.586 7[illegible]	+ 1.566 892	175
180	+ 1.790 896	+ 1.766 285	+ 1.742 258	+ 1.718 792	+ 1.695 880	+ 1.673 494	+ 1.651 620	+ 1.630 247	+ 1.609 350	+ 1.588 922	+ 1.568 945	180
ψ / z	0.31	0.32	0.33	0.34	0.35	0.36	0.37	0.38	0.39	0.40	0.41	ψ / z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.50	0.51	z / ψ
0°	− 9.436 70	− 9.822 05	− 10.232 0	− 10.668 7	− 11.134 5	− 11.631 9	− 12.163 7	− 12.733 3	− 13.344 0	− 14.000 0	− 14.705 6	0°
5	− 9.278 79	− 9.650 16	− 10.044 7	− 10.464 2	− 10.910 8	− 11.386 9	− 11.895 0	− 12.438 0	− 13.019 0	− 13.641 5	− 14.309 5	5
10	− 8.826 58	− 9.159 20	− 9.510 93	− 9.883 18	− 10.277 5	− 10.695 6	− 11.139 3	− 11.610 6	− 12.111 7	− 12.645 0	− 13.213 2	10
15	− 8.138 00	− 8.415 26	− 8.706 35	− 9.012 11	− 9.333 44	− 9.671 29	− 10.026 7	− 10.400 7	− 10.794 5	− 11.209 4	− 11.646 5	15
20	− 7.291 25	− 7.506 54	− 7.730 45	− 7.963 35	− 8.205 58	− 8.457 54	− 8.719 59	− 8.992 14	− 9.275 54	− 9.570 22	− 9.876 54	20
25	− 6.366 26	− 6.521 70	− 6.681 52	− 6.845 77	− 7.014 52	− 7.187 78	− 7.365 58	− 7.547 90	− 7.734 73	− 7.926 02	− 8.121 73	25
30	− 5.431 00	− 5.534 40	− 5.639 27	− 5.745 52	− 5.853 07	− 5.961 79	− 6.071 57	− 6.182 26	− 6.293 71	− 6.405 74	− 6.518 18	30
35	− 4.534 86	− 4.596 62	− 4.658 15	− 4.719 33	− 4.780 06	− 4.840 19	− 4.899 60	− 4.958 14	− 5.015 67	− 5.072 03	− 5.127 05	35
40	− 3.708 41	− 3.739 09	− 3.768 76	− 3.797 33	− 3.824 71	− 3.850 81	− 3.875 53	− 3.898 78	− 3.920 46	− 3.940 48	− 3.958 75	40
45	− 2.966 63	− 2.975 59	− 2.983 35	− 2.989 85	− 2.995 05	− 2.998 89	− 3.001 34	− 3.002 34	− 3.001 87	− 2.999 87	− 2.996 30	45
50	− 2.313 39	− 2.308 21	− 2.301 96	− 2.294 61	− 2.286 15	− 2.276 57	− 2.265 86	− 2.254 01	− 2.241 01	− 2.226 87	− 2.211 58	50
55	− 1.745 54	− 1.731 99	− 1.717 63	− 1.702 46	− 1.686 50	− 1.669 74	− 1.652 20	− 1.633 89	− 1.614 83	− 1.595 02	− 1.574 50	55
60	− 1.256 08	− 1.238 34	− 1.220 08	− 1.201 32	− 1.182 08	− 1.162 36	− 1.142 20	− 1.121 60	− 1.100 60	− 1.079 20	− 1.057 44	60
65	− 0.836 363	− 0.817 339	− 0.798 061	− 0.778 544	− 0.758 804	− 0.738 867	− 0.718 747	− 0.698 465	− 0.678 038	− 0.657 486	− 0.636 832	65
70	− 0.477 450	− 0.459 074	− 0.440 649	− 0.422 190	− 0.403 714	− 0.385 238	− 0.366 774	− 0.348 338	− 0.329 947	− 0.311 614	− 0.293 355	70
75	− 0.170 846	− 0.154 341	− 0.137 934	− 0.121 636	− 0.105 459	− 0.089 414	− 0.073 511	− 0.057 759	− 0.042 170	− 0.026 752	− 0.011 513	75
80	- 0.091 121	+ 0.105 036	+ 0.118 761	+ 0.132 289	+ 0.145 611	+ 0.158 726	+ 0.171 626	+ 0.184 306	+ 0.196 761	+ 0.208 987	+ 0.220 982	80
85	+ 0.315 170	+ 0.326 123	+ 0.336 838	+ 0.347 313	+ 0.357 546	+ 0.367 537	+ 0.377 283	+ 0.386 783	+ 0.396 039	+ 0.405 048	+ 0.413 812	85
90	+ 0.507 073	+ 0.514 924	+ 0.522 526	+ 0.529 881	+ 0.536 990	+ 0.543 854	+ 0.550 477	+ 0.556 860	+ 0.563 006	+ 0.568 915	+ 0.574 596	90
95	+ 0.671 720	+ 0.676 489	+ 0.681 013	+ 0.685 308	+ 0.689 391	+ 0.693 246	+ 0.696 881	+ 0.700 305	+ 0.703 518	+ 0.706 528	+ 0.709 337	95
100	+ 0.813 233	+ 0.815 029	+ 0.816 607	+ 0.818 002	+ 0.819 208	+ 0.820 231	+ 0.821 077	+ 0.821 748	+ 0.822 253	+ 0.822 595	+ 0.822 779	100
105	+ 0.935 047	+ 0.934 030	+ 0.932 853	+ 0.931 535	+ 0.930 112	+ 0.928 484	+ 0.926 752	+ 0.924 912	+ 0.922 949	+ 0.920 870	+ 0.918 679	105
110	+ 1.040 039	+ 1.036 411	+ 1.032 687	+ 1.028 872	+ 1.024 972	+ 1.020 990	+ 1.016 933	+ 1.012 803	+ 1.008 608	+ 1.004 350	+ 1.000 031	110
115	+ 1.130 000	+ 1.124 579	+ 1.118 526	+ 1.112 438	+ 1.106 322	+ 1.100 181	+ 1.094 019	+ 1.087 838	+ 1.081 646	+ 1.075 442	+ 1.069 228	115
120	+ 1.208 717	+ 1.200 528	+ 1.192 364	+ 1.184 226	+ 1.176 117	+ 1.168 039	+ 1.159 994	+ 1.151 984	+ 1.144 013	+ 1.136 081	+ 1.128 190	120
125	+ 1.276 043	+ 1.265 904	+ 1.255 848	+ 1.245 876	+ 1.235 988	+ 1.226 185	+ 1.216 467	+ 1.206 836	+ 1.197 291	+ 1.187 835	+ 1.178 467	125
130	+ 1.333 953	+ 1.322 076	+ 1.310 337	+ 1.298 736	+ 1.287 272	+ 1.275 944	+ 1.264 750	+ 1.253 693	+ 1.242 765	+ 1.231 971	+ 1.221 309	130
135	+ 1.383 590	+ 1.370 176	+ 1.356 954	+ 1.343 919	+ 1.331 070	+ 1.318 406	+ 1.305 919	+ 1.293 613	+ 1.281 480	+ 1.269 519	+ 1.257 731	135
140	+ 1.425 898	+ 1.411 140	+ 1.396 622	+ 1.382 338	+ 1.368 284	+ 1.354 456	+ 1.340 849	+ 1.327 460	+ 1.314 286	+ 1.301 317	+ 1.288 555	140
145	+ 1.461 655	+ 1.445 737	+ 1.430 102	+ 1.414 742	+ 1.399 651	+ 1.384 825	+ 1.370 257	+ 1.355 940	+ 1.341 872	+ 1.328 044	+ 1.314 449	145
150	+ 1.491 502	+ 1.474 598	+ 1.458 014	+ 1.441 742	+ 1.425 774	+ 1.410 103	+ 1.394 722	+ 1.379 622	+ 1.364 800	+ 1.350 248	+ 1.335 955	150
155	+ 1.515 958	+ 1.498 236	+ 1.480 863	+ 1.463 833	+ 1.447 138	+ 1.430 767	+ 1.414 711	+ 1.398 976	+ 1.383 520	+ 1.368 369	+ 1.353 500	155
160	+ 1.535 442	+ 1.517 058	+ 1.499 051	+ 1.481 411	+ 1.464 131	+ 1.447 198	+ 1.430 603	+ 1.414 337	+ 1.398 391	+ 1.382 758	+ 1.367 429	160
165	+ 1.550 278	+ 1.531 386	+ 1.512 892	+ 1.494 784	+ 1.477 055	+ 1.459 691	+ 1.442 681	+ 1.426 018	+ 1.409 691	+ 1.393 687	+ 1.378 006	165
170	+ 1.560 702	+ 1.541 450	+ 1.522 612	+ 1.504 175	+ 1.486 127	+ 1.468 458	+ 1.451 157	+ 1.434 212	+ 1.417 615	+ 1.401 355	+ 1.385 422	170
175	+ 1.566 892	+ 1.547 424	+ 1.528 383	+ 1.509 750	+ 1.491 512	+ 1.473 662	+ 1.456 187	+ 1.439 074	+ 1.422 317	+ 1.405 905	+ 1.389 822	175
180	+ 1.568 945	+ 1.549 407	+ 1.530 295	+ 1.511 595	+ 1.493 296	+ 1.475 386	+ 1.457 852	+ 1.440 684	+ 1.423 873	+ 1.407 408	+ 1.391 277	180
ψ / z	0.41	0.42	0.43	0.44	0.45	0.46	0.47	0.48	0.49	0.50	0.51	ψ / z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{k}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	0.61	z / ψ
0°	− 14.705 6	− 15.465 8	− 16.286 4	− 17.173 5	− 18.134 4	− 19.177 3	− 20.311 4	− 21.547 4	− 22.897 2	− 24.375 0	− 25.996 7	0°
5	− 14.309 5	− 15.027 2	− 15.799 7	− 16.632 4	− 17.531 4	− 18.503 6	− 19.557 0	− 20.700 4	− 21.943 8	− 23.298 6	− 24.778 1	5
10	− 13.213 2	− 13.819 1	− 14.466 0	− 15.157 3	− 15.896 9	− 16.689 0	− 17.538 3	− 18.449 9	− 19.429 5	− 20.483 4	− 21.618 6	10
15	− 11.646 5	− 12.167 4	− 12.563 5	− 13.106 3	− 13.647 5	− 14.218 9	− 14.822 2	− 15.459 5	− 16.132 6	− 16.843 7	− 17.594 9	15
20	− 9.876 54	− 10.184 9	− 10.525 6	− 10.869 0	− 11.225 5	− 11.595 4	− 11.978 8	− 12.376 1	− 12.787 3	− 13.212 6	− 13.652 0	20
25	− 8.121 73	− 8.321 73	− 8.525 92	− 8.734 15	− 8.946 21	− 9.161 87	− 9.380 86	− 9.602 86	− 9.827 49	− 10.054 3	− 10.282 9	25
30	− 6.518 18	− 6.630 80	− 6.743 30	− 6.855 70	− 6.967 46	− 7.078 38	− 7.188 17	− 7.296 49	− 7.403 00	− 7.507 34	− 7.609 13	30
35	− 5.127 05	− 5.180 58	− 5.232 43	− 5.282 42	− 5.330 37	− 5.376 10	− 5.419 41	− 5.460 11	− 5.498 01	− 5.532 90	− 5.564 61	35
40	− 3.958 75	− 3.975 17	− 3.989 64	− 4.002 08	− 4.012 40	− 4.020 50	− 4.026 32	− 4.029 77	− 4.030 77	− 4.029 27	− 4.025 2[illegible]	40
45	− 2.996 30	− 2.991 1[illegible]	− 2.984 37	− 2.975 94	− 2.965 83	− 2.954 03	− 2.940 32	− 2.925 30	− 2.908 35	− 2.889 67	− 2.869 28	45
50	− 2.211 58	− 2.195 14	− 2.177 56	− 2.158 85	− 2.139 01	− 2.118 07	− 2.096 03	− 2.072 93	− 2.048 78	− 2.023 60	− 1.997 43	50
55	− 1.574 50	− 1.553 26	− 1.531 35	− 1.508 76	− 1.485 55	− 1.461 71	− 1.437 29	− 1.412 31	− 1.386 80	− 1.360 79	− 1.334 31	55
60	− 1.057 44	− 1.035 33	− 1.012 89	− 0.990 161	− 0.967 154	− 0.943 895	− 0.920 412	− 0.896 727	− 0.872 867	− 0.848 856	− 0.824 721	60
65	− 0.636 832	− 0.616 062	− 0.595 296	− 0.574 436	− 0.553 557	− 0.532 670	− 0.511 796	− 0.490 951	− 0.470 153	− 0.449 421	− 0.428 771	65
70	− 0.293 355	− 0.275 183	− 0.257 112	− 0.239 155	− 0.221 325	− 0.203 634	− 0.186 096	− 0.168 720	− 0.151 519	− 0.134 500	− 0.117 678	70
75	0.011 513	+ 0.003 537	+ 0.018 392	+ 0.033 043	+ 0.047 484	+ 0.061 707	+ 0.075 711	+ 0.089 485	+ 0.103 025	+ 0.116 328	+ 0.129 389	75
80	+ 0.220 982	+ 0.232 742	+ 0.244 262	+ 0.255 542	+ 0.266 579	+ 0.277 372	+ 0.287 915	+ 0.298 214	+ 0.308 264	+ 0.318 065	+ 0.327 618	80
85	+ 0.413 812	+ 0.422 331	+ 0.430 606	+ 0.438 63[illegible]	+ 0.446 426	+ 0.453 974	+ 0.461 285	+ 0.468 357	+ 0.475 196	+ 0.481 802	+ 0.488 178	85
90	+ 0.574 546	+ 0.580 047	+ 0.585 272	+ 0.590 273	+ 0.595 057	+ 0.599 625	+ 0.603 982	+ 0.608 130	+ 0.612 074	+ 0.615 818	+ 0.619 365	90
95	+ 0.709 337	+ 0.711 95[illegible]	+ 0.714 3[illegible]3	+ 0.716 608	+ 0.718 661	+ 0.720 535	+ 0.722 237	+ 0.723 773	+ 0.725 138	+ 0.726 346	+ 0.727 399	95
100	+ 0.822 779	+ 0.822 806	+ 0.822 692	+ 0.822 433	+ 0.822 034	+ 0.821 502	+ 0.820 840	+ 0.820 054	+ 0.819 148	+ 0.818 126	+ 0.816 993	100
105	+ 0.918 679	+ 0.916 384	+ 0.913 990	+ 0.911 500	+ 0.908 919	+ 0.906 251	+ 0.903 500	+ 0.900 671	+ 0.897 767	+ 0.894 792	+ 0.891 750	105
110	+ 1.000 031	+ 0.995 65[illegible]	+ 0.991 236	+ 0.986 766	+ 0.982 253	+ 0.977 700	+ 0.973 111	+ 0.968 488	+ 0.963 835	+ 0.959 156	+ 0.954 451	110
115	+ 1.069 228	+ 1.063 [illegible]	+ 1.056 79[illegible]	+ 1.050 573	+ 1.044 358	+ 1.038 148	+ 1.031 947	+ 1.025 75[illegible]	+ 1.019 576	+ 1.013 412	+ 1.007 263	115
120	+ 1.128 1[illegible]0	+ 1.120 34[illegible]	+ 1.112 54[illegible]	+ 1.104 783	+ 1.097 054	+ 1.089 414	+ 1.081 803	+ 1.074 243	+ 1.066 736	+ 1.059 281	+ 1.051 879	120
125	+ 1.178 467	+ 1.169 18[illegible]	+ 1.159 9[illegible]4	+ 1.150 891	+ 1.141 876	+ 1.132 950	+ 1.124 112	+ 1.115 362	+ 1.106 701	+ 1.098 127	+ 1.089 641	125
130	+ 1.221 306	+ 1.210 77[illegible]	+ 1.200 370	+ 1.190 003	+ 1.179 942	+ 1.169 915	+ 1.160 013	+ 1.150 232	+ 1.140 573	+ 1.131 032	+ 1.121 611	130
135	+ 1.257 731	+ 1.246 108	+ 1.234 651	+ 1.223 358	+ 1.212 223	+ 1.201 246	+ 1.190 424	+ 1.179 755	+ 1.169 237	+ 1.158 864	+ 1.148 639	135
140	+ 1.288 555	+ 1.275 995	+ 1.263 63[illegible]	+ 1.251 464	+ 1.239 484	+ 1.227 692	+ 1.216 082	+ 1.204 652	+ 1.193 398	+ 1.182 316	+ 1.171 403	140
145	+ 1.3[illegible]	+ 1.30[illegible]	+ 1.287 955	+ 1.275 142	+ 1.262 343	+ 1.249 858	+ 1.237 580	+ 1.225 503	+ 1.213 616	+ 1.201 935	+ 1.190 436	145
150	+ 1.335 955	+ 1.321 921	+ 1.308 137	+ 1.294 598	+ 1.281 297	+ 1.268 230	+ 1.255 391	+ 1.242 773	+ 1.230 375	+ 1.218 189	+ 1.206 209	150
155	+ 1.353 500	+ 1.338 910	+ 1.324 5[illegible]1	+ 1.310 538	+ 1.296 741	+ 1.283 196	+ 1.269 895	+ 1.256 833	+ 1.244 008	+ 1.231 408	+ 1.219 031	155
160	+ 1.367 42[illegible]	+ 1.352 394	+ 1.337 647	+ 1.323 181	+ 1.308 988	+ 1.295 061	+ 1.281 393	+ 1.267 978	+ 1.254 809	+ 1.241 881	+ 1.229 186	160
165	+ 1.378 006	+ 1.362 632	+ 1.347 558	+ 1.332 778	+ 1.318 281	+ 1.304 064	+ 1.290 114	+ 1.276 432	+ 1.263 001	+ 1.249 822	+ 1.236 885	165
170	+ 1.385 4[illegible]2	+ 1.369 808	+ 1.354 504	+ 1.339 501	+ 1.324 790	+ 1.310 369	+ 1.296 223	+ 1.282 348	+ 1.268 736	+ 1.255 380	+ 1.242 273	170
175	+ 1.389 822	+ 1.374 065	+ 1.358 624	+ 1.343 490	+ 1.328 651	+ 1.314 108	+ 1.299 846	+ 1.285 855	+ 1.272 137	+ 1.258 676	+ 1.245 468	175
180	+ 1.391 277	+ 1.375 474	+ 1.359 988	+ 1.344 810	+ 1.329 932	+ 1.315 346	+ 1.301 044	+ 1.287 018	+ 1.273 261	+ 1.259 766	+ 1.246 525	180
ψ / z	0.51	0.52	0.53	0.54	0.55	0.56	0.57	0.58	0.59	0.60	0.61	ψ \ z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $\frac{r}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79	0.80	0.81	z / ψ
0°	−56.3410	−61.8804	−68.2264	−75.5348	−84.0000	−93.8657	−105.441	119.121	−135.417	155.000	−178.758	0°
5	−51.1910	55.7830	60.9662	−66.8382	−73.5156	−81.1390	−89.8780	−99.0387	−111.572	125.084	−140.851	5
10	−39.5929	42.3501	−45.3542	−48.6286	−52.1980	−56.0882	−60.3266	−64.9406	−69.9572	−75.4014	−81.2957	10
15	−27.7987	29.1302	30.5224	−31.9745	−33.4849	35.0509	−36.6684	−38.3319	−40.0343	−41.7663	−43.5169	15
20	−18.7324	−19.28[illegible]9	19.8453	−20.4017	−20.9545	21.5002	−22.0354	−22.5563	−23.0588	−23.5385	−23.9910	20
25	−12.5[illegible]	12.7082	−12.8963	−13.0742	−13.2404	−13.3938	−13.5331	−13.657[illegible]	−13.7645	−13.8542	−13.9251	25
30	−8.39319	−8.43847	−8.47600	8.50540	−8.52633	−8.53846	−8.54151	−8.53524	−8.51946	8.49402	−8.45884	30
35	−5.66764	−5.65337	5.63424	−5.61020	5.58122	5.54732	5.50851	−5.46484	−5.41636	−5.36318	−5.30540	35
40	−3.83486	−3.80068	3.76383	3.72436	−3.68236	−3.63787	3.59100	3.54182	3.49045	−3.43697	−3.38151	40
45	−2.57515	−2.53752	−2.49801	2.45848	−2.41718	−2.37481	−2.33142	2.28[illegible]10	2.24191	−2.19534	−2.14928	45
50	−1.68885	−1.65424	−1.61914	−1.58359	−1.54764	−1.51136	−1.47478	1.43796	−1.40096	−1.36380	−1.32656	50
55	−1.05093	−1.02140	−0.991984	0.962448	−0.932915	−0.903415	−0.873979	0.84463[illegible]	0.815419	−0.786356	0.75746[illegible]	55
60	−0.581897	−0.557980	−0.534215	−0.510623	−0.487221	−0.46402[illegible]	−0.441056	0.418326	0.395851	−0.373645	−0.351721	60
65	−0.230259	−0.211506	−0.193001	0.174752	−0.156766	−0.139052	−0.121618	0.104471	0.087616	−0.071060	−0.054805	65
70	+0.038061	+0.052248	+0.066158	+0.079795	+0.093154	+0.106239	+0.119138	+0.131558	+0.143793	+0.155746	+0.167416	70
75	+0.246[illegible]	+0.256330	+0.266317	+0.276042	+0.285515	+0.294729	+0.303699	+0.312393	+0.320837	+0.329056	+0.337018	75
80	+0.409672	+0.416566	+0.42323[illegible]	+0.429671	+0.435887	+0.441884	+0.447664	+0.453231	+0.458588	+0.463740	+0.468688	80
85	+0.539965	+0.544019	+0.547885	+0.551565	+0.555003	+0.558383	+0.561529	+0.564504	+0.567314	+0.569961	+0.572449	85
90	+0.644920	+0.646574	+0.648080	+0.649443	+0.650667	+0.651754	+0.652710	+0.653538	+0.654242	+0.654826	+0.655293	90
95	+0.730317	+0.729937	+0.729454	+0.728868	+0.728183	+0.727403	+0.726531	+0.725571	+0.724526	+0.723399	+0.722194	95
100	+0.800412	+0.798311	+0.796144	+0.793913	+0.791621	+0.789272	+0.786868	+0.784413	+0.781908	+0.779356	+0.776761	100
105	+0.858386	+0.854822	+0.851227	+0.847605	+0.843956	+0.840285	+0.836592	+0.832880	+0.829150	+0.825406	+0.821647	105
110	+0.906642	+0.901832	+0.897075	+0.892222	+0.887425	+0.882635	+0.877853	+0.873081	+0.868321	+0.863575	+0.858838	110
115	+0.947022	+0.941150	+0.935310	+0.929504	+0.923731	+0.917992	+0.912388	+0.906620	+0.900989	+0.895395	+0.889835	115
120	+0.980941	+0.974163	+0.967443	+0.960782	+0.954179	+0.947634	+0.941147	+0.934719	+0.928349	+0.922037	+0.915782	120
125	+1.009510	+1.001959	+0.994488	+0.987098	+0.979788	+0.972558	+0.965407	+0.958333	+0.951337	+0.944416	+0.937573	125
130	+1.033594	+1.025384	+1.017273	+1.009263	+1.001352	+0.993539	+0.985823	+0.978202	+0.970674	+0.963239	+0.955896	130
135	+1.053881	+1.045110	+1.036455	+1.027919	+1.019498	+1.011192	+1.002995	+0.994911	+0.986934	+0.979063	+0.971297	135
140	+1.070915	+1.061667	+1.052553	+1.043572	+1.034720	+1.025997	+1.017395	+1.008920	+1.000564	+0.992326	+0.984204	140
145	+1.085128	+1.075481	+1.065981	+1.056625	+1.047412	+1.038338	+1.029399	+1.020595	+1.011921	+1.003376	+0.994956	145
150	+1.096864	+1.086885	+1.077064	+1.067398	+1.057885	+1.048520	+1.039302	+1.030225	+1.021288	+1.012488	+1.003822	150
155	+1.106393	+1.096143	+1.086061	+1.076142	+1.066383	+1.056783	+1.047336	+1.038038	+1.028887	+1.019880	+1.011012	155
160	+1.113929	+1.103464	+1.093174	+1.083055	+1.073103	+1.063315	+1.053686	+1.044213	+1.034893	+1.025721	+1.016696	160
165	+1.119636	+1.109007	+1.098560	+1.088289	+1.078190	+1.068262	+1.058491	+1.048887	+1.039440	+1.030143	+1.020997	165
170	+1.123626	+1.112883	+1.102325	+1.091947	+1.081744	+1.071714	+1.061851	+1.052152	+1.042613	+1.033230	+1.024000	170
175	+1.125992	+1.115181	+1.104557	+1.094115	+1.083853	+1.073760	+1.063842	+1.054087	+1.044494	+1.035060	+1.025780	175
180	+1.126772	+1.115940	+1.105294	+1.094832	+1.084548	+1.074439	+1.064500	+1.054727	+1.045117	+1.035665	+1.026368	180
ψ / z	0.71	0.72	0.73	0.74	0.75	0.76	0.77	0.78	0.79	0.80	0.81	ψ \ z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1.00	1.01	z / ψ
0°	−1506.31	−2121.88	−3133.82	−4924.07	−8420.00	−16275.0	38181.5	−127550	1010100	− ∞	−990098	0°
5	−595.531	−699.506	−819.169	−952.860	1095.67	−1238.28	−1366.68	−1463.74	−1513.20	−1505.13	−1439.93	5
10	−161.476	−169.623	−177.020	183.386	188.449	191.962	−193.733	−193.639	−191.646	187.813	−182.288	10
15	−58.090 1	−58.758 1	−59.194 0	−59.383 7	−59.317 9	−58.992 4	−58.408 9	−57.574 7	−56.502 6	−55.210 4	−53.719 9	15
20	−26.060 0	25.946 5	25.768 2	−25.526 1	−25.221 6	−24.857 2	−24.436 0	23.961 9	−23.439 1	−22.872 6	−22.267 1	20
25	−13.457 8	−13.290 6	−13.103 2	−12.896 6	−12.672 0	−12.430 9	−12.174 5	−11.904 2	−11.621 6	−11.328 2	−11.025 7	25
30	−7.591 28	−7.458 88	−7.319 88	−7.174 83	−7.024 28	−6.868 83	−6.709 06	−6.545 56	−6.378 93	−6.209 77	−6.038 64	30
35	−4.512 14	−4.416 11	4.318 06	−4.218 24	−4.116 92	−4.014 39	−3.910 89	−3.806 68	−3.702 01	−3.597 11	−3.492 22	35
40	−2.745 91	−2.677 23	−2.608 17	−2.538 85	−2.469 39	−2.399 94	−2.330 59	−2.261 46	−2.192 67	2.124 32	−2.056 49	40
45	−1.661 44	−1.612 22	−1.563 19	−1.514 43	−1.465 98	−1.417 92	−1.370 28	1.323 13	−1.276 50	−1.230 44	−1.184 99	45
50	−0.958 863	−0.923 446	−0.888 413	−0.853 791	−0.819 606	−0.785 883	−0.752 643	−0.719 908	−0.687 693	−0.656 015	−0.624 897	50
55	−0.483 534	−0.458 054	−0.432 982	0.408 329	−0.384 105	−0.360 318	−0.336 976	−0.314 085	−0.291 651	−0.269 676	−0.248 168	55
60	−0.150 275	−0.132 073	−0.114 244	−0.096 791	0.079 714	−0.063 014	−0.046 694	0.030 752	−0.015 188	+0.000 000	+0.014 813	60
65	+0.090 450	+0.103 221	+0.115 675	+0.127 812	+0.139 636	+0.151 148	+0.162 352	+0.173 249	+0.183 843	+0.194 138	+0.204 138	65
70	+0.268 780	+0.277 428	+0.285 817	+0.293 949	+0.301 829	+0.309 460	+0.316 846	+0.323 992	+0.330 900	+0.337 576	+0.344 023	70
75	+0.403 772	+0.409 238	+0.414 500	+0.419 558	+0.424 423	+0.429 093	+0.433 574	+0.437 871	+0.441 988	+0.445 925	+0.449 691	75
80	+0.507 877	+0.510 852	+0.513 670	+0.516 335	+0.518 852	+0.521 224	+0.523 455	+0.525 549	+0.527 509	+0.529 340	+0.531 044	80
85	+0.588 467	+0.590 464	+0.591 347	+0.592 120	+0.592 785	+0.593 347	+0.593 809	+0.594 173	+0.594 443	+0.594 622	+0.594 713	85
90	+0.654 312	+0.653 722	+0.653 054	+0.652 311	+0.651 498	+0.650 615	+0.649 666	+0.648 653	+0.647 579	+0.646 447	+0.645 258	90
95	+0.706 480	+0.704 602	+0.702 677	+0.700 709	+0.698 699	+0.696 650	+0.694 563	+0.692 441	+0.690 286	+0.688 099	+0.685 881	95
100	+0.748 892	+0.745 959	+0.743 007	+0.740 038	+0.737 052	+0.734 052	+0.731 038	+0.728 013	+0.724 978	+0.721 934	+0.718 882	100
105	+0.783 686	+0.779 883	+0.776 083	+0.772 288	+0.768 499	+0.764 717	+0.760 942	+0.757 176	+0.753 419	+0.749 672	+0.745 935	105
110	+0.812 446	+0.807 920	+0.803 417	+0.798 936	+0.794 483	+0.790 053	+0.785 648	+0.781 260	+0.776 914	+0.772 586	+0.768 284	110
115	+0.836 369	+0.831 239	+0.826 149	+0.821 098	+0.816 089	+0.811 118	+0.806 189	+0.801 298	+0.796 448	+0.791 636	+0.786 864	115
120	+0.856 360	+0.850 724	+0.845 141	+0.839 613	+0.834 138	+0.828 716	+0.823 347	+0.818 029	+0.812 764	+0.807 549	+0.802 386	120
125	+0.873 124	+0.867 062	+0.861 065	+0.855 135	+0.849 270	+0.843 468	+0.837 730	+0.832 055	+0.826 442	+0.820 888	+0.815 396	125
130	+0.887 203	+0.880 781	+0.874 436	+0.868 168	+0.861 974	+0.855 854	+0.849 806	+0.843 830	+0.837 924	+0.832 087	+0.826 319	130
135	+0.899 023	+0.892 300	+0.885 662	+0.879 109	+0.872 639	+0.866 251	+0.859 945	+0.853 714	+0.847 562	+0.841 507	+0.835 500	135
140	+0.908 920	+0.901 944	+0.895 060	+0.888 268	+0.881 567	+0.874 954	+0.868 429	+0.861 997	+0.855 630	+0.849 355	+0.843 161	140
145	+0.917 158	+0.909 971	+0.902 883	+0.895 892	+0.888 998	+0.882 197	+0.875 490	+0.868 873	+0.862 345	+0.855 904	+0.849 550	145
150	+0.923 947	+0.916 585	+0.909 328	+0.902 174	+0.895 120	+0.888 165	+0.881 308	+0.874 546	+0.867 877	+0.861 299	+0.854 811	150
155	+0.929 450	+0.921 946	+0.914 552	+0.907 265	+0.900 083	+0.893 002	+0.886 024	+0.879 144	+0.872 360	+0.865 672	+0.859 076	155
160	+0.933 796	+0.926 181	+0.918 679	+0.911 286	+0.904 002	+0.896 824	+0.889 749	+0.882 775	+0.875 902	+0.869 125	+0.862 445	160
165	+0.937 084	+0.929 386	+0.921 801	+0.914 327	+0.906 967	+0.899 714	+0.892 568	+0.885 522	+0.878 579	+0.871 737	+0.864 904	165
170	+0.939 381	+0.931 622	+0.923 980	+0.916 452	+0.909 037	+0.901 732	+0.894 533	+0.887 440	+0.880 451	+0.873 562	+0.866 772	170
175	+0.940 743	+0.932 947	+0.925 272	+0.917 712	+0.910 264	+0.902 928	+0.895 698	+0.888 577	+0.881 561	+0.874 644	+0.867 827	175
180	+0.941 191	+0.933 386	+0.925 698	+0.918 127	+0.910 669	+0.903 323	+0.896 084	+0.888 953	+0.881 925	+0.875 000	+0.868 174	180
ψ / z	0.91	0.92	0.93	0.94	0.95	0.96	0.97	0.98	0.99	1.00	1.01	ψ \ z

$\frac{1}{m}$ TABLE FOR INTERPOLATING z : $\frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	1.11	1.12	1.13	1.14	1.15	1.16	1.17	1.18	1.19	1.20	1.21	z / ψ
0°	−675.959	−515.807	−401.917	−318.800	−256.779	−209.604	−173.113	−144.464	−121.675	−103.333	−88.413 0	0°
5	−304.962	−256.352	−216.440	−183.609	−156.521	−134.086	−115.424	−99.823 1	−86.732 5	−75.684 2	−66.318 7	5
10	−90.933 5	−82.881 1	−75.425 7	−68.568 0	−62.293 4	−56.576 0	−51.383 4	−46.678 9	−42.424 3	−38.581 6	−35.113 5	10
15	−33.760 5	−31.779 7	−29.866 8	−28.029 5	−26.273 2	−24.601 5	−23.016 1	−21.517 5	−20.164 7	−18.776 0	−17.529 3	15
20	−15.251 1	−14.555 1	−13.874 0	−13.210 1	−12.565 1	−11.940 4	−11.337 1	−10.755 9	−10.197 4	−9.661 73	−9.149 03	20
25	−7.823 92	−7.513 80	−7.209 36	−6.911 26	−6.620 02	−6.336 13	−6.059 93	−5.791 75	−5.531 77	−5.280 18	−5.037 05	25
30	−4.336 72	−4.169 40	−4.011 06	−3.855 91	−3.704 12	−3.555 84	−3.411 21	−3.270 33	−3.133 27	−3.000 08	−2.870 81	30
35	−2.486 25	−2.393 28	−2.302 13	−2.212 89	−2.125 61	−2.040 33	−1.957 10	−1.875 96	−1.796 91	−1.719 97	−1.645 16	35
40	−1.424 22	−1.366 85	−1.310 69	−1.255 78	−1.202 13	−1.149 74	−1.098 64	−1.048 82	−1.000 29	−0.953 043	−0.907 074	40
45	−0.770 468	−0.733 450	−0.697 282	−0.661 968	−0.627 510	−0.593 909	−0.561 158	−0.529 260	−0.498 206	−0.467 989	−0.438 602	45
50	−0.346 175	−0.321 652	−0.297 739	−0.274 433	−0.251 731	−0.229 626	−0.208 115	−0.187 190	−0.166 844	−0.147 071	−0.127 862	50
55	−0.058 747	−0.042 334	−0.026 367	−0.010 840	+0.004 253	+0.018 918	+0.033 160	+0.046 987	+0.060 405	+0.073 421	+0.086 042	55
60	+0.142 970	+0.153 877	+0.164 457	+0.174 715	+0.184 656	+0.194 287	+0.203 613	+0.212 640	+0.221 373	+0.229 820	+0.237 987	60
65	+0.288 788	+0.295 819	+0.302 609	+0.309 164	+0.315 487	+0.321 584	+0.327 461	+0.333 121	+0.338 571	+0.343 815	+0.348 857	65
70	+0.396 873	+0.401 092	+0.405 136	+0.409 007	+0.412 711	+0.416 252	+0.419 633	+0.422 859	+0.425 934	+0.428 862	+0.431 647	70
75	+0.478 730	+0.480 857	+0.482 857	+0.484 733	+0.486 492	+0.488 133	+0.489 663	+0.491 083	+0.492 398	+0.493 607	+0.494 718	75
80	+0.541 882	+0.542 417	+0.542 864	+0.543 226	+0.543 506	+0.543 707	+0.543 831	+0.543 881	+0.543 859	+0.543 768	+0.543 611	80
85	+0.590 391	+0.590 063	+0.589 637	+0.589 127	+0.588 264	+0.587 351	+0.586 389	+0.585 379	+0.584 326	+0.583 229	+0.582 092	85
90	+0.630 750	+0.629 081	+0.627 380	+0.625 647	+0.623 886	+0.622 097	+0.620 281	+0.618 441	+0.616 577	+0.614 691	+0.612 784	90
95	+0.662 421	+0.659 977	+0.657 522	+0.655 054	+0.652 575	+0.650 089	+0.647 594	+0.645 092	+0.642 583	+0.640 068	+0.637 549	95
100	+0.688 174	+0.685 106	+0.682 041	+0.678 980	+0.675 925	+0.672 875	+0.669 832	+0.666 796	+0.663 767	+0.660 746	+0.657 734	100
105	+0.709 304	+0.705 726	+0.702 165	+0.698 621	+0.695 094	+0.691 587	+0.688 097	+0.684 626	+0.681 174	+0.677 741	+0.674 327	105
110	+0.726 773	+0.722 774	+0.718 805	+0.714 864	+0.710 952	+0.707 067	+0.703 207	+0.699 382	+0.695 583	+0.691 811	+0.688 068	110
115	+0.741 303	+0.736 958	+0.732 651	+0.728 381	+0.724 149	+0.719 953	+0.715 795	+0.711 670	+0.707 583	+0.703 532	+0.699 516	115
120	+0.753 450	+0.748 816	+0.744 227	+0.739 683	+0.735 183	+0.730 728	+0.726 316	+0.721 948	+0.717 621	+0.713 337	+0.709 094	120
125	+0.763 638	+0.758 760	+0.753 936	+0.749 163	+0.744 441	+0.739 769	+0.735 146	+0.730 572	+0.726 046	+0.721 568	+0.717 136	125
130	+0.772 189	+0.767 111	+0.762 090	+0.757 126	+0.752 218	+0.747 365	+0.742 566	+0.737 821	+0.733 128	+0.728 487	+0.723 897	130
135	+0.779 371	+0.774 124	+0.768 939	+0.763 815	+0.758 750	+0.753 746	+0.748 799	+0.743 910	+0.739 077	+0.734 300	+0.729 577	135
140	+0.785 386	+0.779 997	+0.774 674	+0.769 415	+0.764 220	+0.759 090	+0.754 020	+0.749 010	+0.744 061	+0.739 170	+0.734 337	140
145	+0.790 392	+0.784 886	+0.779 448	+0.774 079	+0.768 776	+0.763 540	+0.758 367	+0.753 259	+0.748 212	+0.743 228	+0.738 303	145
150	+0.794 518	+0.788 915	+0.783 383	+0.777 922	+0.772 530	+0.767 207	+0.761 951	+0.756 760	+0.751 634	+0.746 571	+0.741 571	150
155	+0.797 861	+0.792 180	+0.786 572	+0.781 037	+0.775 573	+0.770 180	+0.764 855	+0.759 598	+0.754 407	+0.749 282	+0.744 221	155
160	+0.800 503	+0.794 760	+0.789 092	+0.783 499	+0.777 978	+0.772 529	+0.767 150	+0.761 841	+0.756 600	+0.751 425	+0.746 315	160
165	+0.802 502	+0.796 712	+0.790 998	+0.785 360	+0.779 796	+0.774 306	+0.768 888	+0.763 540	+0.758 260	+0.753 049	+0.747 903	165
170	+0.803 896	+0.798 075	+0.792 330	+0.786 662	+0.781 068	+0.775 548	+0.770 100	+0.764 723	+0.759 417	+0.754 179	+0.749 007	170
175	+0.804 725	+0.798 883	+0.793 119	+0.787 432	+0.771 821	+0.776 284	+0.770 822	+0.765 431	+0.760 110	+0.754 857	+0.749 669	175
180	+0.804 998	+0.799 150	+0.793 380	+0.787 686	+0.782 069	+0.776 527	+0.771 057	+0.765 658	+0.760 330	+0.755 071	+0.749 879	180
ψ / z	1.11	1.12	1.13	1.14	1.15	1.16	1.17	1.18	1.19	1.20	1.21	ψ \ z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\mu}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.40	1.41	z / ψ
0°	− 24.860 4	− 22.361 8	− 20.170 3	− 18.240 8	− 16.536 0	− 15.024 6	− 13.680 4	− 12.481 3	− 11.408 6	− 10.446 4	− 9.581 11	0°
5	− 21.334 0	− 19.333 8	− 17.558 9	− 15.979 4	− 14.570 0	− 13.309 0	− 12.178 0	− 11.161 3	− 10.245 1	− 9.417 68	− 8.668 91	5
10	− 14.473 1	− 13.324 7	− 12.280 1	− 11.329 0	− 10.461 8	− 9.670 21	− 8.946 76	− 8.284 79	− 7.678 32	− 7.122 05	− 6.611 23	10
15	− 8.800 96	− 8.222 25	− 7.683 64	− 7.182 30	− 6.715 58	− 6.280 96	− 5.876 13	− 5.498 93	− 5.147 33	− 4.819 48	− 4.513 64	15
20	− 5.196 12	− 4.904 05	− 4.627 73	− 4.366 42	− 4.119 36	− 3.885 82	− 3.665 11	− 3.456 53	− 3.259 45	− 3.073 23	− 2.897 27	20
25	− 3.060 50	− 2.905 06	− 2.756 47	− 2.614 49	− 2.478 88	− 2.349 41	− 2.225 83	− 2.107 92	− 1.995 44	− 1.888 16	1.785 86	25
30	− 1.789 94	− 1.701 92	− 1.617 24	− 1.535 82	− 1.457 56	− 1.382 37	− 1.310 15	− 1.240 81	− 1.174 25	− 1.110 38	− 1.049 11	30
35	− 1.010 86	0.958 220	− 0.907 400	− 0.858 352	− 0.811 036	− 0.765 402	− 0.721 408	− 0.679 005	− 0.638 155	− 0.598 805	− 0.560 912	35
40	− 0.515 142	− 0.482 331	− 0.450 592	− 0.419 902	− 0.390 236	− 0.361 570	− 0.333 878	− 0.307 134	− 0.281 314	− 0.256 392	− 0.232 344	40
45	− 0.188 014	− 0.166 993	− 0.146 649	− 0.126 965	0.107 927	− 0.089 518	− 0.071 723	− 0.054 525	− 0.037 911	− 0.021 863	− 0.006 368	45
50	+ 0.035 225	+ 0.048 852	+ 0.062 034	+ 0.074 780	+ 0.087 102	+ 0.099 010	+ 0.110 514	+ 0.121 625	+ 0.132 352	+ 0.142 708	+ 0.152 700	50
55	+ 0.192 195	+ 0.200 974	+ 0.209 449	+ 0.217 629	+ 0.225 521	+ 0.233 133	+ 0.240 471	+ 0.247 543	+ 0.254 356	+ 0.260 915	+ 0.267 234	55
60	+ 0.305 529	+ 0.311 002	+ 0.316 265	+ 0.321 324	+ 0.326 185	+ 0.330 852	+ 0.335 331	+ 0.339 627	+ 0.343 746	+ 0.347 692	+ 0.351 470	60
65	+ 0.389 283	+ 0.392 427	+ 0.395 426	+ 0.398 281	+ 0.401 000	+ 0.403 586	+ 0.406 040	+ 0.408 369	+ 0.410 576	+ 0.412 664	+ 0.414 638	65
70	+ 0.452 460	+ 0.453 915	+ 0.455 270	+ 0.456 527	+ 0.457 691	+ 0.458 762	+ 0.459 745	+ 0.460 642	+ 0.461 457	+ 0.462 190	+ 0.462 847	70
75	+ 0.500 983	+ 0.501 184	+ 0.501 318	+ 0.501 382	+ 0.501 393	+ 0.501 341	+ 0.501 227	+ 0.501 056	+ 0.500 832	+ 0.500 555	+ 0.500 228	75
80	+ 0.538 848	+ 0.538 098	+ 0.537 305	+ 0.536 472	+ 0.535 599	+ 0.534 689	+ 0.533 743	+ 0.532 763	+ 0.531 749	+ 0.530 704	+ 0.529 629	80
85	+ 0.568 915	+ 0.567 329	+ 0.565 821	+ 0.564 290	+ 0.562 742	+ 0.561 167	+ 0.559 576	+ 0.557 968	+ 0.556 343	+ 0.554 703	+ 0.553 048	85
90	+ 0.592 825	+ 0.590 763	+ 0.588 692	+ 0.586 614	+ 0.584 528	+ 0.582 437	+ 0.580 341	+ 0.578 239	+ 0.576 134	+ 0.574 026	+ 0.571 914	90
95	+ 0.612 273	+ 0.609 752	+ 0.607 233	+ 0.604 719	+ 0.602 209	+ 0.599 704	+ 0.597 205	+ 0.594 710	+ 0.592 222	+ 0.589 741	+ 0.587 266	95
100	+ 0.628 176	+ 0.625 284	+ 0.622 406	+ 0.619 540	+ 0.616 688	+ 0.613 850	+ 0.611 025	+ 0.608 215	+ 0.605 418	+ 0.602 636	+ 0.599 868	100
105	+ 0.641 285	+ 0.638 093	+ 0.634 921	+ 0.631 770	+ 0.628 640	+ 0.625 530	+ 0.622 441	+ 0.619 373	+ 0.616 325	+ 0.613 298	+ 0.610 291	105
110	+ 0.652 167	+ 0.648 727	+ 0.645 315	+ 0.641 929	+ 0.638 570	+ 0.635 239	+ 0.631 932	+ 0.628 652	+ 0.625 398	+ 0.622 170	+ 0.618 967	110
115	+ 0.661 250	+ 0.657 607	+ 0.653 996	+ 0.650 417	+ 0.646 870	+ 0.643 353	+ 0.639 867	+ 0.636 412	+ 0.632 987	+ 0.629 593	+ 0.626 228	115
120	+ 0.668 565	+ 0.665 052	+ 0.661 276	+ 0.657 536	+ 0.653 831	+ 0.650 161	+ 0.646 526	+ 0.642 926	+ 0.639 360	+ 0.635 827	+ 0.632 328	120
125	+ 0.675 268	+ 0.671 313	+ 0.667 398	+ 0.663 523	+ 0.659 687	+ 0.655 890	+ 0.652 129	+ 0.648 409	+ 0.644 724	+ 0.641 076	+ 0.637 464	125
130	+ 0.680 653	+ 0.676 583	+ 0.672 553	+ 0.668 566	+ 0.664 621	+ 0.660 716	+ 0.656 853	+ 0.653 030	+ 0.649 247	+ 0.645 502	+ 0.641 795	130
135	+ 0.685 190	+ 0.681 018	+ 0.676 892	+ 0.672 810	+ 0.668 772	+ 0.664 779	+ 0.660 829	+ 0.656 920	+ 0.653 054	+ 0.649 229	+ 0.645 444	135
140	+ 0.688 991	+ 0.684 737	+ 0.680 530	+ 0.676 369	+ 0.672 255	+ 0.668 188	+ 0.664 165	+ 0.660 186	+ 0.656 251	+ 0.652 358	+ 0.648 507	140
145	+ 0.692 160	+ 0.687 837	+ 0.683 563	+ 0.679 338	+ 0.675 161	+ 0.671 031	+ 0.666 947	+ 0.662 909	+ 0.658 916	+ 0.654 968	+ 0.651 065	145
150	+ 0.694 775	+ 0.690 395	+ 0.686 066	+ 0.681 787	+ 0.677 558	+ 0.673 378	+ 0.669 244	+ 0.665 159	+ 0.661 119	+ 0.657 124	+ 0.653 173	150
155	+ 0.696 896	+ 0.692 471	+ 0.688 097	+ 0.683 776	+ 0.679 504	+ 0.675 282	+ 0.671 108	+ 0.666 983	+ 0.662 903	+ 0.658 873	+ 0.654 887	155
160	+ 0.698 574	+ 0.694 112	+ 0.689 703	+ 0.685 347	+ 0.681 042	+ 0.676 788	+ 0.672 583	+ 0.668 427	+ 0.664 318	+ 0.660 257	+ 0.656 242	160
165	+ 0.699 843	+ 0.695 355	+ 0.690 920	+ 0.686 538	+ 0.682 208	+ 0.677 929	+ 0.673 700	+ 0.669 520	+ 0.665 388	+ 0.661 305	+ 0.657 268	165
170	+ 0.700 731	+ 0.696 223	+ 0.691 769	+ 0.687 369	+ 0.683 021	+ 0.678 725	+ 0.674 479	+ 0.670 284	+ 0.666 137	+ 0.662 038	+ 0.657 986	170
175	+ 0.701 257	+ 0.696 738	+ 0.692 273	+ 0.687 862	+ 0.683 503	+ 0.679 197	+ 0.674 941	+ 0.670 736	+ 0.666 580	+ 0.662 472	+ 0.658 412	175
180	+ 0.701 430	+ 0.696 907	+ 0.692 439	+ 0.688 025	+ 0.683 664	+ 0.679 354	+ 0.675 095	+ 0.670 886	+ 0.666 726	+ 0.662 615	+ 0.658 552	180
ψ / z	1.31	1.32	1.33	1.34	1.35	1.36	1.37	1.38	1.39	1.40	1.41	ψ \ z

$_m^1$ TABLE FOR INTERPOLATING $= \frac{\mu}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	1.41	1.42	1.43	1.44	1.45	1.46	1.47	1.48	1.49	1.50	1.51	z / ψ
0°	− 9.581 11	− 8.801 03	− 8.096 16	− 7.457 84	− 6.878 58	− 6.351 84	5.871 95	− 5.433 95	− 5.033 46	− 4.666 67	− 4.330 18	0°
5	− 8.668 91	− 7.989 94	− 7.373 06	− 6.811 55	− 6.299 53	− 5.831 84	− 5.403 94	5.011 82	− 4.651 93	4.321 15	− 4.016 68	5
10	− 6.611 23	− 6.141 59	− 5.709 33	− 5.311 01	− 4.943 56	− 4.604 24	− 4.290 54	− 4.000 25	− 3.731 33	− 3.481 97	− 3.250 53	10
15	− 4.513 64	− 4.228 20	− 3.961 68	− 3.712 70	− 3.480 00	− 3.262 40	− 3.058 82	− 2.868 26	− 2.689 78	− 2.522 54	− 2.365 75	15
20	2.897 27	− 2.731 00	− 2.573 88	− 2.425 39	− 2.285 05	− 2.152 38	− 2.026 95	− 1.908 35	− 1.796 18	− 1.690 07	− 1.589 67	20
25	− 1.785 86	− 1.688 33	− 1.595 36	1.506 74	− 1.422 28	− 1.341 78	− 1.265 06	− 1.191 95	− 1.122 28	− 1.055 88	− 0.992 612	25
30	1.049 11	0.990 346	− 0.933 992	− 0.879 963	− 0.828 170	− 0.778 529	− 0.730 957	− 0.685 372	− 0.641 698	− 0.599 858	− 0.559 777	30
35	0.560 912	− 0.524 431	0.489 317	− 0.455 525	− 0.423 017	− 0.391 744	− 0.361 668	− 0.332 746	− 0.304 937	− 0.278 207	− 0.252 507	35
40	0.232 344	− 0.209 144	− 0.186 768	− 0.165 191	− 0.144 390	− 0.124 341	− 0.105 019	− 0.086 404	− 0.068 472	− 0.051 202	− 0.034 571	40
45	0.006 368	+ 0.008 591	+ 0.023 029	+ 0.036 959	+ 0.050 397	+ 0.063 357	+ 0.075 853	+ 0.087 899	+ 0.099 509	+ 0.110 695	+ 0.121 471	45
50	+ 0.152 700	+ 0.162 339	+ 0.171 636	+ 0.180 599	+ 0.189 238	+ 0.197 562	+ 0.205 581	+ 0.213 304	+ 0.220 738	+ 0.227 893	+ 0.234 777	50
55	+ 0.267 234	+ 0.273 313	+ 0.279 160	+ 0.284 782	+ 0.290 186	+ 0.295 378	+ 0.300 364	+ 0.305 151	+ 0.309 743	+ 0.314 148	+ 0.318 370	55
60	+ 0.351 470	+ 0.355 086	+ 0.358 543	+ 0.361 847	+ 0.365 002	+ 0.368 012	+ 0.370 882	+ 0.373 617	+ 0.376 219	+ 0.378 694	+ 0.381 044	60
65	+ 0.414 638	+ 0.416 500	+ 0.418 253	+ 0.419 903	+ 0.421 449	+ 0.422 899	+ 0.424 253	+ 0.425 514	+ 0.426 686	+ 0.427 772	+ 0.428 775	65
70	+ 0.462 847	+ 0.463 428	+ 0.463 936	+ 0.464 374	+ 0.464 744	+ 0.465 049	+ 0.465 290	+ 0.465 470	+ 0.465 590	+ 0.465 653	+ 0.465 662	70
75	+ 0.500 228	+ 0.499 848	+ 0.499 430	+ 0.498 962	+ 0.498 451	+ 0.497 898	+ 0.497 305	+ 0.496 672	+ 0.496 004	+ 0.495 299	+ 0.494 560	75
80	+ 0.529 629	+ 0.528 524	+ 0.527 392	+ 0.526 234	+ 0.525 050	+ 0.523 843	+ 0.522 612	+ 0.521 359	+ 0.520 086	+ 0.518 793	+ 0.517 482	80
85	+ 0.553 048	+ 0.551 379	+ 0.549 606	+ 0.548 002	+ 0.546 297	+ 0.544 581	+ 0.542 855	+ 0.541 121	+ 0.539 378	+ 0.537 628	+ 0.535 870	85
90	+ 0.571 914	+ 0.569 801	+ 0.567 686	+ 0.565 570	+ 0.563 454	+ 0.561 337	+ 0.559 222	+ 0.557 107	+ 0.554 994	+ 0.552 882	+ 0.550 773	90
95	+ 0.587 266	+ 0.584 798	+ 0.582 338	+ 0.579 886	+ 0.577 442	+ 0.575 006	+ 0.572 579	+ 0.570 160	+ 0.567 751	+ 0.565 351	+ 0.562 961	95
100	+ 0.599 868	+ 0.597 115	+ 0.594 376	+ 0.591 652	+ 0.588 944	+ 0.586 250	+ 0.583 571	+ 0.580 808	+ 0.578 259	+ 0.575 626	+ 0.573 009	100
105	+ 0.610 291	+ 0.607 305	+ 0.604 343	+ 0.601 395	+ 0.598 471	+ 0.595 567	+ 0.592 683	+ 0.589 820	+ 0.586 976	+ 0.584 153	+ 0.581 350	105
110	+ 0.618 967	+ 0.615 791	+ 0.612 639	+ 0.609 512	+ 0.606 411	+ 0.603 334	+ 0.600 282	+ 0.597 254	+ 0.594 251	+ 0.591 271	+ 0.588 316	110
115	+ 0.626 228	+ 0.622 893	+ 0.619 587	+ 0.616 310	+ 0.613 062	+ 0.609 842	+ 0.606 650	+ 0.603 487	+ 0.600 351	+ 0.597 242	+ 0.594 161	115
120	+ 0.632 328	+ 0.628 861	+ 0.625 427	+ 0.622 025	+ 0.618 655	+ 0.615 317	+ 0.612 009	+ 0.608 732	+ 0.605 485	+ 0.602 269	+ 0.599 082	120
125	+ 0.637 464	+ 0.633 888	+ 0.630 348	+ 0.626 842	+ 0.623 370	+ 0.619 932	+ 0.616 527	+ 0.613 156	+ 0.609 817	+ 0.606 511	+ 0.603 238	125
130	+ 0.641 795	+ 0.638 127	+ 0.634 497	+ 0.630 904	+ 0.627 347	+ 0.623 826	+ 0.620 341	+ 0.616 890	+ 0.613 474	+ 0.610 092	+ 0.606 744	130
135	+ 0.645 444	+ 0.641 700	+ 0.637 995	+ 0.634 328	+ 0.630 699	+ 0.627 109	+ 0.623 556	+ 0.620 039	+ 0.616 558	+ 0.613 113	+ 0.609 703	135
140	+ 0.648 507	+ 0.644 698	+ 0.640 930	+ 0.637 203	+ 0.633 515	+ 0.629 867	+ 0.626 256	+ 0.622 684	+ 0.619 150	+ 0.615 653	+ 0.612 192	140
145	+ 0.651 065	+ 0.647 201	+ 0.643 381	+ 0.639 603	+ 0.635 866	+ 0.632 169	+ 0.628 512	+ 0.624 894	+ 0.621 314	+ 0.617 773	+ 0.614 268	145
150	+ 0.653 173	+ 0.649 266	+ 0.645 405	+ 0.641 586	+ 0.637 808	+ 0.634 071	+ 0.630 375	+ 0.626 720	+ 0.623 103	+ 0.619 526	+ 0.615 987	150
155	+ 0.654 887	+ 0.650 945	+ 0.647 048	+ 0.643 195	+ 0.639 384	+ 0.635 616	+ 0.631 889	+ 0.628 203	+ 0.624 557	+ 0.620 951	+ 0.617 384	155
160	+ 0.656 242	+ 0.652 273	+ 0.648 349	+ 0.644 469	+ 0.640 633	+ 0.636 839	+ 0.633 187	+ 0.629 377	+ 0.625 707	+ 0.622 078	+ 0.618 488	160
165	+ 0.657 268	+ 0.653 278	+ 0.649 334	+ 0.645 434	+ 0.641 578	+ 0.637 765	+ 0.633 994	+ 0.630 266	+ 0.626 578	+ 0.622 931	+ 0.619 323	165
170	+ 0.657 986	+ 0.653 981	+ 0.650 022	+ 0.646 108	+ 0.642 238	+ 0.638 412	+ 0.634 628	+ 0.630 887	+ 0.627 187	+ 0.623 528	+ 0.619 909	170
175	+ 0.658 412	+ 0.654 398	+ 0.650 430	+ 0.646 508	+ 0.642 630	+ 0.638 795	+ 0.635 004	+ 0.631 255	+ 0.627 548	+ 0.623 882	+ 0.620 256	175
180	+ 0.658 552	+ 0.654 536	+ 0.650 565	+ 0.646 640	+ 0.642 760	+ 0.638 923	+ 0.635 129	+ 0.631 378	+ 0.627 669	+ 0.624 000	+ 0.620 373	180
ψ / z	1.41	1.42	1.43	1.44	1.45	1.46	1.47	1.48	1.49	1.50	1.51	z / ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	1.51	1.52	1.53	1.54	1.55	1.56	1.57	1.58	1.59	1.60	1.61	z / ψ
0°	− 4.330 18	− 4.021 03	− 3.736 57	− 3.474 45	− 3.232 59	− 3.009 13	− 2.802 40	− 2.610 92	− 2.433 36	− 2.268 52	− 2.115 31	0°
5	− 4.016 68	− 3.736 06	− 3.477 08	− 3.237 76	− 3.016 34	− 2.811 24	− 2.621 03	− 2.444 45	− 2.280 35	− 2.127 68	− 1.985 51	5
10	− 3.250 53	− 3.035 48	− 2.835 52	− 2.649 41	− 2.476 03	− 2.314 38	2.163 53	− 2.022 65	− 1.890 98	− 1.767 82	− 1.652 53	10
15	− 2.365 75	− 2.218 67	2.080 63	− 1.951 02	− 1.829 24	− 1.714 78	− 1.607 13	− 1.505 84	− 1.410 48	− 1.320 67	− 1.236 04	15
20	− 1.589 67	− 1.494 66	− 1.404 72	− 1.319 57	− 1.238 93	− 1.162 54	− 1.090 17	− 1.021 57	− 0.956 547	− 0.894 888	− 0.836 407	20
25	− 0.992 612	− 0.932 312	− 0.874 840	0.820 062	− 0.767 851	0.718 078	− 0.670 629	− 0.625 391	− 0.582 261	− 0.541 129	− 0.501 904	25
30	− 0.559 777	− 0.521 384	0.484 613	− 0.449 394	− 0.415 664	− 0.383 361	− 0.352 428	− 0.322 804	− 0.294 436	− 0.267 271	− 0.241 256	30
35	− 0.252 507	− 0.227 820	− 0.204 094	− 0.181 295	− 0.159 393	− 0.138 353	− 0.118 144	− 0.098 734	− 0.080 094	− 0.062 196	− 0.045 010	35
40	− 0.034 571	− 0.018 559	− 0.003 146	+ 0.011 688	+ 0.025 963	+ 0.039 698	+ 0.052 911	+ 0.065 620	+ 0.077 841	+ 0.089 593	+ 0.100 892	40
45	+ 0.121 471	+ 0.131 850	+ 0.141 843	+ 0.151 464	+ 0.160 723	+ 0.169 634	+ 0.178 206	+ 0.186 450	+ 0.194 378	+ 0.202 000	+ 0.209 325	45
50	+ 0.234 777	+ 0.241 399	+ 0.247 765	+ 0.253 886	+ 0.259 767	+ 0.265 416	+ 0.270 841	+ 0.276 048	+ 0.281 045	+ 0.285 838	+ 0.290 434	50
55	+ 0.318 370	+ 0.322 416	+ 0.326 289	+ 0.329 996	+ 0.333 542	+ 0.336 933	+ 0.340 172	+ 0.343 264	+ 0.346 215	+ 0.349 028	+ 0.351 709	55
60	+ 0.381 044	+ 0.383 275	+ 0.385 390	+ 0.387 392	+ 0.389 284	+ 0.391 072	+ 0.392 757	+ 0.394 343	+ 0.395 833	+ 0.397 230	+ 0.398 538	60
65	+ 0.428 775	+ 0.429 696	+ 0.430 539	+ 0.431 307	+ 0.432 001	+ 0.432 624	+ 0.433 179	+ 0.433 668	+ 0.434 093	+ 0.434 457	+ 0.434 761	65
70	+ 0.465 662	+ 0.465 617	+ 0.465 521	+ 0.465 376	+ 0.465 186	+ 0.464 943	+ 0.464 660	+ 0.464 333	+ 0.463 966	+ 0.463 559	+ 0.463 114	70
75	+ 0.494 560	+ 0.493 787	+ 0.492 984	+ 0.492 149	+ 0.491 286	+ 0.490 395	+ 0.489 478	+ 0.488 534	+ 0.487 567	+ 0.486 576	+ 0.485 563	75
80	+ 0.517 482	+ 0.516 152	+ 0.514 805	+ 0.513 443	+ 0.512 065	+ 0.510 673	+ 0.509 268	+ 0.507 850	+ 0.506 420	+ 0.504 978	+ 0.503 527	80
85	+ 0.535 870	+ 0.534 106	+ 0.532 337	+ 0.530 563	+ 0.528 784	+ 0.527 001	+ 0.525 214	+ 0.523 425	+ 0.521 634	+ 0.519 840	+ 0.518 045	85
90	+ 0.550 773	+ 0.548 666	+ 0.546 563	+ 0.544 463	+ 0.542 367	+ 0.540 274	+ 0.538 187	+ 0.536 103	+ 0.534 025	+ 0.531 952	+ 0.529 885	90
95	+ 0.562 961	+ 0.560 580	+ 0.558 209	+ 0.555 849	+ 0.553 498	+ 0.551 159	+ 0.548 829	+ 0.546 511	+ 0.544 203	+ 0.541 906	+ 0.539 626	95
100	+ 0.573 009	+ 0.570 406	+ 0.567 820	+ 0.565 248	+ 0.562 692	+ 0.560 152	+ 0.557 627	+ 0.555 118	+ 0.552 624	+ 0.550 145	+ 0.547 682	100
105	+ 0.581 350	+ 0.578 570	+ 0.575 804	+ 0.573 060	+ 0.570 337	+ 0.567 633	+ 0.564 948	+ 0.562 283	+ 0.559 637	+ 0.557 010	+ 0.554 402	105
110	+ 0.588 316	+ 0.585 384	+ 0.582 475	+ 0.579 590	+ 0.576 728	+ 0.573 889	+ 0.571 073	+ 0.568 279	+ 0.565 508	+ 0.562 759	+ 0.560 032	110
115	+ 0.594 161	+ 0.591 106	+ 0.588 077	+ 0.585 074	+ 0.582 097	+ 0.579 147	+ 0.576 221	+ 0.573 321	+ 0.570 446	+ 0.567 596	+ 0.564 770	115
120	+ 0.599 082	+ 0.595 925	+ 0.592 795	+ 0.589 695	+ 0.586 623	+ 0.583 580	+ 0.580 564	+ 0.577 575	+ 0.574 613	+ 0.571 678	+ 0.568 769	120
125	+ 0.603 238	+ 0.599 993	+ 0.596 781	+ 0.593 600	+ 0.590 448	+ 0.587 326	+ 0.584 235	+ 0.581 173	+ 0.578 138	+ 0.575 130	+ 0.572 151	125
130	+ 0.606 744	+ 0.603 429	+ 0.600 146	+ 0.596 896	+ 0.593 678	+ 0.590 491	+ 0.587 336	+ 0.584 211	+ 0.581 115	+ 0.578 050	+ 0.575 014	130
135	+ 0.609 703	+ 0.606 328	+ 0.602 987	+ 0.599 680	+ 0.596 406	+ 0.593 164	+ 0.589 956	+ 0.586 778	+ 0.583 633	+ 0.580 518	+ 0.577 434	135
140	+ 0.612 192	+ 0.608 766	+ 0.605 376	+ 0.602 021	+ 0.598 700	+ 0.595 413	+ 0.592 160	+ 0.588 939	+ 0.585 750	+ 0.582 594	+ 0.579 469	140
145	+ 0.614 268	+ 0.610 803	+ 0.607 372	+ 0.603 977	+ 0.600 618	+ 0.597 293	+ 0.594 002	+ 0.590 745	+ 0.587 523	+ 0.584 331	+ 0.581 172	145
150	+ 0.615 987	+ 0.612 487	+ 0.609 023	+ 0.605 595	+ 0.602 203	+ 0.598 847	+ 0.595 526	+ 0.592 239	+ 0.588 986	+ 0.585 768	+ 0.582 582	150
155	+ 0.617 384	+ 0.613 855	+ 0.610 363	+ 0.606 910	+ 0.603 492	+ 0.600 111	+ 0.596 764	+ 0.593 454	+ 0.590 178	+ 0.586 936	+ 0.583 727	155
160	+ 0.618 488	+ 0.614 937	+ 0.611 425	+ 0.607 950	+ 0.604 512	+ 0.601 111	+ 0.597 746	+ 0.594 416	+ 0.591 121	+ 0.587 861	+ 0.584 635	160
165	+ 0.619 323	+ 0.615 756	+ 0.612 227	+ 0.608 737	+ 0.605 285	+ 0.601 869	+ 0.598 490	+ 0.595 146	+ 0.591 837	+ 0.588 563	+ 0.585 323	165
170	+ 0.619 909	+ 0.616 331	+ 0.612 791	+ 0.609 289	+ 0.605 825	+ 0.602 399	+ 0.599 009	+ 0.595 655	+ 0.592 337	+ 0.589 053	+ 0.585 804	170
175	+ 0.620 256	+ 0.616 672	+ 0.613 125	+ 0.609 617	+ 0.606 147	+ 0.602 714	+ 0.599 318	+ 0.595 958	+ 0.592 634	+ 0.589 344	+ 0.586 089	175
180	+ 0.620 373	+ 0.616 784	+ 0.613 235	+ 0.609 725	+ 0.606 253	+ 0.602 818	+ 0.599 420	+ 0.596 058	+ 0.592 731	+ 0.589 440	+ 0.586 183	180
ψ / z	1.51	1.52	1.53	1.54	1.55	1.56	1.57	1.58	1.59	1.60	1.61	ψ \ z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	1.61	1.62	1.63	1.64	1.65	1.66	1.67	1.68	1.69	1.70	1.71	z / ψ
0°	− 2.115 31	− 1.972 78	− 1.840 03	− 1.716 28	− 1.600 81	− 1.492 96	− 1.392 14	− 1.297 82	− 1.209 50	− 1.126 74	− 1.049 12	0°
5	− 1.985 51	− 1.852 98	− 1.729 33	− 1.613 87	− 1.505 95	− 1.405 00	− 1.310 48	− 1.221 93	− 1.138 90	− 1.060 99	− 0.987 826	5
10	− 1.652 53	− 1.544 53	− 1.443 29	− 1.348 31	− 1.259 16	− 1.175 41	− 1.096 69	− 1.022 65	− 0.952 978	− 0.887 368	− 0.825 550	10
15	− 1.236 04	− 1.156 25	− 1.081 00	− 1.009 98	− 0.942 943	− 0.879 624	− 0.819 795	− 0.763 238	− 0.709 752	− 0.659 151	− 0.611 258	15
20	− 0.836 407	− 0.780 925	− 0.728 276	− 0.678 301	− 0.630 853	− 0.585 792	− 0.542 987	− 0.502 316	− 0.463 663	− 0.426 918	− 0.391 979	20
25	− 0.501 904	− 0.464 492	− 0.428 807	− 0.394 764	− 0.362 286	− 0.331 294	− 0.301 719	− 0.273 495	− 0.246 555	− 0.220 838	− 0.196 285	25
30	− 0.241 256	− 0.216 348	− 0.192 496	− 0.169 655	− 0.147 783	− 0.126 838	− 0.106 782	− 0.087 576	− 0.069 185	− 0.051 572	− 0.034 707	30
35	− 0.045 010	− 0.028 511	− 0.012 670	+ 0.002 535	+ 0.017 130	+ 0.031 138	+ 0.044 581	+ 0.057 482	+ 0.069 861	+ 0.081 738	+ 0.093 133	35
40	+ 0.100 892	+ 0.111 752	+ 0.122 190	+ 0.132 220	+ 0.141 858	+ 0.151 116	+ 0.160 008	+ 0.168 548	+ 0.176 747	+ 0.184 619	+ 0.192 174	40
45	+ 0.209 325	+ 0.216 365	+ 0.223 127	+ 0.229 621	+ 0.235 857	+ 0.241 843	+ 0.247 587	+ 0.253 098	+ 0.258 384	+ 0.263 451	+ 0.268 309	45
50	+ 0.290 434	+ 0.294 838	+ 0.299 058	+ 0.303 099	+ 0.306 967	+ 0.310 668	+ 0.314 207	+ 0.317 589	+ 0.320 820	+ 0.323 905	+ 0.326 848	50
55	+ 0.351 709	+ 0.354 260	+ 0.356 688	+ 0.358 994	+ 0.361 184	+ 0.363 261	+ 0.365 229	+ 0.367 091	+ 0.368 852	+ 0.370 513	+ 0.372 079	55
60	+ 0.398 538	+ 0.399 759	+ 0.400 897	+ 0.401 953	+ 0.402 931	+ 0.403 832	+ 0.404 661	+ 0.405 419	+ 0.406 108	+ 0.406 732	+ 0.407 291	60
65	+ 0.434 761	+ 0.435 007	+ 0.435 198	+ 0.435 336	+ 0.435 420	+ 0.435 456	+ 0.435 444	+ 0.435 385	+ 0.435 281	+ 0.435 134	+ 0.434 946	65
70	+ 0.463 114	+ 0.462 633	+ 0.462 116	+ 0.461 565	+ 0.460 982	+ 0.460 368	+ 0.459 723	+ 0.459 050	+ 0.458 348	+ 0.457 620	+ 0.456 867	70
75	+ 0.485 563	+ 0.484 528	+ 0.483 473	+ 0.482 399	+ 0.481 306	+ 0.480 195	+ 0.479 068	+ 0.477 925	+ 0.476 767	+ 0.475 594	+ 0.474 408	75
80	+ 0.503 527	+ 0.502 065	+ 0.500 594	+ 0.499 115	+ 0.497 628	+ 0.496 133	+ 0.494 632	+ 0.493 124	+ 0.491 611	+ 0.490 092	+ 0.488 569	80
85	+ 0.518 045	+ 0.516 249	+ 0.514 452	+ 0.512 655	+ 0.510 858	+ 0.509 061	+ 0.507 267	+ 0.505 472	+ 0.503 680	+ 0.501 889	+ 0.500 100	85
90	+ 0.529 885	+ 0.527 824	+ 0.525 768	+ 0.523 719	+ 0.521 676	+ 0.519 640	+ 0.517 610	+ 0.515 588	+ 0.513 573	+ 0.511 565	+ 0.509 565	90
95	+ 0.539 620	+ 0.537 345	+ 0.535 081	+ 0.532 830	+ 0.530 589	+ 0.528 360	+ 0.526 143	+ 0.523 937	+ 0.521 742	+ 0.519 560	+ 0.517 389	95
100	+ 0.547 682	+ 0.545 235	+ 0.542 803	+ 0.540 386	+ 0.537 985	+ 0.535 600	+ 0.533 229	+ 0.530 874	+ 0.528 534	+ 0.526 209	+ 0.523 899	100
105	+ 0.554 402	+ 0.551 813	+ 0.549 243	+ 0.546 692	+ 0.544 159	+ 0.541 645	+ 0.539 149	+ 0.536 672	+ 0.534 212	+ 0.531 771	+ 0.529 347	105
110	+ 0.560 032	+ 0.557 327	+ 0.554 642	+ 0.551 980	+ 0.549 339	+ 0.546 718	+ 0.544 119	+ 0.541 541	+ 0.538 983	+ 0.536 445	+ 0.533 928	110
115	+ 0.564 770	+ 0.561 966	+ 0.559 187	+ 0.556 433	+ 0.553 701	+ 0.550 994	+ 0.548 310	+ 0.545 648	+ 0.543 008	+ 0.540 390	+ 0.537 794	115
120	+ 0.568 769	+ 0.565 886	+ 0.563 029	+ 0.560 198	+ 0.557 392	+ 0.554 610	+ 0.551 853	+ 0.549 121	+ 0.546 412	+ 0.543 728	+ 0.541 067	120
125	+ 0.572 151	+ 0.569 201	+ 0.566 278	+ 0.563 383	+ 0.560 514	+ 0.557 671	+ 0.554 854	+ 0.552 064	+ 0.549 300	+ 0.546 558	+ 0.543 841	125
130	+ 0.575 014	+ 0.572 008	+ 0.569 030	+ 0.566 080	+ 0.563 158	+ 0.560 264	+ 0.557 397	+ 0.554 556	+ 0.551 742	+ 0.548 954	+ 0.546 192	130
135	+ 0.577 434	+ 0.574 380	+ 0.571 355	+ 0.568 360	+ 0.565 394	+ 0.562 457	+ 0.559 547	+ 0.556 665	+ 0.553 811	+ 0.550 984	+ 0.548 183	135
140	+ 0.579 469	+ 0.576 376	+ 0.573 313	+ 0.570 280	+ 0.567 276	+ 0.564 303	+ 0.561 358	+ 0.558 442	+ 0.555 554	+ 0.552 694	+ 0.549 861	140
145	+ 0.581 172	+ 0.578 046	+ 0.574 951	+ 0.571 886	+ 0.568 852	+ 0.565 849	+ 0.562 875	+ 0.559 930	+ 0.557 014	+ 0.554 127	+ 0.551 267	145
150	+ 0.582 582	+ 0.579 428	+ 0.576 306	+ 0.573 216	+ 0.570 157	+ 0.567 129	+ 0.564 130	+ 0.561 162	+ 0.558 223	+ 0.555 313	+ 0.552 431	150
155	+ 0.583 727	+ 0.580 551	+ 0.577 407	+ 0.574 297	+ 0.571 218	+ 0.568 169	+ 0.565 151	+ 0.562 164	+ 0.559 207	+ 0.556 278	+ 0.553 379	155
160	+ 0.584 635	+ 0.581 442	+ 0.578 282	+ 0.575 154	+ 0.572 058	+ 0.568 994	+ 0.565 961	+ 0.562 958	+ 0.559 985	+ 0.557 043	+ 0.554 130	160
165	+ 0.585 323	+ 0.582 116	+ 0.578 943	+ 0.575 803	+ 0.572 695	+ 0.569 618	+ 0.566 574	+ 0.563 560	+ 0.560 577	+ 0.557 623	+ 0.554 699	165
170	+ 0.585 804	+ 0.582 589	+ 0.579 408	+ 0.576 258	+ 0.573 141	+ 0.570 057	+ 0.567 004	+ 0.563 982	+ 0.560 991	+ 0.558 030	+ 0.555 097	170
175	+ 0.586 089	+ 0.582 868	+ 0.579 681	+ 0.576 527	+ 0.573 405	+ 0.570 316	+ 0.567 258	+ 0.564 232	+ 0.561 236	+ 0.558 270	+ 0.555 334	175
180	+ 0.586 183	+ 0.582 961	+ 0.579 772	+ 0.576 617	+ 0.573 494	+ 0.570 403	+ 0.567 344	+ 0.564 315	+ 0.561 317	+ 0.558 349	+ 0.555 411	180
ψ / z	1.61	1.62	1.63	1.64	1.65	1.66	1.67	1.68	1.69	1.70	1.71	ψ \ z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{u}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	1.71	1.72	1.73	1.74	1.75	1.76	1.77	1.78	1.79	1.80	1.81	z / ψ
0	− 1.049 12	− 0.976 270	− 0.907 851	− 0.843 547	− 0.783 069	− 0.726 153	− 0.672 555	− 0.622 051	− 0.574 434	− 0.529 514	− 0.487 114	0°
5	− 0.987 826	− 0.919 078	− 0.854 436	− 0.793 612	− 0.736 347	− 0.682 399	− 0.631 547	− 0.583 584	− 0.538 321	− 0.495 584	− 0.455 211	5
10	− 0.825 550	− 0.767 272	− 0.713 303	− 0.660 426	− 0.611 444	− 0.565 171	− 0.521 436	− 0.480 081	− 0.440 959	− 0.403 933	− 0.368 875	10
15	− 0.611 258	− 0.565 912	− 0.522 962	− 0.482 263	− 0.443 686	− 0.407 106	− 0.372 407	− 0.339 481	− 0.308 232	− 0.278 558	− 0.250 374	15
20	− 0.391 979	− 0.358 740	− 0.327 138	− 0.297 059	− 0.268 433	− 0.241 182	− 0.215 236	− 0.190 527	− 0.166 991	− 0.144 569	− 0.123 202	20
25	− 0.196 285	− 0.172 842	− 0.150 458	− 0.129 080	− 0.108 663	− 0.089 161	− 0.070 531	− 0.052 732	− 0.035 725	− 0.019 475	− 0.003 947	25
30	− 0.034 707	− 0.018 556	− 0.003 090	+ 0.011 721	+ 0.025 904	+ 0.039 486	+ 0.052 492	+ 0.064 947	+ 0.076 874	+ 0.088 295	+ 0.099 231	30
35	+ 0.093 133	+ 0.104 065	+ 0.114 551	+ 0.124 608	+ 0.134 254	+ 0.143 503	+ 0.152 373	+ 0.160 876	+ 0.169 028	+ 0.176 841	+ 0.184 330	35
40	+ 0.192 174	+ 0.199 425	+ 0.206 382	+ 0.213 055	+ 0.219 456	+ 0.225 593	+ 0.231 477	+ 0.237 117	+ 0.242 521	+ 0.247 698	+ 0.252 656	40
45	+ 0.268 306	+ 0.272 963	+ 0.277 421	+ 0.281 689	+ 0.285 775	+ 0.289 684	+ 0.293 422	+ 0.296 996	+ 0.300 412	+ 0.303 673	+ 0.306 787	45
50	+ 0.326 848	+ 0.329 654	+ 0.332 327	+ 0.334 875	+ 0.337 298	+ 0.339 602	+ 0.341 791	+ 0.343 868	+ 0.345 838	+ 0.347 703	+ 0.349 469	50
55	+ 0.372 079	+ 0.373 553	+ 0.374 938	+ 0.376 237	+ 0.377 452	+ 0.378 586	+ 0.379 644	+ 0.380 625	+ 0.381 535	+ 0.382 374	+ 0.383 146	55
60	+ 0.407 291	+ 0.407 789	+ 0.408 228	+ 0.408 609	+ 0.408 934	+ 0.409 206	+ 0.409 426	+ 0.409 596	+ 0.409 718	+ 0.409 794	+ 0.409 825	60
65	+ 0.434 946	+ 0.434 717	+ 0.434 450	+ 0.434 146	+ 0.433 806	+ 0.433 431	+ 0.433 024	+ 0.432 584	+ 0.432 114	+ 0.431 614	+ 0.431 086	65
70	+ 0.456 867	+ 0.456 093	+ 0.455 286	+ 0.454 465	+ 0.453 620	+ 0.452 754	+ 0.451 869	+ 0.450 965	+ 0.450 043	+ 0.449 104	+ 0.448 148	70
75	+ 0.474 408	+ 0.473 208	+ 0.471 997	+ 0.470 774	+ 0.469 540	+ 0.468 296	+ 0.467 042	+ 0.465 779	+ 0.464 508	+ 0.463 228	+ 0.461 942	75
80	+ 0.488 569	+ 0.487 042	+ 0.485 510	+ 0.483 976	+ 0.482 439	+ 0.480 899	+ 0.479 357	+ 0.477 813	+ 0.476 268	+ 0.474 722	+ 0.473 176	80
85	+ 0.500 100	+ 0.498 314	+ 0.496 531	+ 0.494 751	+ 0.492 973	+ 0.491 200	+ 0.489 430	+ 0.487 664	+ 0.485 902	+ 0.484 145	+ 0.482 392	85
90	+ 0.509 565	+ 0.507 572	+ 0.505 588	+ 0.503 611	+ 0.501 642	+ 0.499 681	+ 0.497 729	+ 0.495 785	+ 0.493 850	+ 0.491 923	+ 0.490 005	90
95	+ 0.517 389	+ 0.515 230	+ 0.513 083	+ 0.510 948	+ 0.508 824	+ 0.506 712	+ 0.504 613	+ 0.502 525	+ 0.500 449	+ 0.498 385	+ 0.496 333	95
100	+ 0.523 899	+ 0.521 605	+ 0.519 325	+ 0.517 061	+ 0.514 811	+ 0.512 577	+ 0.510 357	+ 0.508 151	+ 0.505 961	+ 0.503 785	+ 0.501 623	100
105	+ 0.529 347	+ 0.526 941	+ 0.524 553	+ 0.522 183	+ 0.519 829	+ 0.517 494	+ 0.515 175	+ 0.512 873	+ 0.510 588	+ 0.508 320	+ 0.506 068	105
110	+ 0.533 928	+ 0.531 430	+ 0.528 952	+ 0.526 494	+ 0.524 054	+ 0.521 635	+ 0.519 234	+ 0.516 852	+ 0.514 489	+ 0.512 145	+ 0.509 818	110
115	+ 0.537 794	+ 0.535 220	+ 0.532 667	+ 0.530 136	+ 0.527 626	+ 0.525 136	+ 0.522 667	+ 0.520 219	+ 0.517 790	+ 0.515 382	+ 0.512 993	115
120	+ 0.541 067	+ 0.538 429	+ 0.535 814	+ 0.533 221	+ 0.530 651	+ 0.528 104	+ 0.525 578	+ 0.523 073	+ 0.520 591	+ 0.518 129	+ 0.515 689	120
125	+ 0.543 841	+ 0.541 149	+ 0.538 482	+ 0.535 838	+ 0.533 218	+ 0.530 622	+ 0.528 049	+ 0.525 497	+ 0.522 969	+ 0.520 463	+ 0.517 979	125
130	+ 0.546 192	+ 0.543 456	+ 0.540 745	+ 0.538 059	+ 0.535 397	+ 0.532 759	+ 0.530 145	+ 0.527 555	+ 0.524 988	+ 0.522 445	+ 0.519 925	130
135	+ 0.548 183	+ 0.545 409	+ 0.542 661	+ 0.539 939	+ 0.537 242	+ 0.534 570	+ 0.531 922	+ 0.529 299	+ 0.526 700	+ 0.524 125	+ 0.521 573	135
140	+ 0.549 861	+ 0.547 056	+ 0.544 277	+ 0.541 525	+ 0.538 798	+ 0.536 098	+ 0.533 423	+ 0.530 772	+ 0.528 146	+ 0.525 544	+ 0.522 966	140
145	+ 0.551 267	+ 0.548 436	+ 0.545 632	+ 0.542 854	+ 0.540 103	+ 0.537 378	+ 0.534 679	+ 0.532 006	+ 0.529 357	+ 0.526 733	+ 0.524 134	145
150	+ 0.552 431	+ 0.549 578	+ 0.546 752	+ 0.543 954	+ 0.541 183	+ 0.538 438	+ 0.535 720	+ 0.533 028	+ 0.530 361	+ 0.527 719	+ 0.525 102	150
155	+ 0.553 379	+ 0.550 508	+ 0.547 665	+ 0.544 850	+ 0.542 063	+ 0.539 302	+ 0.536 568	+ 0.533 860	+ 0.531 178	+ 0.528 522	+ 0.525 891	155
160	+ 0.554 130	+ 0.551 245	+ 0.548 390	+ 0.545 562	+ 0.542 761	+ 0.539 987	+ 0.537 241	+ 0.534 521	+ 0.531 827	+ 0.529 159	+ 0.526 516	160
165	+ 0.554 699	+ 0.551 804	+ 0.548 937	+ 0.546 099	+ 0.543 289	+ 0.540 506	+ 0.537 750	+ 0.535 020	+ 0.532 318	+ 0.529 642	+ 0.526 991	165
170	+ 0.555 097	+ 0.552 195	+ 0.549 322	+ 0.546 477	+ 0.543 660	+ 0.540 870	+ 0.538 108	+ 0.535 373	+ 0.532 664	+ 0.529 981	+ 0.527 324	170
175	+ 0.555 334	+ 0.552 427	+ 0.549 549	+ 0.546 700	+ 0.543 879	+ 0.541 086	+ 0.538 320	+ 0.535 581	+ 0.532 868	+ 0.530 182	+ 0.527 521	175
180	+ 0.555 411	+ 0.552 504	+ 0.549 625	+ 0.546 775	+ 0.543 952	+ 0.541 157	+ 0.538 390	+ 0.535 649	+ 0.532 934	+ 0.530 248	+ 0.527 585	180
ψ / z	1.71	1.72	1.73	1.74	1.75	1.76	1.77	1.78	1.79	1.80	1.81	ψ / z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	1.81	1.82	1.83	1.84	1.85	1.86	1.87	1.88	1.89	1.90	1.91	z / ψ
0°	− 0.487 114	− 0.447 072	− 0.409 236	− 0.373 469	− 0.339 639	− 0.307 628	− 0.277 324	− 0.248 623	− 0.221 430	− 0.195 654	− 0.171 212	0°
5	− 0.455 211	− 0.417 053	− 0.380 970	− 0.346 833	− 0.314 524	− 0.283 929	− 0.254 947	− 0.227 480	− 0.201 440	− 0.176 743	− 0.153 311	5
10	− 0.368 875	− 0.335 667	− 0.304 199	− 0.274 368	− 0.246 078	0.219 240	− 0.193 770	− 0.169 591	− 0.146 630	− 0.124 817	− 0.104 091	10
15	− 0.250 374	− 0.223 599	− 0.198 152	− 0.173 963	0.150 961	− 0.129 084	− 0.108 270	− 0.088 465	0.069 613	0.051 665	0.034 575	15
20	− 0.123 202	− 0.102 839	− 0.083 428	− 0.064 922	− 0.047 275	− 0.030 446	− 0.014 393	+ 0.000 921	+ 0.015 532	+ 0.029 474	+ 0.042 781	20
25	0.003 947	+ 0.010 893	+ 0.025 076	+ 0.038 633	+ 0.051 592	+ 0.063 979	+ 0.075 820	+ 0.087 140	+ 0.097 963	+ 0.108 310	+ 0.118 203	25
30	+ 0.099 231	+ 0.109 703	+ 0.119 729	+ 0.129 330	+ 0.138 521	+ 0.147 321	+ 0.155 745	+ 0.163 809	+ 0.171 529	+ 0.178 917	+ 0.185 988	30
35	+ 0.184 330	+ 0.191 507	+ 0.198 382	+ 0.204 970	+ 0.211 280	+ 0.217 324	+ 0.223 110	+ 0.228 650	+ 0.233 953	+ 0.239 029	+ 0.243 884	35
40	+ 0.252 656	+ 0.257 404	+ 0.261 949	+ 0.266 297	+ 0.270 458	+ 0.274 436	+ 0.278 240	+ 0.281 874	+ 0.285 347	+ 0.288 662	+ 0.291 827	40
45	+ 0.306 787	+ 0.309 758	+ 0.312 591	+ 0.315 291	+ 0.317 862	+ 0.320 310	+ 0.322 638	+ 0.324 851	+ 0.326 952	+ 0.328 945	+ 0.330 837	45
50	+ 0.349 469	+ 0.351 137	+ 0.352 712	+ 0.354 196	+ 0.355 593	+ 0.356 900	+ 0.358 138	+ 0.359 291	+ 0.360 368	+ 0.361 372	+ 0.362 306	50
55	+ 0.383 146	+ 0.383 852	+ 0.384 496	+ 0.385 078	+ 0.385 602	+ 0.386 068	+ 0.386 481	+ 0.386 840	+ 0.387 148	+ 0.387 408	+ 0.387 62[illegible]	55
60	+ 0.409 825	+ 0.409 813	+ 0.409 759	+ 0.409 666	+ 0.409 533	+ 0.409 364	+ 0.409 158	+ 0.408 918	+ 0.408 645	+ 0.408 340	+ 0.408 004	60
65	+ 0.431 086	+ 0.430 531	+ 0.429 951	+ 0.429 343	+ 0.428 713	+ 0.428 059	+ 0.427 385	+ 0.426 68[illegible]	+ 0.425 968	+ 0.425 22[illegible]	+ 0.424 475	65
70	+ 0.448 148	+ 0.447 177	+ 0.446 191	+ 0.445 191	+ 0.444 177	+ 0.443 151	+ 0.442 112	+ 0.441 062	+ 0.440 001	+ 0.438 930	+ 0.437 849	70
75	+ 0.461 942	+ 0.460 648	+ 0.459 348	+ 0.458 041	+ 0.456 730	+ 0.455 414	+ 0.454 092	+ 0.452 768	+ 0.451 438	+ 0.450 105	+ 0.448 769	75
80	+ 0.473 176	+ 0.471 629	+ 0.470 082	+ 0.468 535	+ 0.466 989	+ 0.465 444	+ 0.463 900	+ 0.462 357	+ 0.460 816	+ 0.459 277	+ 0.457 739	80
85	+ 0.482 392	+ 0.480 644	+ 0.478 901	+ 0.477 163	+ 0.475 430	+ 0.473 703	+ 0.471 980	+ 0.470 265	+ 0.468 555	+ 0.466 85[illegible]	+ 0.465 152	85
90	+ 0.490 005	+ 0.488 096	+ 0.486 195	+ 0.484 303	+ 0.482 420	+ 0.480 547	+ 0.478 682	+ 0.476 826	+ 0.474 980	+ 0.473 142	+ 0.471 314	90
95	+ 0.496 333	+ 0.494 293	+ 0.492 264	+ 0.490 248	+ 0.488 244	+ 0.486 251	+ 0.484 270	+ 0.482 301	+ 0.480 343	+ 0.478 398	+ 0.476 464	95
100	+ 0.501 623	+ 0.499 476	+ 0.497 343	+ 0.495 225	+ 0.493 121	+ 0.491 031	+ 0.488 955	+ 0.486 893	+ 0.484 844	+ 0.482 810	+ 0.480 789	100
105	+ 0.506 068	+ 0.503 833	+ 0.501 614	+ 0.499 412	+ 0.497 225	+ 0.495 055	+ 0.492 901	+ 0.490 762	+ 0.488 638	+ 0.486 531	+ 0.484 438	105
110	+ 0.509 818	+ 0.507 510	+ 0.505 220	+ 0.502 948	+ 0.500 693	+ 0.498 456	+ 0.496 237	+ 0.494 034	+ 0.491 849	+ 0.489 680	+ 0.487 528	110
115	+ 0.512 803	+ 0.510 624	+ 0.508 274	+ 0.505 945	+ 0.503 634	+ 0.501 341	+ 0.499 067	+ 0.496 811	+ 0.494 573	+ 0.492 354	+ 0.490 152	115
120	+ 0.515 689	+ 0.513 269	+ 0.510 870	+ 0.508 491	+ 0.506 132	+ 0.503 793	+ 0.501 473	+ 0.499 173	+ 0.496 892	+ 0.494 631	+ 0.492 387	120
125	+ 0.517 979	+ 0.515 517	+ 0.513 076	+ 0.510 656	+ 0.508 257	+ 0.505 878	+ 0.503 520	+ 0.501 182	+ 0.498 863	+ 0.496 565	+ 0.494 287	125
130	+ 0.519 925	+ 0.517 426	+ 0.514 950	+ 0.512 495	+ 0.510 062	+ 0.507 651	+ 0.505 261	+ 0.502 892	+ 0.500 543	+ 0.498 215	+ 0.495 907	130
135	+ 0.521 573	+ 0.519 045	+ 0.516 539	+ 0.514 056	+ 0.511 595	+ 0.509 156	+ 0.506 738	+ 0.504 343	+ 0.501 969	+ 0.499 615	+ 0.497 282	135
140	+ 0.522 966	+ 0.520 412	+ 0.517 881	+ 0.515 374	+ 0.512 890	+ 0.510 429	+ 0.507 988	+ 0.505 570	+ 0.503 174	+ 0.500 799	+ 0.498 445	140
145	+ 0.524 134	+ 0.521 559	+ 0.519 008	+ 0.516 480	+ 0.513 976	+ 0.511 494	+ 0.509 036	+ 0.506 599	+ 0.504 184	+ 0.501 792	+ 0.499 422	145
150	+ 0.525 102	+ 0.522 510	+ 0.519 942	+ 0.517 398	+ 0.514 877	+ 0.512 380	+ 0.509 906	+ 0.507 454	+ 0.505 024	+ 0.502 616	+ 0.500 230	150
155	+ 0.525 891	+ 0.523 284	+ 0.520 702	+ 0.518 145	+ 0.515 611	+ 0.513 101	+ 0.510 614	+ 0.508 150	+ 0.505 708	+ 0.503 28[illegible]	+ 0.500 891	155
160	+ 0.526 516	+ 0.523 898	+ 0.521 306	+ 0.518 738	+ 0.516 194	+ 0.513 673	+ 0.511 176	+ 0.508 702	+ 0.506 252	+ 0.503 823	+ 0.501 417	160
165	+ 0.526 991	+ 0.524 364	+ 0.521 763	+ 0.519 187	+ 0.516 635	+ 0.514 107	+ 0.511 602	+ 0.509 121	+ 0.506 663	+ 0.504 227	+ 0.501 814	165
170	+ 0.527 324	+ 0.524 692	+ 0.522 085	+ 0.519 503	+ 0.516 945	+ 0.514 412	+ 0.511 902	+ 0.509 416	+ 0.506 952	+ 0.504 511	+ 0.502 093	170
175	+ 0.527 521	+ 0.524 886	+ 0.522 276	+ 0.519 660	+ 0.517 129	+ 0.514 592	+ 0.512 079	+ 0.509 590	+ 0.507 124	+ 0.504 680	+ 0.502 259	175
180	+ 0.527 585	+ 0.524 950	+ 0.522 339	+ 0.519 752	+ 0.517 190	+ 0.514 652	+ 0.512 138	+ 0.509 648	+ 0.507 181	+ 0.504 736	+ 0.502 314	180
ψ / z	1.81	1.82	1.83	1.84	1.85	1.86	1.87	1.88	1.89	1.90	1.91	ψ \ z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99	2.00	2.01	z / ψ
0°	− 0.171 212	− 0.148 027	− 0.126 026	− 0.105 140	− 0.085 308	− 0.066 470	− 0.048 570	− 0.031 557	− 0.015 382	+ 0.000 000	+ 0.014 632	0°
5	− 0.153 311	− 0.131 071	− 0.109 957	− 0.089 904	− 0.070 852	− 0.052 747	− 0.035 536	− 0.019 171	− 0.003 605	+ 0.011 203	+ 0.025 294	5
10	− 0.104 091	− 0.084 389	− 0.065 658	− 0.047 843	− 0.030 896	− 0.014 771	+ 0.000 575	+ 0.015 185	+ 0.029 096	+ 0.042 344	+ 0.054 963	10
15	− 0.034 575	− 0.018 296	− 0.002 790	+ 0.011 985	+ 0.026 066	+ 0.039 487	+ 0.052 282	+ 0.064 480	+ 0.076 113	+ 0.087 207	+ 0.097 789	15
20	+ 0.042 781	+ 0.055 480	+ 0.067 603	+ 0.079 176	+ 0.090 226	+ 0.100 776	+ 0.110 849	+ 0.120 469	+ 0.129 657	+ 0.138 431	+ 0.146 810	20
25	+ 0.118 203	+ 0.127 661	+ 0.136 705	+ 0.145 351	+ 0.153 619	+ 0.161 523	+ 0.169 080	+ 0.176 304	+ 0.183 210	+ 0.189 812	+ 0.196 122	25
30	+ 0.185 988	+ 0.192 755	+ 0.199 230	+ 0.205 425	+ 0.211 352	+ 0.217 020	+ 0.222 442	+ 0.227 626	+ 0.232 582	+ 0.237 320	+ 0.241 849	30
35	+ 0.243 884	+ 0.248 528	+ 0.252 970	+ 0.257 217	+ 0.261 276	+ 0.265 155	+ 0.268 860	+ 0.272 398	+ 0.275 776	+ 0.278 999	+ 0.282 073	35
40	+ 0.291 827	+ 0.294 846	+ 0.297 725	+ 0.300 469	+ 0.303 083	+ 0.305 571	+ 0.307 939	+ 0.310 190	+ 0.312 329	+ 0.314 360	+ 0.316 287	40
45	+ 0.330 837	+ 0.332 628	+ 0.334 322	+ 0.335 923	+ 0.337 435	+ 0.338 860	+ 0.340 202	+ 0.341 463	+ 0.342 646	+ 0.343 754	+ 0.344 790	45
50	+ 0.362 306	+ 0.363 171	+ 0.363 970	+ 0.364 706	+ 0.365 381	+ 0.365 997	+ 0.366 556	+ 0.367 060	+ 0.367 511	+ 0.367 910	+ 0.368 261	50
55	+ 0.387 620	+ 0.387 786	+ 0.387 908	+ 0.387 988	+ 0.388 026	+ 0.388 026	+ 0.387 987	+ 0.387 911	+ 0.387 800	+ 0.387 655	+ 0.387 475	55
60	+ 0.408 004	+ 0.407 639	+ 0.407 245	+ 0.406 823	+ 0.406 375	+ 0.405 902	+ 0.405 404	+ 0.404 883	+ 0.404 340	+ 0.403 775	+ 0.403 189	60
65	+ 0.424 475	+ 0.423 701	+ 0.422 911	+ 0.422 103	+ 0.421 280	+ 0.420 442	+ 0.419 590	+ 0.418 724	+ 0.417 845	+ 0.416 953	+ 0.416 049	65
70	+ 0.437 849	+ 0.436 759	+ 0.435 659	+ 0.434 552	+ 0.433 437	+ 0.432 315	+ 0.431 186	+ 0.430 050	+ 0.428 909	+ 0.427 762	+ 0.426 610	70
75	+ 0.448 769	+ 0.447 431	+ 0.446 090	+ 0.444 747	+ 0.443 402	+ 0.442 056	+ 0.440 709	+ 0.439 361	+ 0.438 013	+ 0.436 664	+ 0.435 315	75
80	+ 0.457 739	+ 0.456 205	+ 0.454 672	+ 0.453 142	+ 0.451 616	+ 0.450 092	+ 0.448 571	+ 0.447 054	+ 0.445 541	+ 0.444 031	+ 0.442 525	80
85	+ 0.465 152	+ 0.463 460	+ 0.461 775	+ 0.460 095	+ 0.458 422	+ 0.456 756	+ 0.455 096	+ 0.453 443	+ 0.451 797	+ 0.450 158	+ 0.448 525	85
90	+ 0.471 314	+ 0.469 495	+ 0.467 686	+ 0.465 885	+ 0.464 094	+ 0.462 312	+ 0.460 540	+ 0.458 777	+ 0.457 023	+ 0.455 279	+ 0.453 544	90
95	+ 0.476 464	+ 0.474 542	+ 0.472 631	+ 0.470 732	+ 0.468 845	+ 0.466 969	+ 0.465 104	+ 0.463 251	+ 0.461 409	+ 0.459 579	+ 0.457 752	95
100	+ 0.480 789	+ 0.478 782	+ 0.476 789	+ 0.474 809	+ 0.472 842	+ 0.470 889	+ 0.468 948	+ 0.467 021	+ 0.465 107	+ 0.463 206	+ 0.461 318	100
105	+ 0.484 438	+ 0.482 361	+ 0.480 209	+ 0.478 252	+ 0.476 220	+ 0.474 202	+ 0.472 199	+ 0.470 211	+ 0.468 237	+ 0.466 277	+ 0.464 331	105
110	+ 0.487 528	+ 0.485 393	+ 0.483 273	+ 0.481 171	+ 0.479 084	+ 0.477 013	+ 0.474 958	+ 0.472 918	+ 0.470 895	+ 0.468 885	+ 0.466 892	110
115	+ 0.490 152	+ 0.487 968	+ 0.485 801	+ 0.483 653	+ 0.481 519	+ 0.479 404	+ 0.477 305	+ 0.475 223	+ 0.473 157	+ 0.471 107	+ 0.469 075	115
120	+ 0.492 387	+ 0.490 161	+ 0.487 954	+ 0.485 765	+ 0.483 594	+ 0.481 442	+ 0.479 307	+ 0.477 189	+ 0.475 087	+ 0.473 002	+ 0.470 935	120
125	+ 0.494 287	+ 0.492 028	+ 0.489 788	+ 0.487 567	+ 0.485 364	+ 0.483 179	+ 0.481 012	+ 0.478 863	+ 0.476 732	+ 0.474 620	+ 0.472 525	125
130	+ 0.495 907	+ 0.493 618	+ 0.491 349	+ 0.489 100	+ 0.486 870	+ 0.484 659	+ 0.482 467	+ 0.480 293	+ 0.478 137	+ 0.475 999	+ 0.473 879	130
135	+ 0.497 282	+ 0.494 969	+ 0.492 676	+ 0.490 404	+ 0.488 150	+ 0.485 917	+ 0.483 702	+ 0.481 507	+ 0.479 330	+ 0.477 172	+ 0.475 032	135
140	+ 0.498 445	+ 0.496 111	+ 0.493 799	+ 0.491 507	+ 0.499 235	+ 0.486 982	+ 0.484 749	+ 0.482 536	+ 0.480 342	+ 0.478 166	+ 0.476 009	140
145	+ 0.499 422	+ 0.497 071	+ 0.494 742	+ 0.492 433	+ 0.490 145	+ 0.487 877	+ 0.485 629	+ 0.483 400	+ 0.481 191	+ 0.479 001	+ 0.476 830	145
150	+ 0.500 230	+ 0.497 867	+ 0.495 524	+ 0.493 202	+ 0.490 901	+ 0.488 620	+ 0.486 359	+ 0.484 119	+ 0.481 897	+ 0.479 695	+ 0.477 513	150
155	+ 0.500 891	+ 0.498 516	+ 0.496 161	+ 0.493 829	+ 0.491 517	+ 0.489 226	+ 0.486 954	+ 0.484 703	+ 0.482 472	+ 0.480 261	+ 0.478 068	155
160	+ 0.501 417	+ 0.499 032	+ 0.496 669	+ 0.494 327	+ 0.492 007	+ 0.489 707	+ 0.487 428	+ 0.485 169	+ 0.482 930	+ 0.480 711	+ 0.478 511	160
165	+ 0.501 814	+ 0.499 423	+ 0.497 053	+ 0.494 705	+ 0.492 378	+ 0.490 072	+ 0.487 787	+ 0.485 522	+ 0.483 277	+ 0.481 053	+ 0.478 847	165
170	+ 0.502 093	+ 0.499 607	+ 0.497 323	+ 0.494 970	+ 0.492 639	+ 0.490 329	+ 0.488 040	+ 0.485 771	+ 0.483 521	+ 0.481 292	+ 0.479 084	170
175	+ 0.502 259	+ 0.499 860	+ 0.497 482	+ 0.495 127	+ 0.492 793	+ 0.490 481	+ 0.488 189	+ 0.485 917	+ 0.483 666	+ 0.481 435	+ 0.479 223	175
180	+ 0.502 314	+ 0.499 914	+ 0.497 536	+ 0.495 179	+ 0.492 845	+ 0.490 531	+ 0.488 238	+ 0.485 965	+ 0.483 714	+ 0.481 482	+ 0.479 269	180
ψ / z	1.91	1.92	1.93	1.94	1.95	1.96	1.97	1.98	1.99	2.00	2.01	ψ / z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	2.11	2.12	2.13	2.14	2.15	2.16	2.17	2.18	2.19	2.20	2.21	z / ψ
0°	+0.127 397	+0.135 953	+0.144 108	+0.151 883	+0.159 295	+0.166 362	+0.173 101	+0.179 527	+0.185 655	+0.191 498	+0.197 071	0°
5	+0.134 064	+0.142 328	+0.150 207	+0.157 720	+0.164 884	+0.171 716	+0.178 231	+0.184 444	+0.190 369	+0.196 020	+0.201 410	5
10	+0.152 786	+0.160 248	+0.167 365	+0.174 155	+0.180 631	+0.186 810	+0.192 704	+0.198 327	+0.203 690	+0.208 806	+0.213 686	10
15	+0.180 276	+0.186 599	+0.192 633	+0.198 392	+0.203 887	+0.209 132	+0.214 137	+0.218 912	+0.223 468	+0.227 814	+0.231 959	15
20	+0.212 460	+0.217 511	+0.222 332	+0.226 934	+0.231 327	+0.235 518	+0.239 517	+0.243 332	+0.246 970	+0.250 439	+0.253 746	20
25	+0.245 667	+0.249 478	+0.253 114	+0.256 582	+0.259 889	+0.263 041	+0.266 046	+0.268 909	+0.271 635	+0.274 231	+0.276 701	25
30	+0.277 270	+0.279 973	+0.282 546	+0.284 994	+0.287 323	+0.289 537	+0.291 641	+0.293 638	+0.295 533	+0.297 329	+0.299 031	30
35	+0.305 778	+0.307 540	+0.309 216	+0.310 798	+0.312 294	+0.313 705	+0.315 036	+0.316 289	+0.317 468	+0.318 574	+0.319 611	35
40	+0.330 504	+0.331 597	+0.332 533	+0.333 405	+0.334 215	+0.334 965	+0.335 657	+0.336 294	+0.336 878	+0.337 410	+0.337 892	40
45	+0.351 718	+0.352 100	+0.352 447	+0.352 749	+0.353 001	+0.353 224	+0.353 401	+0.353 540	+0.353 640	+0.353 705	+0.353 739	45
50	+0.369 413	+0.369 325	+0.369 206	+0.369 056	+0.368 877	+0.368 670	+0.368 435	+0.368 175	+0.367 889	+0.367 579	+0.367 246	50
55	+0.384 147	+0.383 681	+0.383 194	+0.382 688	+0.382 163	+0.381 620	+0.381 059	+0.380 481	+0.379 888	+0.379 278	+0.378 655	55
60	+0.396 365	+0.395 601	+0.394 825	+0.394 037	+0.393 238	+0.392 428	+0.391 608	+0.390 779	+0.389 940	+0.389 093	+0.388 237	60
65	+0.406 486	+0.405 488	+0.404 482	+0.403 471	+0.402 456	+0.401 435	+0.400 410	+0.399 380	+0.398 345	+0.397 308	+0.396 267	65
70	+0.414 886	+0.413 701	+0.412 513	+0.411 325	+0.410 136	+0.408 947	+0.407 757	+0.406 567	+0.405 376	+0.404 188	+0.402 997	70
75	+0.421 873	+0.420 538	+0.419 207	+0.417 875	+0.416 547	+0.415 222	+0.413 900	+0.412 581	+0.411 265	+0.409 953	+0.408 643	75
80	+0.427 706	+0.426 251	+0.424 800	+0.423 355	+0.421 916	+0.420 481	+0.419 052	+0.417 629	+0.416 211	+0.414 799	+0.413 392	80
85	+0.432 593	+0.431 040	+0.429 495	+0.427 956	+0.426 426	+0.424 902	+0.423 386	+0.421 878	+0.420 377	+0.418 883	+0.417 398	85
90	+0.436 705	+0.435 073	+0.433 449	+0.431 834	+0.430 229	+0.428 633	+0.427 045	+0.425 467	+0.423 898	+0.422 338	+0.420 787	90
95	+0.440 171	+0.438 474	+0.436 785	+0.435 108	+0.433 442	+0.431 787	+0.430 140	+0.428 503	+0.426 878	+0.425 264	+0.423 659	95
100	+0.443 122	+0.441 369	+0.439 628	+0.437 898	+0.436 180	+0.434 474	+0.432 779	+0.431 096	+0.429 423	+0.427 762	+0.426 112	100
105	+0.445 624	+0.443 825	+0.442 040	+0.440 267	+0.438 506	+0.436 758	+0.435 023	+0.433 300	+0.431 588	+0.429 888	+0.428 201	105
110	+0.447 759	+0.445 923	+0.444 100	+0.442 291	+0.440 495	+0.438 712	+0.436 943	+0.435 186	+0.433 442	+0.431 710	+0.429 992	110
115	+0.449 583	+0.447 715	+0.445 861	+0.444 021	+0.442 195	+0.440 383	+0.438 584	+0.436 800	+0.435 028	+0.433 270	+0.431 525	115
120	+0.451 143	+0.449 249	+0.447 368	+0.445 503	+0.443 652	+0.441 815	+0.439 992	+0.438 183	+0.436 388	+0.434 607	+0.432 839	120
125	+0.452 478	+0.450 560	+0.448 657	+0.446 770	+0.444 899	+0.443 041	+0.441 198	+0.439 369	+0.437 554	+0.435 753	+0.433 967	125
130	+0.453 619	+0.451 682	+0.449 761	+0.447 856	+0.445 967	+0.444 090	+0.442 229	+0.440 383	+0.438 552	+0.436 735	+0.434 932	130
135	+0.454 590	+0.452 638	+0.450 701	+0.448 780	+0.446 875	+0.444 985	+0.443 109	+0.441 250	+0.439 404	+0.437 573	+0.435 757	135
140	+0.455 416	+0.453 450	+0.451 500	+0.449 565	+0.447 647	+0.445 744	+0.443 856	+0.441 984	+0.440 127	+0.438 284	+0.436 457	140
145	+0.456 111	+0.454 133	+0.452 171	+0.450 226	+0.448 297	+0.446 384	+0.444 486	+0.442 604	+0.440 738	+0.438 884	+0.437 047	145
150	+0.456 688	+0.454 701	+0.452 731	+0.450 777	+0.448 839	+0.446 916	+0.445 010	+0.443 119	+0.441 244	+0.439 384	+0.437 539	150
155	+0.457 160	+0.455 165	+0.453 187	+0.451 225	+0.449 280	+0.447 351	+0.445 437	+0.443 540	+0.441 658	+0.439 792	+0.437 940	155
160	+0.457 535	+0.455 534	+0.453 550	+0.451 583	+0.449 632	+0.447 698	+0.445 779	+0.443 876	+0.441 989	+0.440 117	+0.438 260	160
165	+0.457 818	+0.455 813	+0.453 825	+0.451 853	+0.449 898	+0.447 960	+0.446 038	+0.444 131	+0.442 240	+0.440 364	+0.438 503	165
170	+0.458 020	+0.456 012	+0.454 020	+0.452 045	+0.450 087	+0.448 145	+0.446 219	+0.444 310	+0.442 416	+0.440 537	+0.438 674	170
175	+0.458 139	+0.456 129	+0.454 135	+0.452 158	+0.450 198	+0.448 254	+0.446 326	+0.444 415	+0.442 520	+0.440 640	+0.438 775	175
180	+0.458 177	+0.456 167	+0.454 173	+0.452 196	+0.450 235	+0.448 291	+0.446 362	+0.444 450	+0.442 554	+0.440 674	+0.438 808	180
ψ / z	2.11	2.12	2.13	2.14	2.15	2.16	2.17	2.18	2.19	2.20	2.21	ψ / z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ $\frac{1}{m}$	2.21	2.22	2.23	2.24	2.25	2.26	2.27	2.28	2.29	2.30	2.31	$\frac{1}{m}$ / ψ
0°	+ 0.197 071	+ 0.202 384	+ 0.207 451	+ 0.212 282	+ 0.216 889	+ 0.221 280	+ 0.225 467	+ 0.229 457	+ 0.233 260	+ 0.236 884	+ 0.240 337	0°
5	+ 0.201 410	+ 0.206 549	+ 0.211 450	+ 0.216 123	+ 0.220 579	+ 0.224 827	+ 0.228 877	+ 0.232 736	+ 0.236 415	+ 0.239 920	+ 0.243 258	5
10	+ 0.213 686	+ 0.218 340	+ 0.222 779	+ 0.227 011	+ 0.231 046	+ 0.234 893	+ 0.238 560	+ 0.242 054	+ 0.245 383	+ 0.248 554	+ 0.251 574	10
15	+ 0.231 959	+ 0.235 912	+ 0.239 681	+ 0.243 275	+ 0.246 700	+ 0.249 963	+ 0.253 073	+ 0.256 034	+ 0.258 853	+ 0.261 537	+ 0.264 091	15
20	+ 0.253 746	+ 0.256 898	+ 0.259 900	+ 0.262 760	+ 0.265 483	+ 0.268 074	+ 0.270 539	+ 0.272 884	+ 0.275 112	+ 0.277 229	+ 0.279 239	20
25	+ 0.276 701	+ 0.279 050	+ 0.281 283	+ 0.283 405	+ 0.285 419	+ 0.287 331	+ 0.289 144	+ 0.290 862	+ 0.292 488	+ 0.294 027	+ 0.295 482	25
30	+ 0.299 031	+ 0.300 642	+ 0.302 166	+ 0.303 605	+ 0.304 964	+ 0.306 244	+ 0.307 450	+ 0.308 583	+ 0.309 648	+ 0.310 645	+ 0.311 578	30
35	+ 0.319 611	+ 0.320 581	+ 0.321 487	+ 0.322 332	+ 0.323 116	+ 0.323 844	+ 0.324 516	+ 0.325 134	+ 0.325 702	+ 0.326 220	+ 0.326 691	35
40	+ 0.337 892	+ 0.338 327	+ 0.338 715	+ 0.339 060	+ 0.339 361	+ 0.339 621	+ 0.339 841	+ 0.340 024	+ 0.340 169	+ 0.340 279	+ 0.340 355	40
45	+ 0.353 735	+ 0.353 732	+ 0.353 697	+ 0.353 631	+ 0.353 534	+ 0.353 410	+ 0.353 257	+ 0.353 079	+ 0.352 875	+ 0.352 646	+ 0.352 393	45
50	+ 0.367 246	+ 0.366 890	+ 0.366 513	+ 0.366 115	+ 0.365 697	+ 0.365 260	+ 0.364 805	+ 0.364 332	+ 0.363 842	+ 0.363 335	+ 0.362 814	50
55	+ 0.378 655	+ 0.378 017	+ 0.377 365	+ 0.376 701	+ 0.376 024	+ 0.375 335	+ 0.374 636	+ 0.373 925	+ 0.373 204	+ 0.372 473	+ 0.371 733	55
60	+ 0.388 237	+ 0.387 374	+ 0.386 503	+ 0.385 625	+ 0.384 741	+ 0.383 850	+ 0.382 953	+ 0.382 050	+ 0.381 143	+ 0.380 230	+ 0.379 313	60
65	+ 0.396 267	+ 0.395 223	+ 0.394 177	+ 0.393 128	+ 0.392 076	+ 0.391 022	+ 0.389 967	+ 0.388 910	+ 0.387 852	+ 0.386 793	+ 0.385 731	65
70	+ 0.402 997	+ 0.401 808	+ 0.400 620	+ 0.399 433	+ 0.398 247	+ 0.397 062	+ 0.395 879	+ 0.394 697	+ 0.393 517	+ 0.392 339	+ 0.391 163	70
75	+ 0.408 643	+ 0.407 337	+ 0.406 035	+ 0.404 736	+ 0.403 441	+ 0.402 149	+ 0.400 863	+ 0.399 580	+ 0.398 301	+ 0.397 026	+ 0.395 756	75
80	+ 0.413 392	+ 0.411 991	+ 0.410 596	+ 0.409 207	+ 0.407 823	+ 0.406 446	+ 0.405 074	+ 0.403 708	+ 0.402 348	+ 0.400 994	+ 0.399 646	80
85	+ 0.417 398	+ 0.415 919	+ 0.414 448	+ 0.412 984	+ 0.411 528	+ 0.410 080	+ 0.408 639	+ 0.407 205	+ 0.405 779	+ 0.404 360	+ 0.402 948	85
90	+ 0.420 787	+ 0.419 244	+ 0.417 711	+ 0.416 186	+ 0.414 670	+ 0.413 163	+ 0.411 665	+ 0.410 175	+ 0.408 694	+ 0.407 222	+ 0.405 758	90
95	+ 0.423 659	+ 0.422 063	+ 0.420 479	+ 0.418 902	+ 0.417 338	+ 0.415 783	+ 0.414 238	+ 0.412 700	+ 0.411 174	+ 0.409 658	+ 0.408 152	95
100	+ 0.426 112	+ 0.424 473	+ 0.422 845	+ 0.421 228	+ 0.419 621	+ 0.418 025	+ 0.416 440	+ 0.414 865	+ 0.413 300	+ 0.411 746	+ 0.410 202	100
105	+ 0.428 201	+ 0.426 526	+ 0.424 861	+ 0.423 210	+ 0.421 568	+ 0.419 938	+ 0.418 319	+ 0.416 713	+ 0.415 116	+ 0.413 531	+ 0.411 955	105
110	+ 0.429 992	+ 0.428 285	+ 0.426 591	+ 0.424 908	+ 0.423 238	+ 0.421 580	+ 0.419 934	+ 0.418 299	+ 0.416 676	+ 0.415 064	+ 0.413 464	110
115	+ 0.431 525	+ 0.429 792	+ 0.428 072	+ 0.426 365	+ 0.424 670	+ 0.422 988	+ 0.421 318	+ 0.419 660	+ 0.418 015	+ 0.416 382	+ 0.414 760	115
120	+ 0.432 839	+ 0.431 084	+ 0.429 343	+ 0.427 615	+ 0.425 900	+ 0.424 198	+ 0.422 508	+ 0.420 830	+ 0.419 166	+ 0.417 513	+ 0.415 872	120
125	+ 0.433 967	+ 0.432 193	+ 0.430 433	+ 0.428 688	+ 0.426 957	+ 0.425 238	+ 0.423 529	+ 0.421 835	+ 0.420 154	+ 0.418 485	+ 0.416 829	125
130	+ 0.434 932	+ 0.433 143	+ 0.431 368	+ 0.429 607	+ 0.427 859	+ 0.426 125	+ 0.424 404	+ 0.422 697	+ 0.421 002	+ 0.419 320	+ 0.417 651	130
135	+ 0.435 757	+ 0.433 954	+ 0.432 167	+ 0.430 392	+ 0.428 631	+ 0.426 885	+ 0.425 153	+ 0.423 433	+ 0.421 727	+ 0.420 033	+ 0.418 353	135
140	+ 0.436 457	+ 0.434 643	+ 0.432 844	+ 0.431 060	+ 0.429 290	+ 0.427 533	+ 0.425 789	+ 0.424 059	+ 0.422 343	+ 0.420 640	+ 0.418 951	140
145	+ 0.437 047	+ 0.435 224	+ 0.433 417	+ 0.431 622	+ 0.429 842	+ 0.428 077	+ 0.426 325	+ 0.424 588	+ 0.422 864	+ 0.421 153	+ 0.419 455	145
150	+ 0.437 539	+ 0.435 708	+ 0.433 892	+ 0.432 091	+ 0.430 304	+ 0.428 531	+ 0.426 773	+ 0.425 028	+ 0.423 297	+ 0.421 579	+ 0.419 875	150
155	+ 0.437 940	+ 0.436 103	+ 0.434 281	+ 0.432 474	+ 0.430 681	+ 0.428 903	+ 0.427 138	+ 0.425 388	+ 0.423 651	+ 0.421 929	+ 0.420 219	155
160	+ 0.438 260	+ 0.436 418	+ 0.434 592	+ 0.432 780	+ 0.430 982	+ 0.429 199	+ 0.427 430	+ 0.425 675	+ 0.423 934	+ 0.422 207	+ 0.420 493	160
165	+ 0.438 503	+ 0.436 657	+ 0.434 826	+ 0.433 010	+ 0.431 209	+ 0.429 423	+ 0.427 651	+ 0.425 893	+ 0.424 149	+ 0.422 418	+ 0.420 701	165
170	+ 0.438 674	+ 0.436 826	+ 0.434 993	+ 0.433 174	+ 0.431 370	+ 0.429 581	+ 0.427 806	+ 0.426 046	+ 0.424 300	+ 0.422 567	+ 0.420 848	170
175	+ 0.438 775	+ 0.436 925	+ 0.435 091	+ 0.433 271	+ 0.431 467	+ 0.429 675	+ 0.427 899	+ 0.426 137	+ 0.424 390	+ 0.422 656	+ 0.420 935	175
180	+ 0.438 808	+ 0.436 958	+ 0.435 123	+ 0.433 303	+ 0.431 497	+ 0.429 706	+ 0.427 929	+ 0.426 167	+ 0.424 419	+ 0.422 684	+ 0.420 963	180
ψ / z	2.21	2.22	2.23	2.24	2.25	2.26	2.27	2.28	2.29	2.30	2.31	z / ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING z: $\frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	2.31	2.32	2.33	2.34	2.35	2.36	2.37	2.38	2.39	2.40	2.41	z / ψ
0°	+ 0.240 337	+ 0.243 626	+ 0.246 758	+ 0.249 739	+ 0.252 578	+ 0.255 279	+ 0.257 848	+ 0.260 291	+ 0.262 614	+ 0.264 820	+ 0.266 916	0°
5	+ 0.243 258	+ 0.246 439	+ 0.249 467	+ 0.252 349	+ 0.255 093	+ 0.257 703	+ 0.260 185	+ 0.262 545	+ 0.264 788	+ 0.266 918	+ 0.268 941	5
10	+ 0.251 574	+ 0.254 449	+ 0.257 186	+ 0.259 789	+ 0.262 266	+ 0.264 620	+ 0.266 858	+ 0.268 984	+ 0.271 002	+ 0.272 917	+ 0.274 733	10
15	+ 0.264 091	+ 0.266 518	+ 0.268 827	+ 0.271 021	+ 0.273 104	+ 0.275 082	+ 0.276 959	+ 0.278 738	+ 0.280 424	+ 0.282 021	+ 0.283 531	15
20	+ 0.279 239	+ 0.281 146	+ 0.282 955	+ 0.284 669	+ 0.286 293	+ 0.287 829	+ 0.289 281	+ 0.290 653	+ 0.291 947	+ 0.293 167	+ 0.294 315	20
25	+ 0.295 482	+ 0.296 855	+ 0.298 151	+ 0.299 372	+ 0.300 521	+ 0.301 601	+ 0.302 614	+ 0.303 563	+ 0.304 451	+ 0.305 279	+ 0.306 050	25
30	+ 0.311 578	+ 0.312 450	+ 0.313 262	+ 0.314 016	+ 0.314 716	+ 0.315 362	+ 0.315 957	+ 0.316 504	+ 0.317 003	+ 0.317 456	+ 0.317 866	30
35	+ 0.326 691	+ 0.327 115	+ 0.327 496	+ 0.327 834	+ 0.328 132	+ 0.328 390	+ 0.328 160	+ 0.328 794	+ 0.328 942	+ 0.329 057	+ 0.329 135	35
40	+ 0.340 355	+ 0.340 398	+ 0.340 409	+ 0.340 390	+ 0.340 341	+ 0.340 264	+ 0.340 160	+ 0.340 030	+ 0.339 875	+ 0.339 695	+ 0.339 492	40
45	+ 0.352 393	+ 0.352 117	+ 0.351 820	+ 0.351 501	+ 0.351 163	+ 0.350 804	+ 0.350 427	+ 0.350 033	+ 0.349 621	+ 0.349 192	+ 0.348 747	45
50	+ 0.362 814	+ 0.362 277	+ 0.361 726	+ 0.361 161	+ 0.360 583	+ 0.359 993	+ 0.359 390	+ 0.358 776	+ 0.358 150	+ 0.357 515	+ 0.356 869	50
55	+ 0.371 733	+ 0.370 984	+ 0.370 226	+ 0.369 460	+ 0.368 687	+ 0.367 906	+ 0.367 118	+ 0.366 324	+ 0.365 523	+ 0.364 717	+ 0.363 905	55
60	+ 0.379 313	+ 0.378 392	+ 0.377 466	+ 0.376 537	+ 0.375 605	+ 0.374 669	+ 0.373 730	+ 0.372 789	+ 0.371 845	+ 0.370 899	+ 0.369 951	60
65	+ 0.385 731	+ 0.384 672	+ 0.383 611	+ 0.382 549	+ 0.381 487	+ 0.380 425	+ 0.379 364	+ 0.378 302	+ 0.377 243	+ 0.376 181	+ 0.375 122	65
70	+ 0.391 163	+ 0.389 988	+ 0.388 817	+ 0.387 647	+ 0.386 480	+ 0.385 315	+ 0.384 153	+ 0.382 994	+ 0.381 840	+ 0.380 683	+ 0.379 533	70
75	+ 0.395 756	+ 0.394 489	+ 0.393 227	+ 0.391 969	+ 0.390 716	+ 0.389 467	+ 0.388 223	+ 0.386 984	+ 0.385 750	+ 0.384 518	+ 0.383 293	75
80	+ 0.399 646	+ 0.398 304	+ 0.396 968	+ 0.395 638	+ 0.394 315	+ 0.392 997	+ 0.391 685	+ 0.390 380	+ 0.389 080	+ 0.387 787	+ 0.386 500	80
85	+ 0.402 948	+ 0.401 544	+ 0.400 148	+ 0.398 758	+ 0.397 376	+ 0.396 002	+ 0.394 634	+ 0.393 274	+ 0.391 921	+ 0.390 576	+ 0.389 238	85
90	+ 0.405 758	+ 0.404 302	+ 0.402 855	+ 0.401 417	+ 0.399 987	+ 0.398 565	+ 0.397 151	+ 0.395 746	+ 0.394 349	+ 0.392 960	+ 0.391 579	90
95	+ 0.408 152	+ 0.406 652	+ 0.405 164	+ 0.403 685	+ 0.402 214	+ 0.400 753	+ 0.399 301	+ 0.397 859	+ 0.396 427	+ 0.394 999	+ 0.393 582	95
100	+ 0.410 202	+ 0.408 668	+ 0.407 145	+ 0.405 631	+ 0.404 127	+ 0.402 633	+ 0.401 149	+ 0.399 675	+ 0.398 210	+ 0.396 754	+ 0.395 308	100
105	+ 0.411 955	+ 0.410 392	+ 0.408 839	+ 0.407 296	+ 0.405 765	+ 0.404 243	+ 0.402 732	+ 0.401 230	+ 0.399 739	+ 0.398 260	+ 0.396 789	105
110	+ 0.413 464	+ 0.411 875	+ 0.410 298	+ 0.408 731	+ 0.407 175	+ 0.405 630	+ 0.404 096	+ 0.402 573	+ 0.401 060	+ 0.399 557	+ 0.398 065	110
115	+ 0.414 760	+ 0.413 149	+ 0.411 549	+ 0.409 962	+ 0.408 386	+ 0.406 821	+ 0.405 269	+ 0.403 725	+ 0.402 194	+ 0.400 674	+ 0.399 164	115
120	+ 0.415 872	+ 0.414 244	+ 0.412 628	+ 0.411 024	+ 0.409 431	+ 0.407 849	+ 0.406 280	+ 0.404 721	+ 0.403 173	+ 0.401 637	+ 0.400 112	120
125	+ 0.416 829	+ 0.415 186	+ 0.413 554	+ 0.411 935	+ 0.410 328	+ 0.408 732	+ 0.407 148	+ 0.405 576	+ 0.404 015	+ 0.402 466	+ 0.400 927	125
130	+ 0.417 651	+ 0.415 994	+ 0.414 350	+ 0.412 719	+ 0.411 099	+ 0.409 491	+ 0.407 896	+ 0.406 312	+ 0.404 740	+ 0.403 179	+ 0.401 630	130
135	+ 0.418 353	+ 0.416 686	+ 0.415 031	+ 0.413 388	+ 0.411 758	+ 0.410 140	+ 0.408 534	+ 0.406 941	+ 0.405 359	+ 0.403 789	+ 0.402 230	135
140	+ 0.418 951	+ 0.417 274	+ 0.415 610	+ 0.413 959	+ 0.412 320	+ 0.410 693	+ 0.409 079	+ 0.407 477	+ 0.405 887	+ 0.404 309	+ 0.402 743	140
145	+ 0.419 455	+ 0.417 770	+ 0.416 099	+ 0.414 440	+ 0.412 794	+ 0.411 160	+ 0.409 539	+ 0.407 930	+ 0.406 333	+ 0.404 748	+ 0.403 175	145
150	+ 0.419 875	+ 0.418 184	+ 0.416 506	+ 0.414 841	+ 0.413 189	+ 0.411 550	+ 0.409 923	+ 0.408 308	+ 0.406 706	+ 0.405 115	+ 0.403 537	150
155	+ 0.420 219	+ 0.418 523	+ 0.416 839	+ 0.415 169	+ 0.413 513	+ 0.411 868	+ 0.410 237	+ 0.408 617	+ 0.407 010	+ 0.405 415	+ 0.403 832	155
160	+ 0.420 493	+ 0.418 792	+ 0.417 105	+ 0.415 431	+ 0.413 770	+ 0.412 122	+ 0.410 486	+ 0.408 863	+ 0.407 252	+ 0.405 654	+ 0.404 068	160
165	+ 0.420 701	+ 0.418 997	+ 0.417 306	+ 0.415 630	+ 0.413 967	+ 0.412 316	+ 0.410 677	+ 0.409 051	+ 0.407 437	+ 0.405 836	+ 0.404 246	165
170	+ 0.420 848	+ 0.419 142	+ 0.417 449	+ 0.415 770	+ 0.414 104	+ 0.412 451	+ 0.410 811	+ 0.409 183	+ 0.407 567	+ 0.405 964	+ 0.404 373	170
175	+ 0.420 935	+ 0.419 228	+ 0.417 534	+ 0.415 854	+ 0.414 187	+ 0.412 532	+ 0.410 890	+ 0.409 261	+ 0.407 645	+ 0.406 041	+ 0.404 449	175
180	+ 0.420 963	+ 0.419 256	+ 0.417 562	+ 0.415 881	+ 0.414 213	+ 0.412 558	+ 0.410 916	+ 0.409 287	+ 0.407 670	+ 0.406 066	+ 0.404 473	180
ψ / z	2.31	2.32	2.33	2.34	2.35	2.36	2.37	2.38	2.39	2.40	2.41	z \ ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	2.41	2.42	2.43	2.44	2.45	2.46	2.47	2.48	2.49	2.50	2.51	z / ψ
0°	+ 0.266 916	+ 0.268 905	+ 0.270 793	+ 0.272 583	+ 0.274 279	+ 0.275 885	+ 0.277 405	+ 0.278 842	+ 0.280 200	+ 0.281 481	+ 0.282 690	0°
5	+ 0.268 941	+ 0.270 860	+ 0.272 681	+ 0.274 406	+ 0.276 040	+ 0.277 587	+ 0.279 050	+ 0.280 433	+ 0.281 738	+ 0.282 969	+ 0.284 129	5
10	+ 0.274 733	+ 0.276 454	+ 0.278 085	+ 0.279 628	+ 0.281 087	+ 0.282 466	+ 0.283 768	+ 0.284 995	+ 0.286 152	+ 0.287 240	+ 0.288 262	10
15	+ 0.283 531	+ 0.284 959	+ 0.286 307	+ 0.287 579	+ 0.288 778	+ 0.289 907	+ 0.290 968	+ 0.291 965	+ 0.292 898	+ 0.293 772	+ 0.294 589	15
20	+ 0.294 315	+ 0.295 395	+ 0.296 408	+ 0.297 359	+ 0.298 248	+ 0.299 078	+ 0.299 853	+ 0.300 572	+ 0.301 240	+ 0.301 858	+ 0.302 427	20
25	+ 0.306 050	+ 0.306 766	+ 0.307 430	+ 0.308 043	+ 0.308 607	+ 0.309 124	+ 0.309 596	+ 0.310 024	+ 0.310 411	+ 0.310 757	+ 0.311 064	25
30	+ 0.317 866	+ 0.318 233	+ 0.318 560	+ 0.318 847	+ 0.319 097	+ 0.319 311	+ 0.319 490	+ 0.319 635	+ 0.319 748	+ 0.319 830	+ 0.319 881	30
35	+ 0.329 135	+ 0.329 189	+ 0.329 210	+ 0.329 201	+ 0.329 164	+ 0.329 101	+ 0.329 011	+ 0.328 897	+ 0.328 758	+ 0.328 597	+ 0.328 413	35
40	+ 0.339 492	+ 0.339 266	+ 0.339 019	+ 0.338 751	+ 0.338 463	+ 0.338 156	+ 0.337 830	+ 0.337 487	+ 0.337 127	+ 0.336 750	+ 0.336 357	40
45	+ 0.348 747	+ 0.348 287	+ 0.347 812	+ 0.347 325	+ 0.346 822	+ 0.346 308	+ 0.345 780	+ 0.345 240	+ 0.344 689	+ 0.344 126	+ 0.343 554	45
50	+ 0.356 869	+ 0.356 213	+ 0.355 548	+ 0.354 874	+ 0.354 192	+ 0.353 502	+ 0.352 804	+ 0.352 099	+ 0.351 387	+ 0.350 669	+ 0.349 944	50
55	+ 0.363 905	+ 0.363 088	+ 0.362 266	+ 0.361 439	+ 0.360 608	+ 0.359 772	+ 0.358 933	+ 0.358 091	+ 0.357 245	+ 0.356 396	+ 0.355 544	55
60	+ 0.369 951	+ 0.369 001	+ 0.368 050	+ 0.367 097	+ 0.366 143	+ 0.365 188	+ 0.364 232	+ 0.363 276	+ 0.362 319	+ 0.361 362	+ 0.360 404	60
65	+ 0.375 122	+ 0.374 063	+ 0.373 006	+ 0.371 950	+ 0.370 895	+ 0.369 841	+ 0.368 789	+ 0.367 738	+ 0.366 690	+ 0.365 643	+ 0.364 599	65
70	+ 0.379 533	+ 0.378 385	+ 0.377 241	+ 0.376 100	+ 0.374 962	+ 0.373 827	+ 0.372 696	+ 0.371 568	+ 0.370 444	+ 0.369 323	+ 0.368 206	70
75	+ 0.383 293	+ 0.382 072	+ 0.380 856	+ 0.379 645	+ 0.378 439	+ 0.377 237	+ 0.376 041	+ 0.374 849	+ 0.373 662	+ 0.372 480	+ 0.371 304	75
80	+ 0.386 500	+ 0.385 219	+ 0.383 944	+ 0.382 675	+ 0.381 412	+ 0.380 155	+ 0.378 904	+ 0.377 660	+ 0.376 421	+ 0.375 189	+ 0.373 962	80
85	+ 0.389 238	+ 0.387 906	+ 0.386 583	+ 0.385 266	+ 0.383 956	+ 0.382 653	+ 0.381 358	+ 0.380 069	+ 0.378 788	+ 0.377 513	+ 0.376 245	85
90	+ 0.391 579	+ 0.390 207	+ 0.388 842	+ 0.387 485	+ 0.386 136	+ 0.384 795	+ 0.383 462	+ 0.382 137	+ 0.380 819	+ 0.379 509	+ 0.378 207	90
95	+ 0.393 582	+ 0.392 175	+ 0.390 778	+ 0.389 388	+ 0.388 005	+ 0.386 632	+ 0.385 269	+ 0.383 912	+ 0.382 564	+ 0.381 224	+ 0.379 894	95
100	+ 0.395 308	+ 0.393 872	+ 0.392 444	+ 0.391 026	+ 0.389 617	+ 0.388 217	+ 0.386 827	+ 0.385 445	+ 0.384 071	+ 0.382 707	+ 0.381 351	100
105	+ 0.396 789	+ 0.395 328	+ 0.393 875	+ 0.392 433	+ 0.391 002	+ 0.389 580	+ 0.388 166	+ 0.386 763	+ 0.385 368	+ 0.383 984	+ 0.382 606	105
110	+ 0.398 065	+ 0.396 583	+ 0.395 111	+ 0.393 650	+ 0.392 198	+ 0.390 756	+ 0.389 324	+ 0.387 902	+ 0.386 488	+ 0.385 086	+ 0.383 692	110
115	+ 0.399 164	+ 0.397 664	+ 0.396 175	+ 0.394 697	+ 0.393 229	+ 0.391 771	+ 0.390 323	+ 0.388 885	+ 0.387 457	+ 0.386 039	+ 0.384 630	115
120	+ 0.400 112	+ 0.398 598	+ 0.397 094	+ 0.395 601	+ 0.394 118	+ 0.392 646	+ 0.391 185	+ 0.389 734	+ 0.388 292	+ 0.386 861	+ 0.385 440	120
125	+ 0.400 927	+ 0.399 401	+ 0.397 885	+ 0.396 380	+ 0.394 886	+ 0.393 402	+ 0.391 929	+ 0.390 466	+ 0.389 013	+ 0.387 571	+ 0.386 139	125
130	+ 0.401 630	+ 0.400 092	+ 0.398 565	+ 0.397 050	+ 0.395 546	+ 0.394 052	+ 0.392 569	+ 0.391 096	+ 0.389 634	+ 0.388 182	+ 0.386 741	130
135	+ 0.402 230	+ 0.400 683	+ 0.399 148	+ 0.397 624	+ 0.396 110	+ 0.394 608	+ 0.393 117	+ 0.391 636	+ 0.390 165	+ 0.388 707	+ 0.387 259	135
140	+ 0.402 743	+ 0.401 188	+ 0.399 645	+ 0.398 113	+ 0.396 592	+ 0.395 083	+ 0.393 585	+ 0.392 097	+ 0.390 621	+ 0.389 155	+ 0.387 699	140
145	+ 0.403 175	+ 0.401 614	+ 0.400 065	+ 0.398 527	+ 0.397 000	+ 0.395 485	+ 0.393 980	+ 0.392 487	+ 0.391 005	+ 0.389 533	+ 0.388 072	145
150	+ 0.403 537	+ 0.401 971	+ 0.400 416	+ 0.398 872	+ 0.397 341	+ 0.395 821	+ 0.394 312	+ 0.392 813	+ 0.391 326	+ 0.389 850	+ 0.388 385	150
155	+ 0.403 832	+ 0.402 261	+ 0.400 702	+ 0.399 155	+ 0.397 619	+ 0.396 095	+ 0.394 582	+ 0.393 080	+ 0.391 589	+ 0.390 109	+ 0.388 640	155
160	+ 0.404 068	+ 0.402 494	+ 0.400 932	+ 0.399 381	+ 0.397 842	+ 0.396 314	+ 0.394 798	+ 0.393 293	+ 0.391 799	+ 0.390 316	+ 0.388 844	160
165	+ 0.404 246	+ 0.402 669	+ 0.401 104	+ 0.399 551	+ 0.398 010	+ 0.396 480	+ 0.394 962	+ 0.393 454	+ 0.391 958	+ 0.390 472	+ 0.388 998	165
170	+ 0.404 373	+ 0.402 795	+ 0.401 228	+ 0.399 673	+ 0.398 129	+ 0.396 598	+ 0.395 077	+ 0.393 568	+ 0.392 070	+ 0.390 583	+ 0.389 107	170
175	+ 0.404 449	+ 0.402 869	+ 0.401 301	+ 0.399 746	+ 0.398 201	+ 0.396 668	+ 0.395 146	+ 0.393 636	+ 0.392 137	+ 0.390 649	+ 0.389 172	175
180	+ 0.404 473	+ 0.402 893	+ 0.401 325	+ 0.399 768	+ 0.398 223	+ 0.396 690	+ 0.395 168	+ 0.393 658	+ 0.392 158	+ 0.390 670	+ 0.389 193	180
ψ / z	2.41	2.42	2.43	2.44	2.45	2.46	2.47	2.48	2.49	2.50	2.51	ψ \ z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $\frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	2.51	2.52	2.53	2.54	2.55	2.56	2.57	2.58	2.59	2.60	2.61	z / ψ
0	+0.282 690	+0.283 828	+0.284 809	+0.285 904	+0.286 848	+0.287 732	+0.288 558	+0.289 330	+0.290 048	+0.290 715	+0.291 333	0°
5	+0.284 129	+0.285 220	+0.286 246	+0.287 209	+0.288 111	+0.288 955	+0.289 743	+0.290 478	+0.291 160	+0.291 794	+0.292 379	5
10	+0.288 262	+0.289 221	+0.290 120	+0.290 960	+0.291 744	+0.292 475	+0.293 154	+0.293 783	+0.294 364	+0.294 900	+0.295 391	10
15	+0.294 589	+0.295 351	+0.296 058	+0.296 713	+0.297 321	+0.297 881	+0.298 396	+0.298 867	+0.299 295	+0.299 681	+0.300 033	15
20	+0.302 427	+0.302 950	+0.303 428	+0.303 863	+0.304 257	+0.304 611	+0.304 927	+0.305 206	+0.305 449	+0.305 659	+0.305 836	20
25	+0.311 064	+0.311 334	+0.311 568	+0.311 768	+0.311 934	+0.312 068	+0.312 172	+0.312 246	+0.312 291	+0.312 309	+0.312 301	25
30	+0.319 881	+0.319 904	+0.319 900	+0.319 868	+0.319 811	+0.319 729	+0.319 623	+0.319 495	+0.319 344	+0.319 172	+0.318 980	30
35	+0.328 413	+0.328 208	+0.327 982	+0.327 736	+0.327 471	+0.327 188	+0.326 888	+0.326 570	+0.326 236	+0.325 887	+0.325 522	35
40	+0.336 357	+0.335 949	+0.335 527	+0.335 090	+0.334 641	+0.334 178	+0.333 703	+0.333 217	+0.332 718	+0.332 209	+0.331 690	40
45	+0.343 554	+0.342 971	+0.342 379	+0.341 778	+0.341 167	+0.340 549	+0.339 922	+0.339 288	+0.338 647	+0.337 998	+0.337 343	45
50	+0.349 944	+0.349 213	+0.348 477	+0.347 735	+0.346 989	+0.346 238	+0.345 482	+0.344 722	+0.343 958	+0.343 19[illegible]	+0.342 419	50
55	+0.355 544	+0.354 689	+0.353 832	+0.352 973	+0.352 112	+0.351 249	+0.350 384	+0.349 518	+0.348 651	+0.347 782	+0.346 913	55
60	+0.360 404	+0.359 447	+0.358 489	+0.357 532	+0.356 575	+0.355 619	+0.354 664	+0.353 709	+0.352 755	+0.351 802	+0.350 850	60
65	+0.364 590	+0.363 556	+0.362 516	+0.361 477	+0.360 441	+0.359 407	+0.358 376	+0.357 347	+0.356 321	+0.355 297	+0.354 277	65
70	+0.368 206	+0.367 093	+0.365 984	+0.364 878	+0.363 776	+0.362 678	+0.361 583	+0.360 493	+0.359 407	+0.358 324	+0.357 246	70
75	+0.371 304	+0.370 132	+0.368 960	+0.367 804	+0.366 647	+0.365 495	+0.364 348	+0.363 207	+0.362 071	+0.360 939	+0.359 813	75
80	+0.373 962	+0.372 742	+0.371 528	+0.370 319	+0.369 117	+0.367 921	+0.366 730	+0.365 546	+0.364 368	+0.363 195	+0.362 029	80
85	+0.376 245	+0.374 984	+0.373 730	+0.372 483	+0.371 243	+0.370 009	+0.368 783	+0.367 563	+0.366 349	+0.365 142	+0.363 942	85
90	+0.378 207	+0.376 913	+0.375 625	+0.374 346	+0.373 074	+0.371 809	+0.370 552	+0.369 301	+0.368 059	+0.366 823	+0.365 595	90
95	+0.379 804	+0.378 571	+0.377 256	+0.375 950	+0.374 651	+0.373 360	+0.372 076	+0.370 800	+0.369 534	+0.368 274	+0.367 020	95
100	+0.381 351	+0.380 004	+0.378 666	+0.377 336	+0.376 015	+0.374 701	+0.373 397	+0.372 100	+0.370 811	+0.369 531	+0.368 259	100
105	+0.382 606	+0.381 239	+0.379 881	+0.378 532	+0.377 191	+0.375 859	+0.374 537	+0.373 222	+0.371 914	+0.370 618	+0.369 330	105
110	+0.383 692	+0.382 308	+0.380 933	+0.379 567	+0.378 210	+0.376 862	+0.375 524	+0.374 193	+0.372 872	+0.371 560	+0.370 256	110
115	+0.384 630	+0.383 231	+0.381 842	+0.380 462	+0.379 091	+0.377 730	+0.376 377	+0.375 034	+0.373 700	+0.372 375	+0.371 059	115
120	+0.385 440	+0.384 029	+0.382 627	+0.381 235	+0.379 853	+0.378 480	+0.377 117	+0.375 763	+0.374 418	+0.373 081	+0.371 754	120
125	+0.386 139	+0.384 717	+0.383 305	+0.381 903	+0.380 510	+0.379 128	+0.377 755	+0.376 391	+0.375 036	+0.373 692	+0.372 355	125
130	+0.386 741	+0.385 311	+0.383 890	+0.382 479	+0.381 078	+0.379 687	+0.378 306	+0.376 934	+0.375 572	+0.374 219	+0.372 875	130
135	+0.387 259	+0.385 820	+0.384 391	+0.382 973	+0.381 565	+0.380 167	+0.378 778	+0.377 400	+0.376 031	+0.374 672	+0.373 322	135
140	+0.387 699	+0.386 254	+0.384 820	+0.383 396	+0.381 981	+0.380 577	+0.379 183	+0.377 799	+0.376 424	+0.375 058	+0.373 702	140
145	+0.388 072	+0.386 622	+0.385 182	+0.383 752	+0.382 333	+0.380 924	+0.379 524	+0.378 135	+0.376 755	+0.375 386	+0.374 026	145
150	+0.388 385	+0.386 930	+0.385 485	+0.384 050	+0.382 627	+0.381 214	+0.379 811	+0.378 417	+0.377 034	+0.375 660	+0.374 296	150
155	+0.388 640	+0.387 181	+0.385 733	+0.384 295	+0.382 868	+0.381 451	+0.380 044	+0.378 647	+0.377 260	+0.375 884	+0.374 516	155
160	+0.388 844	+0.387 382	+0.385 931	+0.384 490	+0.383 060	+0.381 641	+0.380 231	+0.378 832	+0.377 442	+0.376 063	+0.374 694	160
165	+0.388 998	+0.387 534	+0.386 081	+0.384 638	+0.383 207	+0.381 785	+0.380 373	+0.378 972	+0.377 581	+0.376 200	+0.374 828	165
170	+0.389 107	+0.387 642	+0.386 187	+0.384 743	+0.383 310	+0.381 887	+0.380 474	+0.379 071	+0.377 678	+0.376 296	+0.374 923	170
175	+0.389 172	+0.387 706	+0.386 251	+0.384 805	+0.383 371	+0.381 947	+0.380 533	+0.379 130	+0.377 737	+0.376 353	+0.374 980	175
180	+0.389 193	+0.387 727	+0.386 271	+0.384 826	+0.383 391	+0.381 967	+0.380 553	+0.379 149	+0.377 755	+0.376 372	+0.374 998	180
ψ / z	2.51	2.52	2.53	2.54	2.55	2.56	2.57	2.58	2.59	2.60	2.61	z \ ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	2.61	2.62	2.63	2.64	2.65	2.66	2.67	2.68	2.69	2.70	2.71	z / ψ
0°	+ 0.291 333	+ 0.291 903	+ 0.292 431	+ 0.292 913	+ 0.293 354	+ 0.293 755	+ 0.294 115	+ 0.294 441	+ 0.294 730	+ 0.294 985	+ 0.295 206	0°
5	+ 0.292 379	+ 0.292 917	+ 0.293 414	+ 0.293 867	+ 0.294 279	+ 0.294 652	+ 0.294 987	+ 0.295 286	+ 0.295 551	+ 0.295 782	+ 0.295 980	5
10	+ 0.295 391	+ 0.295 840	+ 0.296 248	+ 0.296 617	+ 0.296 948	+ 0.297 242	+ 0.297 502	+ 0.297 728	+ 0.297 922	+ 0.298 084	+ 0.298 217	10
15	+ 0.300 033	+ 0.300 345	+ 0.300 620	+ 0.300 862	+ 0.301 069	+ 0.301 245	+ 0.301 390	+ 0.301 505	+ 0.301 591	+ 0.301 651	+ 0.301 683	15
20	+ 0.305 836	+ 0.305 981	+ 0.306 096	+ 0.306 182	+ 0.306 239	+ 0.306 270	+ 0.306 274	+ 0.306 254	+ 0.306 209	+ 0.306 141	+ 0.306 051	20
25	+ 0.312 301	+ 0.312 267	+ 0.312 209	+ 0.312 127	+ 0.312 022	+ 0.311 896	+ 0.311 748	+ 0.311 580	+ 0.311 393	+ 0.311 187	+ 0.310 963	25
30	+ 0.318 980	+ 0.318 768	+ 0.318 537	+ 0.318 288	+ 0.318 021	+ 0.317 738	+ 0.317 439	+ 0.317 124	+ 0.316 794	+ 0.316 450	+ 0.316 092	30
35	+ 0.325 522	+ 0.325 144	+ 0.324 752	+ 0.324 345	+ 0.323 926	+ 0.323 495	+ 0.323 052	+ 0.322 599	+ 0.322 133	+ 0.321 657	+ 0.321 172	35
40	+ 0.331 690	+ 0.331 161	+ 0.330 622	+ 0.330 074	+ 0.329 517	+ 0.328 952	+ 0.328 378	+ 0.327 798	+ 0.327 210	+ 0.326 614	+ 0.326 013	40
45	+ 0.337 343	+ 0.336 682	+ 0.336 015	+ 0.335 343	+ 0.334 664	+ 0.333 981	+ 0.333 292	+ 0.332 600	+ 0.331 903	+ 0.331 202	+ 0.330 498	45
50	+ 0.342 419	+ 0.341 645	+ 0.340 868	+ 0.340 088	+ 0.339 305	+ 0.338 520	+ 0.337 733	+ 0.336 943	+ 0.336 152	+ 0.335 359	+ 0.334 565	50
55	+ 0.346 913	+ 0.346 043	+ 0.345 172	+ 0.344 301	+ 0.343 429	+ 0.342 559	+ 0.341 686	+ 0.340 813	+ 0.339 942	+ 0.339 070	+ 0.338 199	55
60	+ 0.350 850	+ 0.349 899	+ 0.348 950	+ 0.348 002	+ 0.347 055	+ 0.346 110	+ 0.345 167	+ 0.344 225	+ 0.343 285	+ 0.342 347	+ 0.341 411	60
65	+ 0.354 277	+ 0.353 258	+ 0.352 243	+ 0.351 231	+ 0.350 221	+ 0.349 215	+ 0.348 212	+ 0.347 211	+ 0.346 214	+ 0.345 219	+ 0.344 228	65
70	+ 0.357 246	+ 0.356 171	+ 0.355 101	+ 0.354 035	+ 0.352 973	+ 0.351 915	+ 0.350 861	+ 0.349 811	+ 0.348 766	+ 0.347 724	+ 0.346 687	70
75	+ 0.359 813	+ 0.358 691	+ 0.357 575	+ 0.356 464	+ 0.355 358	+ 0.354 256	+ 0.353 160	+ 0.352 069	+ 0.350 983	+ 0.349 902	+ 0.348 826	75
80	+ 0.362 029	+ 0.360 868	+ 0.359 713	+ 0.358 564	+ 0.357 421	+ 0.356 284	+ 0.355 152	+ 0.354 027	+ 0.352 907	+ 0.351 792	+ 0.350 684	80
85	+ 0.363 942	+ 0.362 749	+ 0.361 561	+ 0.360 381	+ 0.359 207	+ 0.358 039	+ 0.356 878	+ 0.355 723	+ 0.354 574	+ 0.353 432	+ 0.352 296	85
90	+ 0.365 595	+ 0.364 373	+ 0.363 159	+ 0.361 952	+ 0.360 752	+ 0.359 558	+ 0.358 372	+ 0.357 192	+ 0.356 020	+ 0.354 853	+ 0.353 694	90
95	+ 0.367 020	+ 0.365 777	+ 0.364 540	+ 0.363 311	+ 0.362 088	+ 0.360 873	+ 0.359 666	+ 0.358 466	+ 0.357 272	+ 0.356 086	+ 0.354 906	95
100	+ 0.368 259	+ 0.366 994	+ 0.365 738	+ 0.364 489	+ 0.363 248	+ 0.362 015	+ 0.360 789	+ 0.359 571	+ 0.358 361	+ 0.357 158	+ 0.355 962	100
105	+ 0.369 330	+ 0.368 047	+ 0.366 774	+ 0.365 509	+ 0.364 253	+ 0.363 004	+ 0.361 762	+ 0.360 530	+ 0.359 305	+ 0.358 086	+ 0.356 878	105
110	+ 0.370 256	+ 0.368 960	+ 0.367 673	+ 0.366 394	+ 0.365 124	+ 0.363 862	+ 0.362 608	+ 0.361 362	+ 0.360 124	+ 0.358 895	+ 0.357 673	110
115	+ 0.371 059	+ 0.369 751	+ 0.368 452	+ 0.367 162	+ 0.365 880	+ 0.364 606	+ 0.363 341	+ 0.362 085	+ 0.360 837	+ 0.359 597	+ 0.358 365	115
120	+ 0.371 754	+ 0.370 436	+ 0.369 127	+ 0.367 827	+ 0.366 535	+ 0.365 252	+ 0.363 978	+ 0.362 712	+ 0.361 454	+ 0.360 205	+ 0.358 963	120
125	+ 0.372 355	+ 0.371 029	+ 0.369 711	+ 0.368 403	+ 0.367 104	+ 0.365 812	+ 0.364 529	+ 0.363 255	+ 0.361 988	+ 0.360 732	+ 0.359 483	125
130	+ 0.372 875	+ 0.371 541	+ 0.370 215	+ 0.368 899	+ 0.367 592	+ 0.366 294	+ 0.365 005	+ 0.363 724	+ 0.362 452	+ 0.361 188	+ 0.359 933	130
135	+ 0.373 322	+ 0.371 981	+ 0.370 649	+ 0.369 327	+ 0.368 014	+ 0.366 709	+ 0.365 414	+ 0.364 127	+ 0.362 849	+ 0.361 580	+ 0.360 319	135
140	+ 0.373 702	+ 0.372 356	+ 0.371 019	+ 0.369 692	+ 0.368 373	+ 0.367 064	+ 0.365 763	+ 0.364 472	+ 0.363 190	+ 0.361 916	+ 0.360 650	140
145	+ 0.374 026	+ 0.372 676	+ 0.371 335	+ 0.370 003	+ 0.368 680	+ 0.367 365	+ 0.366 060	+ 0.364 764	+ 0.363 477	+ 0.362 199	+ 0.360 930	145
150	+ 0.374 296	+ 0.372 941	+ 0.371 596	+ 0.370 260	+ 0.368 934	+ 0.367 617	+ 0.366 309	+ 0.365 009	+ 0.363 719	+ 0.362 437	+ 0.361 165	150
155	+ 0.374 516	+ 0.373 159	+ 0.371 810	+ 0.370 472	+ 0.369 142	+ 0.367 823	+ 0.366 512	+ 0.365 210	+ 0.363 917	+ 0.362 633	+ 0.361 358	155
160	+ 0.374 694	+ 0.373 334	+ 0.371 983	+ 0.370 642	+ 0.369 310	+ 0.367 988	+ 0.366 675	+ 0.365 371	+ 0.364 076	+ 0.362 789	+ 0.361 512	160
165	+ 0.374 828	+ 0.373 466	+ 0.372 114	+ 0.370 771	+ 0.369 438	+ 0.368 114	+ 0.366 799	+ 0.365 493	+ 0.364 196	+ 0.362 908	+ 0.361 629	165
170	+ 0.374 923	+ 0.373 560	+ 0.372 206	+ 0.370 862	+ 0.369 527	+ 0.368 202	+ 0.366 886	+ 0.365 579	+ 0.364 281	+ 0.362 992	+ 0.361 712	170
175	+ 0.374 980	+ 0.373 615	+ 0.372 261	+ 0.370 915	+ 0.369 580	+ 0.368 254	+ 0.366 938	+ 0.365 630	+ 0.364 331	+ 0.363 041	+ 0.361 761	175
180	+ 0.374 998	+ 0.373 634	+ 0.372 279	+ 0.370 934	+ 0.369 597	+ 0.368 271	+ 0.366 954	+ 0.365 647	+ 0.364 347	+ 0.363 058	+ 0.361 777	180
ψ / z	2.61	2.62	2.63	2.64	2.65	2.66	2.67	2.68	2.69	2.70	2.71	z \ ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING : $\frac{\nu}{k}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	2.91	2.92	2.93	2.94	2.95	2.96	2.97	2.98	2.99	3.00	3.01	z / ψ
0	+ 0.294 324	+ 0.294 080	+ 0.293 822	+ 0.293 551	+ 0.293 267	+ 0.292 969	+ 0.292 661	+ 0.292 340	+ 0.292 009	+ 0.291 667	+ 0.291 314	0°
5	+ 0.294 770	+ 0.294 515	+ 0.294 246	+ 0.293 963	+ 0.293 668	+ 0.293 361	+ 0.293 043	+ 0.292 713	+ 0.292 372	+ 0.292 021	+ 0.291 661	5
10	+ 0.296 065	+ 0.295 776	+ 0.295 475	+ 0.295 162	+ 0.294 837	+ 0.294 500	+ 0.294 153	+ 0.293 796	+ 0.293 428	+ 0.293 051	+ 0.292 665	10
15	+ 0.298 089	+ 0.297 749	+ 0.297 399	+ 0.297 038	+ 0.296 666	+ 0.296 285	+ 0.295 894	+ 0.295 494	+ 0.295 085	+ 0.294 668	+ 0.294 243	15
20	+ 0.300 673	+ 0.300 270	+ 0.299 858	+ 0.299 436	+ 0.299 007	+ 0.298 569	+ 0.298 123	+ 0.297 670	+ 0.297 210	+ 0.296 742	+ 0.296 269	20
25	+ 0.303 624	+ 0.303 151	+ 0.302 670	+ 0.302 183	+ 0.301 688	+ 0.301 187	+ 0.300 680	+ 0.300 168	+ 0.299 650	+ 0.299 126	+ 0.298 597	25
30	+ 0.306 760	+ 0.306 214	+ 0.305 663	+ 0.305 106	+ 0.304 545	+ 0.303 979	+ 0.303 409	+ 0.302 835	+ 0.302 256	+ 0.301 674	+ 0.301 088	30
35	+ 0.309 924	+ 0.309 307	+ 0.308 687	+ 0.308 063	+ 0.307 437	+ 0.306 807	+ 0.306 175	+ 0.305 540	+ 0.304 902	+ 0.304 264	+ 0.303 622	35
40	+ 0.312 996	+ 0.312 313	+ 0.311 628	+ 0.310 942	+ 0.310 254	+ 0.309 564	+ 0.308 874	+ 0.308 183	+ 0.307 490	+ 0.306 797	+ 0.306 103	40
45	+ 0.315 894	+ 0.315 151	+ 0.314 408	+ 0.313 664	+ 0.312 920	+ 0.312 177	+ 0.311 433	+ 0.310 690	+ 0.309 947	+ 0.309 204	+ 0.308 462	45
50	+ 0.318 570	+ 0.317 773	+ 0.316 978	+ 0.316 183	+ 0.315 390	+ 0.314 597	+ 0.313 806	+ 0.313 017	+ 0.312 229	+ 0.311 442	+ 0.310 656	50
55	+ 0.321 000	+ 0.320 157	+ 0.319 316	+ 0.318 477	+ 0.317 640	+ 0.316 805	+ 0.315 972	+ 0.315 142	+ 0.314 314	+ 0.313 488	+ 0.312 664	55
60	+ 0.323 181	+ 0.322 297	+ 0.321 417	+ 0.320 539	+ 0.319 664	+ 0.318 792	+ 0.317 924	+ 0.317 058	+ 0.316 195	+ 0.315 335	+ 0.314 478	60
65	+ 0.325 122	+ 0.324 203	+ 0.323 288	+ 0.322 377	+ 0.321 470	+ 0.320 566	+ 0.319 666	+ 0.318 769	+ 0.317 877	+ 0.316 988	+ 0.316 103	65
70	+ 0.326 837	+ 0.325 889	+ 0.324 945	+ 0.324 005	+ 0.323 070	+ 0.322 139	+ 0.321 212	+ 0.320 289	+ 0.319 371	+ 0.318 457	+ 0.317 547	70
75	+ 0.328 346	+ 0.327 373	+ 0.326 405	+ 0.325 440	+ 0.324 482	+ 0.323 527	+ 0.322 577	+ 0.321 632	+ 0.320 692	+ 0.319 756	+ 0.318 825	75
80	+ 0.329 671	+ 0.328 676	+ 0.327 687	+ 0.326 702	+ 0.325 723	+ 0.324 748	+ 0.323 779	+ 0.322 815	+ 0.321 855	+ 0.320 901	+ 0.319 951	80
85	+ 0.330 832	+ 0.329 818	+ 0.328 811	+ 0.327 809	+ 0.326 812	+ 0.325 821	+ 0.324 835	+ 0.323 854	+ 0.322 878	+ 0.321 908	+ 0.320 943	85
90	+ 0.331 847	+ 0.330 818	+ 0.329 796	+ 0.328 778	+ 0.327 767	+ 0.326 761	+ 0.325 760	+ 0.324 766	+ 0.323 776	+ 0.322 792	+ 0.321 814	90
95	+ 0.332 736	+ 0.331 694	+ 0.330 656	+ 0.329 626	+ 0.328 602	+ 0.327 585	+ 0.326 571	+ 0.325 563	+ 0.324 564	+ 0.323 569	+ 0.322 579	95
100	+ 0.333 512	+ 0.332 458	+ 0.331 411	+ 0.330 370	+ 0.329 335	+ 0.328 305	+ 0.327 282	+ 0.326 265	+ 0.325 254	+ 0.324 249	+ 0.323 249	100
105	+ 0.334 190	+ 0.333 127	+ 0.332 071	+ 0.331 018	+ 0.329 974	+ 0.328 935	+ 0.327 904	+ 0.326 879	+ 0.325 858	+ 0.324 844	+ 0.323 836	105
110	+ 0.334 783	+ 0.333 711	+ 0.332 646	+ 0.331 587	+ 0.330 535	+ 0.329 489	+ 0.328 449	+ 0.327 415	+ 0.326 388	+ 0.325 366	+ 0.324 351	110
115	+ 0.335 301	+ 0.334 222	+ 0.333 149	+ 0.332 083	+ 0.331 024	+ 0.329 971	+ 0.328 925	+ 0.327 884	+ 0.326 850	+ 0.325 823	+ 0.324 801	115
120	+ 0.335 751	+ 0.334 666	+ 0.333 588	+ 0.332 516	+ 0.331 451	+ 0.330 392	+ 0.329 340	+ 0.328 294	+ 0.327 255	+ 0.326 222	+ 0.325 195	120
125	+ 0.336 144	+ 0.335 054	+ 0.333 971	+ 0.332 894	+ 0.331 823	+ 0.330 759	+ 0.329 702	+ 0.328 651	+ 0.327 607	+ 0.326 569	+ 0.325 538	125
130	+ 0.336 484	+ 0.335 389	+ 0.334 301	+ 0.333 219	+ 0.332 143	+ 0.331 077	+ 0.330 016	+ 0.328 961	+ 0.327 913	+ 0.326 871	+ 0.325 835	130
135	+ 0.336 777	+ 0.335 679	+ 0.334 587	+ 0.333 502	+ 0.332 424	+ 0.331 352	+ 0.330 277	+ 0.329 229	+ 0.328 177	+ 0.327 132	+ 0.326 093	135
140	+ 0.337 031	+ 0.335 928	+ 0.334 833	+ 0.333 744	+ 0.332 663	+ 0.331 588	+ 0.330 520	+ 0.329 459	+ 0.328 404	+ 0.327 356	+ 0.326 314	140
145	+ 0.337 245	+ 0.336 140	+ 0.335 042	+ 0.333 950	+ 0.332 866	+ 0.331 789	+ 0.330 719	+ 0.329 654	+ 0.328 598	+ 0.327 546	+ 0.326 502	145
150	+ 0.337 425	+ 0.336 317	+ 0.335 217	+ 0.334 123	+ 0.333 037	+ 0.331 957	+ 0.330 885	+ 0.329 819	+ 0.328 759	+ 0.327 706	+ 0.326 660	150
155	+ 0.337 571	+ 0.336 462	+ 0.335 359	+ 0.334 265	+ 0.333 177	+ 0.332 096	+ 0.331 022	+ 0.329 954	+ 0.328 893	+ 0.327 838	+ 0.326 790	155
160	+ 0.337 688	+ 0.336 579	+ 0.335 476	+ 0.334 379	+ 0.333 289	+ 0.332 206	+ 0.331 131	+ 0.330 062	+ 0.329 000	+ 0.327 943	+ 0.326 894	160
165	+ 0.337 780	+ 0.336 668	+ 0.335 562	+ 0.334 464	+ 0.333 374	+ 0.332 291	+ 0.331 214	+ 0.330 144	+ 0.329 081	+ 0.328 024	+ 0.326 974	165
170	+ 0.337 844	+ 0.336 731	+ 0.335 626	+ 0.334 527	+ 0.333 434	+ 0.332 350	+ 0.331 271	+ 0.330 201	+ 0.329 137	+ 0.328 080	+ 0.327 029	170
175	+ 0.337 882	+ 0.336 768	+ 0.335 662	+ 0.334 562	+ 0.333 470	+ 0.332 385	+ 0.331 307	+ 0.330 236	+ 0.329 172	+ 0.328 114	+ 0.327 062	175
180	+ 0.337 894	+ 0.336 780	+ 0.335 674	+ 0.334 575	+ 0.333 483	+ 0.332 398	+ 0.331 320	+ 0.330 247	+ 0.329 183	+ 0.328 125	+ 0.327 074	180
ψ / z	2.91	2.92	2.93	2.94	2.95	2.96	2.97	2.98	2.99	3.00	3.01	ψ / z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	3.01	3.02	3.03	3.04	3.05	3.06	3.07	3.08	3.09	3.10	3.11	z / ψ
0°	+ 0.291 314	+ 0.290 953	+ 0.290 581	+ 0.290 201	+ 0.289 812	+ 0.289 414	+ 0.289 009	+ 0.288 596	+ 0.288 176	+ 0.287 748	+ 0.287 315	0°
5	+ 0.291 661	+ 0.291 289	+ 0.290 910	+ 0.290 521	+ 0.290 125	+ 0.289 720	+ 0.289 307	+ 0.288 887	+ 0.288 460	+ 0.288 026	+ 0.287 586	5
10	+ 0.292 665	+ 0.292 270	+ 0.291 867	+ 0.291 456	+ 0.291 037	+ 0.290 610	+ 0.290 176	+ 0.289 736	+ 0.289 289	+ 0.288 835	+ 0.288 376	10
15	+ 0.294 243	+ 0.293 811	+ 0.293 371	+ 0.292 924	+ 0.292 470	+ 0.292 010	+ 0.291 543	+ 0.291 071	+ 0.290 592	+ 0.290 109	+ 0.289 621	15
20	+ 0.296 269	+ 0.295 788	+ 0.295 302	+ 0.294 810	+ 0.294 313	+ 0.293 810	+ 0.293 302	+ 0.292 790	+ 0.292 273	+ 0.291 751	+ 0.291 226	20
25	+ 0.298 597	+ 0.298 064	+ 0.297 526	+ 0.296 984	+ 0.296 437	+ 0.295 886	+ 0.295 332	+ 0.294 774	+ 0.294 213	+ 0.293 649	+ 0.293 082	25
30	+ 0.301 089	+ 0.300 500	+ 0.299 909	+ 0.299 314	+ 0.298 716	+ 0.298 116	+ 0.297 513	+ 0.296 909	+ 0.296 302	+ 0.295 692	+ 0.295 082	30
35	+ 0.303 622	+ 0.302 980	+ 0.302 336	+ 0.301 688	+ 0.301 041	+ 0.300 392	+ 0.299 740	+ 0.299 090	+ 0.298 436	+ 0.297 785	+ 0.297 129	35
40	+ 0.306 103	+ 0.305 409	+ 0.304 714	+ 0.304 018	+ 0.303 323	+ 0.302 627	+ 0.301 931	+ 0.301 235	+ 0.300 539	+ 0.299 844	+ 0.299 148	40
45	+ 0.308 462	+ 0.307 721	+ 0.306 980	+ 0.306 239	+ 0.305 500	+ 0.304 761	+ 0.304 023	+ 0.303 286	+ 0.302 550	+ 0.301 815	+ 0.301 082	45
50	+ 0.310 656	+ 0.309 872	+ 0.309 090	+ 0.308 309	+ 0.307 530	+ 0.306 753	+ 0.305 978	+ 0.305 204	+ 0.304 432	+ 0.303 662	+ 0.302 894	50
55	+ 0.312 664	+ 0.311 843	+ 0.311 024	+ 0.310 208	+ 0.309 394	+ 0.308 583	+ 0.307 774	+ 0.306 967	+ 0.306 163	+ 0.305 362	+ 0.304 563	55
60	+ 0.314 478	+ 0.313 625	+ 0.312 774	+ 0.311 926	+ 0.311 082	+ 0.310 241	+ 0.309 403	+ 0.308 567	+ 0.307 736	+ 0.306 907	+ 0.306 081	60
65	+ 0.316 103	+ 0.315 221	+ 0.314 342	+ 0.313 468	+ 0.312 597	+ 0.311 730	+ 0.310 866	+ 0.310 006	+ 0.309 150	+ 0.308 297	+ 0.307 448	65
70	+ 0.317 547	+ 0.316 641	+ 0.315 738	+ 0.314 841	+ 0.313 947	+ 0.313 057	+ 0.312 172	+ 0.311 290	+ 0.310 413	+ 0.309 539	+ 0.308 670	70
75	+ 0.318 825	+ 0.317 898	+ 0.316 975	+ 0.316 058	+ 0.315 144	+ 0.314 235	+ 0.313 331	+ 0.312 431	+ 0.311 535	+ 0.310 644	+ 0.309 757	75
80	+ 0.319 951	+ 0.319 007	+ 0.318 067	+ 0.317 132	+ 0.316 202	+ 0.315 276	+ 0.314 356	+ 0.313 440	+ 0.312 528	+ 0.311 622	+ 0.310 720	80
85	+ 0.320 943	+ 0.319 984	+ 0.319 029	+ 0.318 079	+ 0.317 134	+ 0.316 194	+ 0.315 260	+ 0.314 330	+ 0.313 405	+ 0.312 486	+ 0.311 570	85
90	+ 0.321 814	+ 0.320 841	+ 0.319 873	+ 0.318 911	+ 0.317 954	+ 0.317 002	+ 0.316 056	+ 0.315 114	+ 0.314 178	+ 0.313 247	+ 0.312 321	90
95	+ 0.322 579	+ 0.321 594	+ 0.320 614	+ 0.319 643	+ 0.318 674	+ 0.317 712	+ 0.316 756	+ 0.315 804	+ 0.314 858	+ 0.313 917	+ 0.312 982	95
100	+ 0.323 249	+ 0.322 255	+ 0.321 267	+ 0.320 285	+ 0.319 308	+ 0.318 336	+ 0.317 371	+ 0.316 410	+ 0.315 456	+ 0.314 506	+ 0.313 562	100
105	+ 0.323 836	+ 0.322 834	+ 0.321 838	+ 0.320 847	+ 0.319 864	+ 0.318 884	+ 0.317 911	+ 0.316 943	+ 0.315 980	+ 0.315 024	+ 0.314 074	105
110	+ 0.324 351	+ 0.323 342	+ 0.322 339	+ 0.321 341	+ 0.320 350	+ 0.319 364	+ 0.318 384	+ 0.317 410	+ 0.316 442	+ 0.315 479	+ 0.314 522	110
115	+ 0.324 801	+ 0.323 786	+ 0.322 777	+ 0.321 774	+ 0.320 777	+ 0.319 786	+ 0.318 800	+ 0.317 820	+ 0.316 846	+ 0.315 878	+ 0.314 915	115
120	+ 0.325 195	+ 0.324 175	+ 0.323 160	+ 0.322 151	+ 0.321 149	+ 0.320 153	+ 0.319 162	+ 0.318 178	+ 0.317 200	+ 0.316 227	+ 0.315 260	120
125	+ 0.325 538	+ 0.324 513	+ 0.323 494	+ 0.322 481	+ 0.321 474	+ 0.320 473	+ 0.319 478	+ 0.318 490	+ 0.317 506	+ 0.316 529	+ 0.315 560	125
130	+ 0.325 835	+ 0.324 806	+ 0.323 783	+ 0.322 767	+ 0.321 757	+ 0.320 753	+ 0.319 754	+ 0.318 762	+ 0.317 775	+ 0.316 795	+ 0.315 820	130
135	+ 0.326 093	+ 0.325 061	+ 0.324 035	+ 0.323 015	+ 0.322 001	+ 0.320 994	+ 0.319 992	+ 0.318 997	+ 0.318 007	+ 0.317 024	+ 0.316 047	135
140	+ 0.326 314	+ 0.325 279	+ 0.324 250	+ 0.323 228	+ 0.322 211	+ 0.321 201	+ 0.320 197	+ 0.319 200	+ 0.318 208	+ 0.317 222	+ 0.316 242	140
145	+ 0.326 502	+ 0.325 464	+ 0.324 433	+ 0.323 408	+ 0.322 390	+ 0.321 377	+ 0.320 371	+ 0.319 371	+ 0.318 377	+ 0.317 389	+ 0.316 407	145
150	+ 0.326 660	+ 0.325 620	+ 0.324 587	+ 0.323 560	+ 0.322 540	+ 0.321 526	+ 0.320 518	+ 0.319 516	+ 0.318 520	+ 0.317 531	+ 0.316 547	150
155	+ 0.326 790	+ 0.325 749	+ 0.324 714	+ 0.323 686	+ 0.322 664	+ 0.321 648	+ 0.320 638	+ 0.319 635	+ 0.318 637	+ 0.317 646	+ 0.316 661	155
160	+ 0.326 894	+ 0.325 851	+ 0.324 815	+ 0.323 786	+ 0.322 762	+ 0.321 745	+ 0.320 734	+ 0.319 730	+ 0.318 732	+ 0.317 740	+ 0.316 753	160
165	+ 0.326 974	+ 0.325 930	+ 0.324 893	+ 0.323 863	+ 0.322 839	+ 0.321 820	+ 0.320 808	+ 0.319 802	+ 0.318 803	+ 0.317 810	+ 0.316 823	165
170	+ 0.327 029	+ 0.325 985	+ 0.324 947	+ 0.323 916	+ 0.322 891	+ 0.321 873	+ 0.320 860	+ 0.319 854	+ 0.318 854	+ 0.317 861	+ 0.316 873	170
175	+ 0.327 062	+ 0.326 017	+ 0.324 979	+ 0.323 947	+ 0.322 922	+ 0.321 903	+ 0.320 891	+ 0.319 885	+ 0.318 884	+ 0.317 890	+ 0.316 902	175
180	+ 0.327 074	+ 0.326 029	+ 0.324 991	+ 0.323 959	+ 0.322 933	+ 0.321 914	+ 0.320 901	+ 0.319 895	+ 0.318 894	+ 0.317 900	+ 0.316 912	180
ψ / z	3.01	3.02	3.03	3.04	3.05	3.06	3.07	3.08	3.09	3.10	3.11	ψ / z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	3.11	3.12	3.13	3.14	3.15	3.16	3.17	3.18	3.19	3.20	3.21	z / ψ
0°	+ 0.287 315	+ 0.286 874	+ 0.286 428	+ 0.285 975	+ 0.285 517	+ 0.285 054	+ 0.284 586	+ 0.284 112	+ 0.283 634	+ 0.283 152	+ 0.282 665	0°
5	+ 0.287 586	+ 0.287 139	+ 0.286 686	+ 0.286 228	+ 0.285 764	+ 0.285 295	+ 0.284 821	+ 0.284 343	+ 0.283 858	+ 0.283 370	+ 0.282 880	5
10	+ 0.288 376	+ 0.287 911	+ 0.287 441	+ 0.286 965	+ 0.286 484	+ 0.285 999	+ 0.285 509	+ 0.285 015	+ 0.284 517	+ 0.284 014	+ 0.283 508	10
15	+ 0.289 621	+ 0.289 127	+ 0.288 629	+ 0.288 127	+ 0.287 620	+ 0.287 109	+ 0.286 594	+ 0.286 076	+ 0.285 554	+ 0.285 029	+ 0.284 501	15
20	+ 0.291 226	+ 0.290 696	+ 0.290 163	+ 0.289 627	+ 0.289 087	+ 0.288 543	+ 0.287 997	+ 0.287 448	+ 0.286 896	+ 0.286 342	+ 0.285 786	20
25	+ 0.293 082	+ 0.292 512	+ 0.291 939	+ 0.291 364	+ 0.290 786	+ 0.290 206	+ 0.289 624	+ 0.289 040	+ 0.288 455	+ 0.287 868	+ 0.287 279	25
30	+ 0.295 082	+ 0.294 469	+ 0.293 855	+ 0.293 239	+ 0.292 622	+ 0.292 004	+ 0.291 385	+ 0.290 764	+ 0.290 143	+ 0.289 521	+ 0.288 898	30
35	+ 0.297 129	+ 0.296 473	+ 0.295 817	+ 0.295 162	+ 0.294 507	+ 0.293 850	+ 0.293 194	+ 0.292 536	+ 0.291 879	+ 0.291 223	+ 0.290 567	35
40	+ 0.299 148	+ 0.298 453	+ 0.297 758	+ 0.297 064	+ 0.296 370	+ 0.295 677	+ 0.294 985	+ 0.294 293	+ 0.293 602	+ 0.292 912	+ 0.292 223	40
45	+ 0.301 082	+ 0.300 350	+ 0.299 619	+ 0.298 889	+ 0.298 160	+ 0.297 433	+ 0.296 707	+ 0.295 983	+ 0.295 260	+ 0.294 539	+ 0.293 820	45
50	+ 0.302 894	+ 0.302 128	+ 0.301 364	+ 0.300 602	+ 0.299 842	+ 0.299 084	+ 0.298 328	+ 0.297 574	+ 0.296 822	+ 0.296 073	+ 0.295 325	50
55	+ 0.304 563	+ 0.303 767	+ 0.302 973	+ 0.302 183	+ 0.301 395	+ 0.300 609	+ 0.299 826	+ 0.299 046	+ 0.298 268	+ 0.297 494	+ 0.296 721	55
60	+ 0.306 081	+ 0.305 259	+ 0.304 439	+ 0.303 623	+ 0.302 810	+ 0.302 000	+ 0.301 193	+ 0.300 390	+ 0.299 589	+ 0.298 792	+ 0.297 998	60
65	+ 0.307 448	+ 0.306 603	+ 0.305 761	+ 0.304 923	+ 0.304 088	+ 0.303 257	+ 0.302 429	+ 0.301 605	+ 0.300 785	+ 0.299 968	+ 0.299 155	65
70	+ 0.308 670	+ 0.307 805	+ 0.306 944	+ 0.306 086	+ 0.305 232	+ 0.304 383	+ 0.303 537	+ 0.302 695	+ 0.301 858	+ 0.301 024	+ 0.300 194	70
75	+ 0.309 757	+ 0.308 875	+ 0.307 997	+ 0.307 122	+ 0.306 252	+ 0.305 387	+ 0.304 526	+ 0.303 668	+ 0.302 816	+ 0.301 967	+ 0.301 122	75
80	+ 0.310 720	+ 0.309 822	+ 0.308 930	+ 0.308 041	+ 0.307 157	+ 0.306 278	+ 0.305 403	+ 0.304 533	+ 0.303 667	+ 0.302 806	+ 0.301 948	80
85	+ 0.311 570	+ 0.310 660	+ 0.309 755	+ 0.308 854	+ 0.307 959	+ 0.307 067	+ 0.306 181	+ 0.305 299	+ 0.304 422	+ 0.303 549	+ 0.302 681	85
90	+ 0.312 321	+ 0.311 399	+ 0.310 483	+ 0.309 572	+ 0.308 666	+ 0.307 765	+ 0.306 868	+ 0.305 976	+ 0.305 089	+ 0.304 207	+ 0.303 330	90
95	+ 0.312 982	+ 0.312 051	+ 0.311 126	+ 0.310 205	+ 0.309 290	+ 0.308 380	+ 0.307 475	+ 0.306 574	+ 0.305 679	+ 0.304 788	+ 0.303 903	95
100	+ 0.313 562	+ 0.312 624	+ 0.311 690	+ 0.310 762	+ 0.309 839	+ 0.308 921	+ 0.308 009	+ 0.307 101	+ 0.306 198	+ 0.305 301	+ 0.304 408	100
105	+ 0.314 074	+ 0.313 128	+ 0.312 186	+ 0.311 252	+ 0.310 322	+ 0.309 398	+ 0.308 479	+ 0.307 565	+ 0.306 656	+ 0.305 752	+ 0.304 852	105
110	+ 0.314 522	+ 0.313 570	+ 0.312 624	+ 0.311 683	+ 0.310 747	+ 0.309 817	+ 0.308 892	+ 0.307 973	+ 0.307 059	+ 0.306 150	+ 0.305 246	110
115	+ 0.314 915	+ 0.313 958	+ 0.313 006	+ 0.312 060	+ 0.311 119	+ 0.310 185	+ 0.309 255	+ 0.308 331	+ 0.307 412	+ 0.306 499	+ 0.305 590	115
120	+ 0.315 260	+ 0.314 298	+ 0.313 344	+ 0.312 391	+ 0.311 446	+ 0.310 508	+ 0.309 573	+ 0.308 645	+ 0.307 722	+ 0.306 805	+ 0.305 892	120
125	+ 0.315 560	+ 0.314 594	+ 0.313 634	+ 0.312 680	+ 0.311 731	+ 0.310 789	+ 0.309 851	+ 0.308 920	+ 0.307 993	+ 0.307 072	+ 0.306 156	125
130	+ 0.315 820	+ 0.314 852	+ 0.313 889	+ 0.312 932	+ 0.311 980	+ 0.311 034	+ 0.310 093	+ 0.309 159	+ 0.308 229	+ 0.307 306	+ 0.306 386	130
135	+ 0.316 047	+ 0.315 076	+ 0.314 110	+ 0.313 150	+ 0.312 195	+ 0.311 247	+ 0.310 303	+ 0.309 366	+ 0.308 434	+ 0.307 508	+ 0.306 586	135
140	+ 0.316 242	+ 0.315 268	+ 0.314 300	+ 0.313 338	+ 0.312 381	+ 0.311 430	+ 0.310 484	+ 0.309 545	+ 0.308 610	+ 0.307 682	+ 0.306 758	140
145	+ 0.316 407	+ 0.315 432	+ 0.314 462	+ 0.313 498	+ 0.312 539	+ 0.311 586	+ 0.310 638	+ 0.309 697	+ 0.308 760	+ 0.307 830	+ 0.306 905	145
150	+ 0.316 547	+ 0.315 570	+ 0.314 597	+ 0.313 631	+ 0.312 670	+ 0.311 716	+ 0.310 767	+ 0.309 825	+ 0.308 886	+ 0.307 954	+ 0.307 027	150
155	+ 0.316 661	+ 0.315 682	+ 0.314 709	+ 0.313 742	+ 0.312 781	+ 0.311 826	+ 0.310 875	+ 0.309 931	+ 0.308 991	+ 0.308 058	+ 0.307 129	155
160	+ 0.316 753	+ 0.315 773	+ 0.314 798	+ 0.313 830	+ 0.312 867	+ 0.311 910	+ 0.310 958	+ 0.310 013	+ 0.309 073	+ 0.308 140	+ 0.307 210	160
165	+ 0.316 823	+ 0.315 842	+ 0.314 867	+ 0.313 898	+ 0.312 934	+ 0.311 977	+ 0.311 025	+ 0.310 079	+ 0.309 138	+ 0.308 203	+ 0.307 273	165
170	+ 0.316 873	+ 0.315 891	+ 0.314 915	+ 0.313 945	+ 0.312 981	+ 0.312 023	+ 0.311 070	+ 0.310 124	+ 0.309 182	+ 0.308 247	+ 0.307 316	170
175	+ 0.316 902	+ 0.315 920	+ 0.314 944	+ 0.313 974	+ 0.313 009	+ 0.312 051	+ 0.311 098	+ 0.310 151	+ 0.309 209	+ 0.308 273	+ 0.307 342	175
180	+ 0.316 912	+ 0.315 930	+ 0.314 953	+ 0.313 983	+ 0.313 018	+ 0.312 060	+ 0.311 107	+ 0.310 160	+ 0.309 218	+ 0.308 282	+ 0.307 351	180
ψ / z	3.11	3.12	3.13	3.14	3.15	3.16	3.17	3.18	3.19	3.20	3.21	z \ ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	3.21	3.22	3.23	3.24	3.25	3.26	3.27	3.28	3.29	3.30	3.31	z / ψ
0°	+ 0.282 665	+ 0.282 174	+ 0.281 680	+ 0.281 181	+ 0.280 680	+ 0.280 174	+ 0.279 666	+ 0.279 155	+ 0.278 641	+ 0.278 124	+ 0.277 605	0°
5	+ 0.282 880	+ 0.282 385	+ 0.281 885	+ 0.281 383	+ 0.280 876	+ 0.280 367	+ 0.279 854	+ 0.279 339	+ 0.278 821	+ 0.278 301	+ 0.277 778	5
10	+ 0.283 508	+ 0.282 999	+ 0.282 486	+ 0.281 970	+ 0.281 451	+ 0.280 929	+ 0.280 405	+ 0.279 877	+ 0.279 348	+ 0.278 816	+ 0.278 282	10
15	+ 0.284 501	+ 0.283 970	+ 0.283 436	+ 0.282 899	+ 0.282 360	+ 0.281 819	+ 0.281 275	+ 0.280 729	+ 0.280 182	+ 0.279 632	+ 0.279 081	15
20	+ 0.285 786	+ 0.285 227	+ 0.284 666	+ 0.284 103	+ 0.283 539	+ 0.282 972	+ 0.282 404	+ 0.281 835	+ 0.281 264	+ 0.280 692	+ 0.280 119	20
25	+ 0.287 279	+ 0.286 689	+ 0.286 097	+ 0.285 505	+ 0.284 911	+ 0.284 316	+ 0.283 720	+ 0.283 124	+ 0.282 527	+ 0.281 930	+ 0.281 332	25
30	+ 0.288 898	+ 0.288 275	+ 0.287 651	+ 0.287 027	+ 0.286 403	+ 0.285 778	+ 0.285 153	+ 0.284 528	+ 0.283 903	+ 0.283 278	+ 0.282 654	30
35	+ 0.290 567	+ 0.289 910	+ 0.289 254	+ 0.288 599	+ 0.287 943	+ 0.287 287	+ 0.286 633	+ 0.285 980	+ 0.285 328	+ 0.284 674	+ 0.284 023	35
40	+ 0.292 223	+ 0.291 534	+ 0.290 847	+ 0.290 161	+ 0.289 476	+ 0.288 792	+ 0.288 109	+ 0.287 428	+ 0.286 748	+ 0.286 069	+ 0.285 391	40
45	+ 0.293 820	+ 0.293 102	+ 0.292 385	+ 0.291 671	+ 0.290 958	+ 0.290 247	+ 0.289 538	+ 0.288 830	+ 0.288 124	+ 0.287 420	+ 0.286 718	45
50	+ 0.295 325	+ 0.294 580	+ 0.293 838	+ 0.293 097	+ 0.292 359	+ 0.291 623	+ 0.290 889	+ 0.290 158	+ 0.289 429	+ 0.288 702	+ 0.287 978	50
55	+ 0.296 721	+ 0.295 952	+ 0.295 185	+ 0.294 422	+ 0.293 660	+ 0.292 902	+ 0.292 146	+ 0.291 394	+ 0.290 643	+ 0.289 896	+ 0.289 152	55
60	+ 0.297 998	+ 0.297 207	+ 0.296 420	+ 0.295 635	+ 0.294 854	+ 0.294 075	+ 0.293 300	+ 0.292 528	+ 0.291 760	+ 0.290 994	+ 0.290 231	60
65	+ 0.299 155	+ 0.298 345	+ 0.297 539	+ 0.296 736	+ 0.295 937	+ 0.295 141	+ 0.294 349	+ 0.293 560	+ 0.292 774	+ 0.291 993	+ 0.291 214	65
70	+ 0.300 194	+ 0.299 367	+ 0.298 545	+ 0.297 727	+ 0.296 912	+ 0.296 101	+ 0.295 294	+ 0.294 490	+ 0.293 690	+ 0.292 894	+ 0.292 102	70
75	+ 0.301 122	+ 0.300 282	+ 0.299 445	+ 0.298 613	+ 0.297 785	+ 0.296 961	+ 0.296 140	+ 0.295 324	+ 0.294 512	+ 0.293 703	+ 0.292 899	75
80	+ 0.301 948	+ 0.301 096	+ 0.300 247	+ 0.299 403	+ 0.298 563	+ 0.297 727	+ 0.296 895	+ 0.296 068	+ 0.295 245	+ 0.294 425	+ 0.293 610	80
85	+ 0.302 681	+ 0.301 818	+ 0.300 959	+ 0.300 104	+ 0.299 254	+ 0.298 408	+ 0.297 567	+ 0.296 730	+ 0.295 897	+ 0.295 068	+ 0.294 244	85
90	+ 0.303 330	+ 0.302 457	+ 0.301 589	+ 0.300 725	+ 0.299 866	+ 0.299 012	+ 0.298 162	+ 0.297 317	+ 0.296 476	+ 0.295 639	+ 0.294 807	90
95	+ 0.303 903	+ 0.303 022	+ 0.302 146	+ 0.301 275	+ 0.300 408	+ 0.299 546	+ 0.298 689	+ 0.297 836	+ 0.296 989	+ 0.296 145	+ 0.295 306	95
100	+ 0.304 408	+ 0.303 521	+ 0.302 638	+ 0.301 760	+ 0.300 887	+ 0.300 018	+ 0.299 155	+ 0.298 296	+ 0.297 442	+ 0.296 592	+ 0.295 747	100
105	+ 0.304 852	+ 0.303 960	+ 0.303 072	+ 0.302 188	+ 0.301 311	+ 0.300 434	+ 0.299 565	+ 0.298 701	+ 0.297 842	+ 0.296 987	+ 0.296 136	105
110	+ 0.305 246	+ 0.304 347	+ 0.303 453	+ 0.302 564	+ 0.301 681	+ 0.300 802	+ 0.299 928	+ 0.299 059	+ 0.298 195	+ 0.297 336	+ 0.296 481	110
115	+ 0.305 590	+ 0.304 687	+ 0.303 790	+ 0.302 896	+ 0.302 008	+ 0.301 125	+ 0.300 247	+ 0.299 374	+ 0.298 506	+ 0.297 642	+ 0.296 784	115
120	+ 0.305 892	+ 0.304 986	+ 0.304 083	+ 0.303 187	+ 0.302 294	+ 0.301 408	+ 0.300 526	+ 0.299 650	+ 0.298 777	+ 0.297 911	+ 0.297 050	120
125	+ 0.306 156	+ 0.305 246	+ 0.304 340	+ 0.303 441	+ 0.302 545	+ 0.301 656	+ 0.300 771	+ 0.299 892	+ 0.299 016	+ 0.298 147	+ 0.297 283	125
130	+ 0.306 386	+ 0.305 473	+ 0.304 565	+ 0.303 663	+ 0.302 765	+ 0.301 873	+ 0.300 985	+ 0.300 103	+ 0.299 225	+ 0.298 353	+ 0.297 485	130
135	+ 0.306 586	+ 0.305 670	+ 0.304 759	+ 0.303 855	+ 0.302 954	+ 0.302 060	+ 0.301 171	+ 0.300 286	+ 0.299 407	+ 0.298 532	+ 0.297 663	135
140	+ 0.306 758	+ 0.305 840	+ 0.304 927	+ 0.304 021	+ 0.303 118	+ 0.302 222	+ 0.301 330	+ 0.300 444	+ 0.299 562	+ 0.298 686	+ 0.297 814	140
145	+ 0.306 905	+ 0.305 986	+ 0.305 071	+ 0.304 163	+ 0.303 259	+ 0.302 361	+ 0.301 467	+ 0.300 579	+ 0.299 695	+ 0.298 818	+ 0.297 944	145
150	+ 0.307 027	+ 0.306 107	+ 0.305 191	+ 0.304 281	+ 0.303 375	+ 0.302 476	+ 0.301 581	+ 0.300 692	+ 0.299 807	+ 0.298 928	+ 0.298 053	150
155	+ 0.307 129	+ 0.306 208	+ 0.305 290	+ 0.304 379	+ 0.303 472	+ 0.302 572	+ 0.301 675	+ 0.300 785	+ 0.299 900	+ 0.299 020	+ 0.298 144	155
160	+ 0.307 210	+ 0.306 287	+ 0.305 369	+ 0.304 457	+ 0.303 549	+ 0.302 648	+ 0.301 751	+ 0.300 860	+ 0.299 973	+ 0.299 093	+ 0.298 216	160
165	+ 0.307 273	+ 0.306 349	+ 0.305 430	+ 0.304 517	+ 0.303 608	+ 0.302 706	+ 0.301 808	+ 0.300 917	+ 0.300 029	+ 0.299 148	+ 0.298 271	165
170	+ 0.307 316	+ 0.306 392	+ 0.305 472	+ 0.304 559	+ 0.303 650	+ 0.302 747	+ 0.301 850	+ 0.300 957	+ 0.300 070	+ 0.299 188	+ 0.298 310	170
175	+ 0.307 342	+ 0.306 417	+ 0.305 497	+ 0.304 584	+ 0.303 675	+ 0.302 772	+ 0.301 873	+ 0.300 981	+ 0.300 093	+ 0.299 211	+ 0.298 333	175
180	+ 0.307 351	+ 0.306 426	+ 0.305 506	+ 0.304 593	+ 0.303 684	+ 0.302 782	+ 0.301 882	+ 0.300 989	+ 0.300 101	+ 0.299 219	+ 0.298 341	180
ψ / z	3.21	3.22	3.23	3.24	3.25	3.26	3.27	3.28	3.29	3.30	3.31	z \ ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	3.31	3.32	3.33	3.34	3.35	3.36	3.37	3.38	3.39	3.40	3.41	z / ψ
0°	+ 0.277 605	+ 0.277 084	+ 0.276 560	+ 0.276 034	+ 0.275 506	+ 0.274 976	+ 0.274 445	+ 0.273 912	+ 0.273 378	+ 0.272 842	+ 0.272 305	0°
5	+ 0.277 778	+ 0.277 253	+ 0.276 725	+ 0.276 196	+ 0.275 665	+ 0.275 132	+ 0.274 597	+ 0.274 061	+ 0.273 523	+ 0.272 984	+ 0.272 444	5
10	+ 0.278 282	+ 0.277 746	+ 0.277 208	+ 0.276 669	+ 0.276 128	+ 0.275 585	+ 0.275 041	+ 0.274 496	+ 0.273 949	+ 0.273 402	+ 0.272 853	10
15	+ 0.279 081	+ 0.278 528	+ 0.277 974	+ 0.277 419	+ 0.276 862	+ 0.276 304	+ 0.275 746	+ 0.275 186	+ 0.274 625	+ 0.274 062	+ 0.273 502	15
20	+ 0.280 119	+ 0.279 545	+ 0.278 970	+ 0.278 394	+ 0.277 818	+ 0.277 241	+ 0.276 663	+ 0.276 085	+ 0.275 506	+ 0.274 927	+ 0.274 348	20
25	+ 0.281 332	+ 0.280 733	+ 0.280 134	+ 0.279 535	+ 0.278 936	+ 0.278 337	+ 0.277 737	+ 0.277 138	+ 0.276 539	+ 0.275 940	+ 0.275 341	25
30	+ 0.282 654	+ 0.282 029	+ 0.281 405	+ 0.280 781	+ 0.280 157	+ 0.279 534	+ 0.278 912	+ 0.278 290	+ 0.277 668	+ 0.277 048	+ 0.276 428	30
35	+ 0.284 023	+ 0.283 372	+ 0.282 722	+ 0.282 072	+ 0.281 425	+ 0.280 779	+ 0.280 133	+ 0.279 489	+ 0.278 845	+ 0.278 200	+ 0.277 560	35
40	+ 0.285 391	+ 0.284 715	+ 0.284 040	+ 0.283 367	+ 0.282 696	+ 0.282 025	+ 0.281 357	+ 0.280 690	+ 0.280 024	+ 0.279 360	+ 0.278 698	40
45	+ 0.286 718	+ 0.286 018	+ 0.285 320	+ 0.284 624	+ 0.283 930	+ 0.283 238	+ 0.282 548	+ 0.281 860	+ 0.281 174	+ 0.280 490	+ 0.279 808	45
50	+ 0.287 978	+ 0.287 256	+ 0.286 536	+ 0.285 819	+ 0.285 104	+ 0.284 392	+ 0.283 682	+ 0.282 974	+ 0.282 269	+ 0.281 567	+ 0.280 867	50
55	+ 0.289 152	+ 0.288 410	+ 0.287 671	+ 0.286 935	+ 0.286 201	+ 0.285 471	+ 0.284 743	+ 0.284 018	+ 0.283 295	+ 0.282 576	+ 0.281 859	55
60	+ 0.290 231	+ 0.289 472	+ 0.288 716	+ 0.287 963	+ 0.287 212	+ 0.286 466	+ 0.285 722	+ 0.284 981	+ 0.284 243	+ 0.283 509	+ 0.282 777	60
65	+ 0.291 214	+ 0.290 439	+ 0.289 668	+ 0.288 900	+ 0.288 135	+ 0.287 373	+ 0.286 615	+ 0.285 859	+ 0.285 109	+ 0.284 361	+ 0.283 617	65
70	+ 0.292 102	+ 0.291 313	+ 0.290 528	+ 0.289 747	+ 0.288 969	+ 0.288 195	+ 0.287 424	+ 0.286 657	+ 0.285 894	+ 0.285 134	+ 0.284 378	70
75	+ 0.292 899	+ 0.292 098	+ 0.291 301	+ 0.290 509	+ 0.289 719	+ 0.288 934	+ 0.288 152	+ 0.287 375	+ 0.286 601	+ 0.285 831	+ 0.285 064	75
80	+ 0.293 610	+ 0.292 799	+ 0.291 993	+ 0.291 190	+ 0.290 391	+ 0.289 596	+ 0.288 805	+ 0.288 018	+ 0.287 235	+ 0.286 455	+ 0.285 680	80
85	+ 0.294 244	+ 0.293 424	+ 0.292 608	+ 0.291 797	+ 0.290 989	+ 0.290 186	+ 0.289 387	+ 0.288 592	+ 0.287 801	+ 0.287 013	+ 0.286 230	85
90	+ 0.294 807	+ 0.293 979	+ 0.293 156	+ 0.292 337	+ 0.291 522	+ 0.290 711	+ 0.289 905	+ 0.289 102	+ 0.288 304	+ 0.287 510	+ 0.286 720	90
95	+ 0.295 306	+ 0.294 471	+ 0.293 641	+ 0.292 816	+ 0.291 994	+ 0.291 177	+ 0.290 364	+ 0.289 556	+ 0.288 752	+ 0.287 952	+ 0.287 156	95
100	+ 0.295 747	+ 0.294 907	+ 0.294 071	+ 0.293 239	+ 0.292 412	+ 0.291 590	+ 0.290 772	+ 0.289 958	+ 0.289 148	+ 0.288 343	+ 0.287 542	100
105	+ 0.296 136	+ 0.295 291	+ 0.294 450	+ 0.293 617	+ 0.292 782	+ 0.291 956	+ 0.291 132	+ 0.290 315	+ 0.289 500	+ 0.288 690	+ 0.287 884	105
110	+ 0.296 481	+ 0.295 631	+ 0.294 786	+ 0.293 945	+ 0.293 109	+ 0.292 277	+ 0.291 450	+ 0.290 628	+ 0.289 810	+ 0.288 996	+ 0.288 187	110
115	+ 0.296 784	+ 0.295 930	+ 0.295 081	+ 0.294 237	+ 0.293 397	+ 0.292 562	+ 0.291 731	+ 0.290 905	+ 0.290 084	+ 0.289 267	+ 0.288 454	115
120	+ 0.297 050	+ 0.296 193	+ 0.295 341	+ 0.294 493	+ 0.293 650	+ 0.292 812	+ 0.291 978	+ 0.291 149	+ 0.290 325	+ 0.289 505	+ 0.288 690	120
125	+ 0.297 283	+ 0.296 423	+ 0.295 568	+ 0.294 718	+ 0.293 872	+ 0.293 031	+ 0.292 195	+ 0.291 363	+ 0.290 536	+ 0.289 714	+ 0.288 896	125
130	+ 0.297 485	+ 0.296 624	+ 0.295 765	+ 0.294 913	+ 0.294 065	+ 0.293 222	+ 0.292 384	+ 0.291 550	+ 0.290 721	+ 0.289 896	+ 0.289 076	130
135	+ 0.297 663	+ 0.296 798	+ 0.295 939	+ 0.295 084	+ 0.294 234	+ 0.293 389	+ 0.292 548	+ 0.291 712	+ 0.290 881	+ 0.290 054	+ 0.289 232	135
140	+ 0.297 814	+ 0.296 948	+ 0.296 086	+ 0.295 230	+ 0.294 379	+ 0.293 532	+ 0.292 690	+ 0.291 852	+ 0.291 019	+ 0.290 191	+ 0.289 367	140
145	+ 0.297 944	+ 0.297 077	+ 0.296 214	+ 0.295 357	+ 0.294 504	+ 0.293 656	+ 0.292 812	+ 0.291 973	+ 0.291 138	+ 0.290 308	+ 0.289 483	145
150	+ 0.298 053	+ 0.297 184	+ 0.296 319	+ 0.295 461	+ 0.294 606	+ 0.293 757	+ 0.292 912	+ 0.292 072	+ 0.291 237	+ 0.290 406	+ 0.289 580	150
155	+ 0.298 144	+ 0.297 274	+ 0.296 408	+ 0.295 548	+ 0.294 693	+ 0.293 842	+ 0.292 996	+ 0.292 155	+ 0.291 319	+ 0.290 487	+ 0.289 660	155
160	+ 0.298 216	+ 0.297 345	+ 0.296 479	+ 0.295 618	+ 0.294 762	+ 0.293 910	+ 0.293 064	+ 0.292 222	+ 0.291 385	+ 0.290 552	+ 0.289 724	160
165	+ 0.298 271	+ 0.297 400	+ 0.296 532	+ 0.295 671	+ 0.294 813	+ 0.293 962	+ 0.293 115	+ 0.292 273	+ 0.291 435	+ 0.290 602	+ 0.289 774	165
170	+ 0.298 310	+ 0.297 438	+ 0.296 570	+ 0.295 709	+ 0.294 852	+ 0.293 999	+ 0.293 152	+ 0.292 309	+ 0.291 471	+ 0.290 637	+ 0.289 808	170
175	+ 0.298 333	+ 0.297 461	+ 0.296 593	+ 0.295 731	+ 0.294 874	+ 0.294 021	+ 0.293 173	+ 0.292 330	+ 0.291 492	+ 0.290 658	+ 0.289 828	175
180	+ 0.298 341	+ 0.297 469	+ 0.296 600	+ 0.295 738	+ 0.294 881	+ 0.294 028	+ 0.293 180	+ 0.292 337	+ 0.291 499	+ 0.290 665	+ 0.289 838	180
ψ / z	3.31	3.32	3.33	3.34	3.35	3.36	3.37	3.38	3.39	3.40	3.41	ψ \ z

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	3.41	3.42	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50	3.51	z / ψ
0°	+ 0.272 305	+ 0.271 766	+ 0.271 227	+ 0.270 686	+ 0.270 145	+ 0.269 603	+ 0.269 060	+ 0.268 517	+ 0.267 973	+ 0.267 429	+ 0.266 884	0°
5	+ 0.272 444	+ 0.271 903	+ 0.271 361	+ 0.270 818	+ 0.270 274	+ 0.269 729	+ 0.269 184	+ 0.268 638	+ 0.268 092	+ 0.267 545	+ 0.266 998	5
10	+ 0.272 853	+ 0.272 303	+ 0.271 753	+ 0.271 202	+ 0.270 651	+ 0.270 098	+ 0.269 546	+ 0.268 993	+ 0.268 439	+ 0.267 886	+ 0.267 332	10
15	+ 0.273 502	+ 0.272 939	+ 0.272 376	+ 0.271 813	+ 0.271 249	+ 0.270 685	+ 0.270 121	+ 0.269 556	+ 0.268 992	+ 0.268 427	+ 0.267 863	15
20	+ 0.274 348	+ 0.273 769	+ 0.273 189	+ 0.272 610	+ 0.272 031	+ 0.271 451	+ 0.270 872	+ 0.270 293	+ 0.269 715	+ 0.269 136	+ 0.268 558	20
25	+ 0.275 341	+ 0.274 742	+ 0.274 144	+ 0.273 546	+ 0.272 949	+ 0.272 352	+ 0.271 756	+ 0.271 160	+ 0.270 565	+ 0.269 971	+ 0.269 378	25
30	+ 0.276 428	+ 0.275 809	+ 0.275 190	+ 0.274 573	+ 0.273 956	+ 0.273 340	+ 0.272 726	+ 0.272 112	+ 0.271 499	+ 0.270 888	+ 0.270 277	30
35	+ 0.277 560	+ 0.276 919	+ 0.276 282	+ 0.275 644	+ 0.275 007	+ 0.274 374	+ 0.273 738	+ 0.273 107	+ 0.272 478	+ 0.271 848	+ 0.271 220	35
40	+ 0.278 698	+ 0.278 038	+ 0.277 379	+ 0.276 722	+ 0.276 066	+ 0.275 413	+ 0.274 761	+ 0.274 111	+ 0.273 463	+ 0.272 817	+ 0.272 172	40
45	+ 0.279 808	+ 0.279 128	+ 0.278 450	+ 0.277 775	+ 0.277 101	+ 0.276 430	+ 0.275 761	+ 0.275 094	+ 0.274 429	+ 0.273 766	+ 0.273 106	45
50	+ 0.280 867	+ 0.280 169	+ 0.279 474	+ 0.278 781	+ 0.278 091	+ 0.277 403	+ 0.276 718	+ 0.276 035	+ 0.275 354	+ 0.274 676	+ 0.274 001	50
55	+ 0.281 859	+ 0.281 145	+ 0.280 434	+ 0.279 726	+ 0.279 020	+ 0.278 317	+ 0.277 617	+ 0.276 920	+ 0.276 225	+ 0.275 534	+ 0.274 844	55
60	+ 0.282 777	+ 0.282 049	+ 0.281 323	+ 0.280 601	+ 0.279 881	+ 0.279 165	+ 0.278 452	+ 0.277 741	+ 0.277 034	+ 0.276 329	+ 0.275 628	60
65	+ 0.283 617	+ 0.282 875	+ 0.282 137	+ 0.281 402	+ 0.280 670	+ 0.279 942	+ 0.279 217	+ 0.278 495	+ 0.277 776	+ 0.277 060	+ 0.276 348	65
70	+ 0.284 378	+ 0.283 625	+ 0.282 876	+ 0.282 130	+ 0.281 387	+ 0.280 648	+ 0.279 913	+ 0.279 181	+ 0.278 452	+ 0.277 726	+ 0.277 004	70
75	+ 0.285 064	+ 0.284 302	+ 0.283 543	+ 0.282 787	+ 0.282 035	+ 0.281 286	+ 0.280 542	+ 0.279 801	+ 0.279 063	+ 0.278 328	+ 0.277 598	75
80	+ 0.285 680	+ 0.284 908	+ 0.284 141	+ 0.283 377	+ 0.282 616	+ 0.281 860	+ 0.281 107	+ 0.280 358	+ 0.279 612	+ 0.278 871	+ 0.278 132	80
85	+ 0.286 230	+ 0.285 451	+ 0.284 676	+ 0.283 904	+ 0.283 137	+ 0.282 373	+ 0.281 613	+ 0.280 857	+ 0.280 105	+ 0.279 356	+ 0.278 612	85
90	+ 0.286 720	+ 0.285 934	+ 0.285 153	+ 0.284 375	+ 0.283 601	+ 0.282 831	+ 0.282 065	+ 0.281 303	+ 0.280 545	+ 0.279 790	+ 0.279 040	90
95	+ 0.287 156	+ 0.286 364	+ 0.285 577	+ 0.284 793	+ 0.284 014	+ 0.283 239	+ 0.282 467	+ 0.281 700	+ 0.280 937	+ 0.280 177	+ 0.279 422	95
100	+ 0.287 542	+ 0.286 746	+ 0.285 953	+ 0.285 165	+ 0.284 381	+ 0.283 601	+ 0.282 825	+ 0.282 053	+ 0.281 285	+ 0.280 521	+ 0.279 761	100
105	+ 0.287 884	+ 0.287 084	+ 0.286 287	+ 0.285 493	+ 0.284 706	+ 0.283 923	+ 0.283 142	+ 0.282 366	+ 0.281 594	+ 0.280 826	+ 0.280 063	105
110	+ 0.288 187	+ 0.287 382	+ 0.286 582	+ 0.285 786	+ 0.284 994	+ 0.284 206	+ 0.283 423	+ 0.282 643	+ 0.281 868	+ 0.281 097	+ 0.280 330	110
115	+ 0.288 454	+ 0.287 646	+ 0.286 844	+ 0.286 043	+ 0.285 248	+ 0.284 457	+ 0.283 670	+ 0.282 888	+ 0.282 110	+ 0.281 336	+ 0.280 566	115
120	+ 0.288 690	+ 0.287 878	+ 0.287 072	+ 0.286 270	+ 0.285 472	+ 0.284 678	+ 0.283 889	+ 0.283 104	+ 0.282 323	+ 0.281 547	+ 0.280 774	120
125	+ 0.288 896	+ 0.288 082	+ 0.287 273	+ 0.286 468	+ 0.285 668	+ 0.284 872	+ 0.284 081	+ 0.283 294	+ 0.282 511	+ 0.281 732	+ 0.280 957	125
130	+ 0.289 076	+ 0.288 260	+ 0.287 449	+ 0.286 642	+ 0.285 840	+ 0.285 042	+ 0.284 248	+ 0.283 459	+ 0.282 674	+ 0.281 894	+ 0.281 117	130
135	+ 0.289 232	+ 0.288 414	+ 0.287 601	+ 0.286 793	+ 0.285 989	+ 0.285 190	+ 0.284 394	+ 0.283 603	+ 0.282 815	+ 0.282 034	+ 0.281 256	135
140	+ 0.289 367	+ 0.288 548	+ 0.287 734	+ 0.286 924	+ 0.286 118	+ 0.285 317	+ 0.284 520	+ 0.283 728	+ 0.282 941	+ 0.282 158	+ 0.281 377	140
145	+ 0.289 483	+ 0.288 662	+ 0.287 846	+ 0.287 035	+ 0.286 227	+ 0.285 424	+ 0.284 627	+ 0.283 834	+ 0.283 045	+ 0.282 260	+ 0.281 479	145
150	+ 0.289 580	+ 0.288 759	+ 0.287 942	+ 0.287 129	+ 0.286 321	+ 0.285 518	+ 0.284 719	+ 0.283 924	+ 0.283 133	+ 0.282 347	+ 0.281 565	150
155	+ 0.289 660	+ 0.288 838	+ 0.288 020	+ 0.287 207	+ 0.286 398	+ 0.285 594	+ 0.284 794	+ 0.283 998	+ 0.283 207	+ 0.282 420	+ 0.281 637	155
160	+ 0.289 724	+ 0.288 901	+ 0.288 083	+ 0.287 269	+ 0.286 459	+ 0.285 654	+ 0.284 854	+ 0.284 058	+ 0.283 266	+ 0.282 478	+ 0.281 695	160
165	+ 0.289 774	+ 0.288 950	+ 0.288 131	+ 0.287 316	+ 0.286 506	+ 0.285 700	+ 0.284 899	+ 0.284 102	+ 0.283 310	+ 0.282 522	+ 0.281 739	165
170	+ 0.289 808	+ 0.288 984	+ 0.288 165	+ 0.287 350	+ 0.286 539	+ 0.285 733	+ 0.284 932	+ 0.284 135	+ 0.283 342	+ 0.282 554	+ 0.281 770	170
175	+ 0.289 828	+ 0.289 004	+ 0.288 184	+ 0.287 369	+ 0.286 558	+ 0.285 752	+ 0.284 951	+ 0.284 154	+ 0.283 361	+ 0.282 573	+ 0.281 788	175
180	+ 0.289 838	+ 0.289 012	+ 0.288 192	+ 0.287 376	+ 0.286 565	+ 0.285 759	+ 0.284 957	+ 0.284 160	+ 0.283 367	+ 0.282 578	+ 0.281 794	180
ψ / z	3.41	3.42	3.43	3.44	3.45	3.46	3.47	3.48	3.49	3.50	3.51	z \ ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $\frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	3.71	3.72	3.73	3.74	3.75	3.76	3.77	3.78	3.79	3.80	3.81	z / ψ
0	0.255 999	0.255 459	0.254 920	0.254 382	0.253 844	0.253 308	0.252 772	0.252 237	0.251 703	0.251 170	0.250 638	0°
5	0.256 076	0.255 535	0.254 995	0.254 455	0.253 916	0.253 379	0.252 842	0.252 305	0.251 770	0.251 236	0.250 703	5
10	0.256 302	0.255 759	0.255 215	0.254 671	0.254 129	0.253 587	0.253 046	0.252 506	0.251 968	0.251 430	0.250 893	10
15	0.256 669	0.256 117	0.255 566	0.255 017	0.254 468	0.253 920	0.253 373	0.252 828	0.252 284	0.251 740	0.251 198	15
20	0.257 148	0.256 588	0.256 029	0.255 471	0.254 914	0.254 359	0.253 804	0.253 251	0.252 700	0.252 149	0.251 600	20
25	0.257 716	0.257 146	0.256 577	0.256 010	0.255 444	0.254 880	0.254 316	0.253 755	0.253 195	0.252 636	0.252 079	25
30	0.258 347	0.257 766	0.257 187	0.256 609	0.256 033	0.255 458	0.254 885	0.254 314	0.253 744	0.253 176	0.252 610	30
35	0.259 010	0.258 420	0.257 830	0.257 241	0.256 655	0.256 070	0.255 488	0.254 906	0.254 327	0.253 749	0.253 174	35
40	0.259 690	0.259 087	0.258 486	0.257 887	0.257 290	0.256 695	0.256 102	0.255 511	0.254 922	0.254 336	0.253 751	40
45	0.260 361	0.259 748	0.259 136	0.258 527	0.257 920	0.257 315	0.256 712	0.256 112	0.255 514	0.254 918	0.254 324	45
50	0.261 011	0.260 387	0.259 768	0.259 147	0.258 531	0.257 916	0.257 305	0.256 695	0.256 088	0.255 484	0.254 881	50
55	0.261 629	0.260 996	0.260 365	0.259 738	0.259 112	0.258 489	0.257 869	0.257 251	0.256 636	0.256 023	0.255 413	55
60	0.262 208	0.261 566	0.260 928	0.260 291	0.259 658	0.259 027	0.258 399	0.257 774	0.257 151	0.256 531	0.255 914	60
65	0.262 744	0.262 095	0.261 447	0.260 806	0.260 165	0.259 527	0.258 892	0.258 260	0.257 630	0.257 004	0.256 379	65
70	0.263 237	0.262 581	0.261 928	0.261 278	0.260 631	0.259 987	0.259 345	0.258 707	0.258 071	0.257 439	0.256 809	70
75	0.263 686	0.263 024	0.262 365	0.261 709	0.261 056	0.260 407	0.259 759	0.259 116	0.258 474	0.257 837	0.257 202	75
80	0.264 093	0.263 426	0.262 761	0.262 100	0.261 442	0.260 787	0.260 136	0.259 487	0.258 841	0.258 198	0.257 558	80
85	0.264 460	0.263 788	0.263 119	0.262 454	0.261 791	0.261 132	0.260 476	0.259 823	0.259 172	0.258 525	0.257 882	85
90	0.264 790	0.264 114	0.263 441	0.262 771	0.262 105	0.261 442	0.260 782	0.260 125	0.259 471	0.258 820	0.258 173	90
95	0.265 080	0.264 406	0.263 730	0.263 057	0.262 387	0.261 720	0.261 056	0.260 396	0.259 739	0.259 085	0.258 434	95
100	0.265 351	0.264 668	0.263 988	0.263 312	0.262 639	0.261 969	0.261 302	0.260 639	0.259 979	0.259 322	0.258 669	100
105	0.265 588	0.264 900	0.264 218	0.263 540	0.262 862	0.262 190	0.261 522	0.260 857	0.260 194	0.259 534	0.258 877	105
110	0.265 796	0.265 108	0.264 423	0.263 742	0.263 064	0.262 389	0.261 717	0.261 049	0.260 384	0.259 723	0.259 065	110
115	0.265 982	0.265 292	0.264 605	0.263 922	0.263 241	0.262 565	0.261 891	0.261 221	0.260 554	0.259 891	0.259 230	115
120	0.266 147	0.265 455	0.264 765	0.264 080	0.263 398	0.262 720	0.262 044	0.261 373	0.260 704	0.260 040	0.259 377	120
125	0.266 292	0.265 598	0.264 907	0.264 220	0.263 536	0.262 857	0.262 180	0.261 507	0.260 837	0.260 171	0.259 507	125
130	0.266 419	0.265 724	0.265 032	0.264 344	0.263 659	0.262 978	0.262 299	0.261 625	0.260 953	0.260 286	0.259 621	130
135	0.266 530	0.265 834	0.265 140	0.264 451	0.263 765	0.263 083	0.262 403	0.261 728	0.261 055	0.260 387	0.259 721	135
140	0.266 627	0.265 930	0.265 235	0.264 544	0.263 857	0.263 174	0.262 493	0.261 817	0.261 143	0.260 474	0.259 807	140
145	0.266 708	0.266 010	0.265 315	0.264 624	0.263 936	0.263 252	0.262 570	0.261 893	0.261 218	0.260 548	0.259 880	145
150	0.266 778	0.266 079	0.265 383	0.264 691	0.264 002	0.263 317	0.262 635	0.261 957	0.261 282	0.260 611	0.259 943	150
155	0.266 836	0.266 136	0.265 440	0.264 747	0.264 057	0.263 372	0.262 689	0.262 011	0.261 335	0.260 664	0.259 995	155
160	0.266 882	0.266 182	0.265 484	0.264 791	0.264 101	0.263 416	0.262 732	0.262 053	0.261 377	0.260 706	0.260 036	160
165	0.266 916	0.266 216	0.265 518	0.264 825	0.264 136	0.263 448	0.262 767	0.262 086	0.261 410	0.260 738	0.260 068	165
170	0.266 942	0.266 241	0.265 543	0.264 850	0.264 159	0.263 473	0.262 789	0.262 110	0.261 433	0.260 760	0.260 090	170
175	0.266 955	0.266 256	0.265 558	0.264 864	0.264 173	0.263 487	0.262 803	0.262 123	0.261 446	0.260 774	0.260 104	175
180	0.266 962	0.266 261	0.265 563	0.264 869	0.264 178	0.263 491	0.262 807	0.262 128	0.261 451	0.260 778	0.260 108	180
ψ / z	3.71	3.72	3.73	3.74	3.75	3.76	3.77	3.78	3.79	3.80	3.81	z \ ψ

$\frac{1}{m}$ TABLE FOR INTERPOLATING $z = \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	3.81	3.82	3.83	3.84	3.85	3.86	3.87	3.88	3.89	3.90	3.91	z / ψ
0°	+ 0.250 638	+ 0.250 107	+ 0.249 577	+ 0.249 048	+ 0.248 520	+ 0.247 993	+ 0.247 467	+ 0.246 943	+ 0.246 419	+ 0.245 897	+ 0.245 376	0°
5	+ 0.250 703	+ 0.250 171	+ 0.249 639	+ 0.249 109	+ 0.248 580	+ 0.248 052	+ 0.247 526	+ 0.247 000	+ 0.246 476	+ 0.245 952	+ 0.245 430	5
10	+ 0.250 893	+ 0.250 358	+ 0.249 823	+ 0.249 290	+ 0.248 758	+ 0.248 227	+ 0.247 697	+ 0.247 169	+ 0.246 641	+ 0.246 115	+ 0.245 590	10
15	+ 0.251 198	+ 0.250 659	+ 0.250 118	+ 0.249 580	+ 0.249 043	+ 0.248 507	+ 0.247 972	+ 0.247 439	+ 0.246 907	+ 0.246 376	+ 0.245 847	15
20	+ 0.251 600	+ 0.251 053	+ 0.250 506	+ 0.249 961	+ 0.249 418	+ 0.248 876	+ 0.248 335	+ 0.247 795	+ 0.247 258	+ 0.246 721	+ 0.246 186	20
25	+ 0.252 079	+ 0.251 523	+ 0.250 968	+ 0.250 415	+ 0.249 864	+ 0.249 315	+ 0.248 767	+ 0.248 220	+ 0.247 675	+ 0.247 132	+ 0.246 590	25
30	+ 0.252 610	+ 0.252 046	+ 0.251 483	+ 0.250 922	+ 0.250 362	+ 0.249 804	+ 0.249 248	+ 0.248 694	+ 0.248 142	+ 0.247 591	+ 0.247 042	30
35	+ 0.253 174	+ 0.252 600	+ 0.252 028	+ 0.251 458	+ 0.250 890	+ 0.250 324	+ 0.249 761	+ 0.249 198	+ 0.248 638	+ 0.248 078	+ 0.247 522	35
40	+ 0.253 751	+ 0.253 168	+ 0.252 587	+ 0.252 009	+ 0.251 432	+ 0.250 857	+ 0.250 285	+ 0.249 714	+ 0.249 146	+ 0.248 579	+ 0.248 015	40
45	+ 0.254 324	+ 0.253 732	+ 0.253 143	+ 0.252 556	+ 0.251 971	+ 0.251 388	+ 0.250 808	+ 0.250 229	+ 0.249 653	+ 0.249 079	+ 0.248 507	45
50	+ 0.254 881	+ 0.254 281	+ 0.253 684	+ 0.253 089	+ 0.252 496	+ 0.251 905	+ 0.251 317	+ 0.250 731	+ 0.250 147	+ 0.249 566	+ 0.248 987	50
55	+ 0.255 413	+ 0.254 806	+ 0.254 200	+ 0.253 598	+ 0.252 998	+ 0.252 400	+ 0.251 804	+ 0.251 212	+ 0.250 621	+ 0.250 033	+ 0.249 447	55
60	+ 0.255 914	+ 0.255 299	+ 0.254 687	+ 0.254 077	+ 0.253 470	+ 0.252 866	+ 0.252 264	+ 0.251 665	+ 0.251 068	+ 0.250 474	+ 0.249 882	60
65	+ 0.256 379	+ 0.255 759	+ 0.255 140	+ 0.254 524	+ 0.253 911	+ 0.253 301	+ 0.252 692	+ 0.252 087	+ 0.251 485	+ 0.250 885	+ 0.250 288	65
70	+ 0.256 809	+ 0.256 182	+ 0.255 558	+ 0.254 936	+ 0.254 317	+ 0.253 702	+ 0.253 088	+ 0.252 478	+ 0.251 870	+ 0.251 265	+ 0.250 663	70
75	+ 0.257 202	+ 0.256 569	+ 0.255 940	+ 0.255 314	+ 0.254 690	+ 0.254 069	+ 0.253 451	+ 0.252 836	+ 0.252 224	+ 0.251 614	+ 0.251 007	75
80	+ 0.257 558	+ 0.256 922	+ 0.256 288	+ 0.255 657	+ 0.255 029	+ 0.254 404	+ 0.253 782	+ 0.253 162	+ 0.252 546	+ 0.251 932	+ 0.251 321	80
85	+ 0.257 882	+ 0.257 241	+ 0.256 603	+ 0.255 968	+ 0.255 336	+ 0.254 707	+ 0.254 081	+ 0.253 458	+ 0.252 838	+ 0.252 221	+ 0.251 606	85
90	+ 0.258 173	+ 0.257 528	+ 0.256 887	+ 0.256 249	+ 0.255 613	+ 0.254 981	+ 0.254 352	+ 0.253 725	+ 0.253 102	+ 0.252 481	+ 0.251 864	90
95	+ 0.258 434	+ 0.257 787	+ 0.257 142	+ 0.256 501	+ 0.255 863	+ 0.255 227	+ 0.254 595	+ 0.253 966	+ 0.253 340	+ 0.252 716	+ 0.252 096	95
100	+ 0.258 669	+ 0.258 019	+ 0.257 371	+ 0.256 727	+ 0.256 086	+ 0.255 448	+ 0.254 813	+ 0.254 182	+ 0.253 553	+ 0.252 927	+ 0.252 304	100
105	+ 0.258 877	+ 0.258 225	+ 0.257 577	+ 0.256 930	+ 0.256 286	+ 0.255 645	+ 0.255 009	+ 0.254 374	+ 0.253 744	+ 0.253 116	+ 0.252 491	105
110	+ 0.259 065	+ 0.258 410	+ 0.257 758	+ 0.257 110	+ 0.256 464	+ 0.255 822	+ 0.255 183	+ 0.254 547	+ 0.253 914	+ 0.253 284	+ 0.252 657	110
115	+ 0.259 230	+ 0.258 574	+ 0.257 920	+ 0.257 270	+ 0.256 622	+ 0.255 979	+ 0.255 337	+ 0.254 700	+ 0.254 065	+ 0.253 434	+ 0.252 805	115
120	+ 0.259 377	+ 0.258 719	+ 0.258 063	+ 0.257 412	+ 0.256 763	+ 0.256 118	+ 0.255 475	+ 0.254 836	+ 0.254 200	+ 0.253 567	+ 0.252 936	120
125	+ 0.259 507	+ 0.258 848	+ 0.258 191	+ 0.257 538	+ 0.256 887	+ 0.256 241	+ 0.255 596	+ 0.254 956	+ 0.254 318	+ 0.253 684	+ 0.253 052	125
130	+ 0.259 621	+ 0.258 960	+ 0.258 302	+ 0.257 648	+ 0.256 996	+ 0.256 348	+ 0.255 703	+ 0.255 062	+ 0.254 423	+ 0.253 788	+ 0.253 155	130
135	+ 0.259 721	+ 0.259 059	+ 0.258 399	+ 0.257 744	+ 0.257 091	+ 0.256 443	+ 0.255 796	+ 0.255 154	+ 0.254 514	+ 0.253 879	+ 0.253 245	135
140	+ 0.259 807	+ 0.259 144	+ 0.258 484	+ 0.257 828	+ 0.257 174	+ 0.256 525	+ 0.255 877	+ 0.255 234	+ 0.254 593	+ 0.253 957	+ 0.253 322	140
145	+ 0.259 880	+ 0.259 217	+ 0.258 556	+ 0.257 899	+ 0.257 245	+ 0.256 595	+ 0.255 947	+ 0.255 303	+ 0.254 661	+ 0.254 024	+ 0.253 388	145
150	+ 0.259 943	+ 0.259 279	+ 0.258 617	+ 0.257 959	+ 0.257 304	+ 0.256 654	+ 0.256 005	+ 0.255 361	+ 0.254 719	+ 0.254 081	+ 0.253 445	150
155	+ 0.259 995	+ 0.259 330	+ 0.258 668	+ 0.258 010	+ 0.257 354	+ 0.256 703	+ 0.256 054	+ 0.255 409	+ 0.254 766	+ 0.254 128	+ 0.253 491	155
160	+ 0.260 036	+ 0.259 371	+ 0.258 708	+ 0.258 050	+ 0.257 394	+ 0.256 742	+ 0.256 092	+ 0.255 447	+ 0.254 804	+ 0.254 165	+ 0.253 528	160
165	+ 0.260 068	+ 0.259 403	+ 0.258 740	+ 0.258 081	+ 0.257 424	+ 0.256 772	+ 0.256 122	+ 0.255 476	+ 0.254 833	+ 0.254 194	+ 0.253 557	165
170	+ 0.260 090	+ 0.259 424	+ 0.258 761	+ 0.258 102	+ 0.257 445	+ 0.256 793	+ 0.256 143	+ 0.255 498	+ 0.254 854	+ 0.254 215	+ 0.253 577	170
175	+ 0.260 104	+ 0.259 438	+ 0.258 775	+ 0.258 116	+ 0.257 459	+ 0.256 807	+ 0.256 157	+ 0.255 511	+ 0.254 867	+ 0.254 227	+ 0.253 589	175
180	+ 0.260 108	+ 0.259 442	+ 0.258 779	+ 0.258 120	+ 0.257 463	+ 0.256 810	+ 0.256 160	+ 0.255 514	+ 0.254 870	+ 0.254 231	+ 0.253 593	180
ψ / z	3.81	3.82	3.83	3.84	3.85	3.86	3.87	3.88	3.89	3.90	3.91	z \ ψ

TABLE FOR INTERPOLATING $z - \frac{\rho}{R}$ WITH ARGUMENTS ψ AND $\frac{1}{m}$.

ψ \ z	3.91	3.92	3.93	3.94	3.95	3.96	3.97	3.98	3.99	4.00	z / ψ
0°	+ 0.245 376	+ 0.244 856	+ 0.244 337	+ 0.243 819	+ 0.243 303	+ 0.242 788	+ 0.242 274	+ 0.241 762	+ 0.241 251	+ 0.240 741	0°
5	+ 0.245 430	+ 0.244 909	+ 0.244 390	+ 0.243 871	+ 0.243 354	+ 0.242 838	+ 0.242 324	+ 0.241 810	+ 0.241 298	+ 0.240 788	5
10	+ 0.245 590	+ 0.245 067	+ 0.244 544	+ 0.244 023	+ 0.243 504	+ 0.242 985	+ 0.242 468	+ 0.241 953	+ 0.241 438	+ 0.240 925	10
15	+ 0.245 847	+ 0.245 319	+ 0.244 792	+ 0.244 267	+ 0.243 744	+ 0.243 220	+ 0.242 701	+ 0.242 179	+ 0.241 663	+ 0.241 148	15
20	+ 0.246 186	+ 0.245 653	+ 0.245 121	+ 0.244 590	+ 0.244 061	+ 0.243 533	+ 0.243 007	+ 0.242 483	+ 0.241 960	+ 0.241 438	20
25	+ 0.246 590	+ 0.246 050	+ 0.245 512	+ 0.244 975	+ 0.244 440	+ 0.243 906	+ 0.243 374	+ 0.242 844	+ 0.242 315	+ 0.241 788	25
30	+ 0.247 042	+ 0.246 494	+ 0.245 949	+ 0.245 405	+ 0.244 863	+ 0.244 322	+ 0.243 784	+ 0.243 247	+ 0.242 712	+ 0.242 179	30
35	+ 0.247 522	+ 0.246 966	+ 0.246 413	+ 0.245 864	+ 0.245 313	+ 0.244 767	+ 0.244 221	+ 0.243 676	+ 0.243 136	+ 0.242 596	35
40	+ 0.248 015	+ 0.247 452	+ 0.246 892	+ 0.246 334	+ 0.245 777	+ 0.245 223	+ 0.244 671	+ 0.244 120	+ 0.243 572	+ 0.243 026	40
45	+ 0.248 507	+ 0.247 937	+ 0.247 369	+ 0.246 804	+ 0.246 241	+ 0.245 680	+ 0.245 121	+ 0.244 564	+ 0.244 009	+ 0.243 456	45
50	+ 0.248 987	+ 0.248 410	+ 0.247 836	+ 0.247 264	+ 0.246 694	+ 0.246 126	+ 0.245 560	+ 0.244 997	+ 0.244 436	+ 0.243 877	50
55	+ 0.249 447	+ 0.248 864	+ 0.248 283	+ 0.247 704	+ 0.247 128	+ 0.246 554	+ 0.245 983	+ 0.245 414	+ 0.244 847	+ 0.244 282	55
60	+ 0.249 882	+ 0.249 292	+ 0.248 706	+ 0.248 121	+ 0.247 539	+ 0.246 960	+ 0.246 383	+ 0.245 808	+ 0.245 236	+ 0.244 666	60
65	+ 0.250 288	+ 0.249 693	+ 0.249 100	+ 0.248 511	+ 0.247 924	+ 0.247 339	+ 0.246 757	+ 0.246 178	+ 0.245 601	+ 0.245 025	65
70	+ 0.250 663	+ 0.250 063	+ 0.249 466	+ 0.248 871	+ 0.248 280	+ 0.247 690	+ 0.247 104	+ 0.246 520	+ 0.245 938	+ 0.245 359	70
75	+ 0.251 007	+ 0.250 403	+ 0.249 802	+ 0.249 202	+ 0.248 607	+ 0.248 013	+ 0.247 423	+ 0.246 835	+ 0.246 249	+ 0.245 666	75
80	+ 0.251 321	+ 0.250 713	+ 0.250 108	+ 0.249 505	+ 0.248 905	+ 0.248 308	+ 0.247 714	+ 0.247 122	+ 0.246 533	+ 0.245 947	80
85	+ 0.251 606	+ 0.250 994	+ 0.250 386	+ 0.249 780	+ 0.249 177	+ 0.248 576	+ 0.247 979	+ 0.247 384	+ 0.246 792	+ 0.246 202	85
90	+ 0.251 864	+ 0.251 249	+ 0.250 637	+ 0.250 028	+ 0.249 422	+ 0.248 819	+ 0.248 218	+ 0.247 621	+ 0.247 026	+ 0.246 433	90
95	+ 0.252 096	+ 0.251 478	+ 0.250 864	+ 0.250 252	+ 0.249 644	+ 0.249 038	+ 0.248 434	+ 0.247 834	+ 0.247 237	+ 0.246 642	95
100	+ 0.252 304	+ 0.251 684	+ 0.251 067	+ 0.250 453	+ 0.249 842	+ 0.249 234	+ 0.248 629	+ 0.248 026	+ 0.247 427	+ 0.246 830	100
105	+ 0.252 491	+ 0.251 869	+ 0.251 250	+ 0.250 634	+ 0.250 020	+ 0.249 410	+ 0.248 803	+ 0.248 199	+ 0.247 598	+ 0.246 999	105
110	+ 0.252 657	+ 0.252 033	+ 0.251 412	+ 0.250 795	+ 0.250 180	+ 0.249 567	+ 0.248 958	+ 0.248 352	+ 0.247 749	+ 0.247 152	110
115	+ 0.252 805	+ 0.252 180	+ 0.251 557	+ 0.250 938	+ 0.250 321	+ 0.249 708	+ 0.249 097	+ 0.248 491	+ 0.247 886	+ 0.247 285	115
120	+ 0.252 936	+ 0.252 310	+ 0.251 685	+ 0.251 065	+ 0.250 446	+ 0.249 832	+ 0.249 219	+ 0.248 611	+ 0.248 004	+ 0.247 402	120
125	+ 0.253 052	+ 0.252 425	+ 0.251 799	+ 0.251 178	+ 0.250 558	+ 0.249 943	+ 0.249 329	+ 0.248 720	+ 0.248 112	+ 0.247 508	125
130	+ 0.253 155	+ 0.252 526	+ 0.251 900	+ 0.251 277	+ 0.250 656	+ 0.250 040	+ 0.249 425	+ 0.248 814	+ 0.248 205	+ 0.247 600	130
135	+ 0.253 245	+ 0.252 615	+ 0.251 988	+ 0.251 365	+ 0.250 743	+ 0.250 126	+ 0.249 510	+ 0.248 899	+ 0.248 288	+ 0.247 682	135
140	+ 0.253 322	+ 0.252 691	+ 0.252 063	+ 0.251 439	+ 0.250 816	+ 0.250 198	+ 0.249 581	+ 0.248 969	+ 0.248 358	+ 0.247 752	140
145	+ 0.253 388	+ 0.252 757	+ 0.252 128	+ 0.251 503	+ 0.250 880	+ 0.250 261	+ 0.249 644	+ 0.249 031	+ 0.248 420	+ 0.247 813	145
150	+ 0.253 445	+ 0.252 813	+ 0.252 183	+ 0.251 557	+ 0.250 933	+ 0.250 314	+ 0.249 696	+ 0.249 083	+ 0.248 471	+ 0.247 864	150
155	+ 0.253 491	+ 0.252 859	+ 0.252 229	+ 0.251 603	+ 0.250 978	+ 0.250 358	+ 0.249 740	+ 0.249 127	+ 0.248 515	+ 0.247 907	155
160	+ 0.253 528	+ 0.252 896	+ 0.252 265	+ 0.251 639	+ 0.251 014	+ 0.250 394	+ 0.249 775	+ 0.249 161	+ 0.248 549	+ 0.247 941	160
165	+ 0.253 557	+ 0.252 924	+ 0.252 293	+ 0.251 667	+ 0.251 042	+ 0.250 421	+ 0.249 802	+ 0.249 188	+ 0.248 575	+ 0.247 967	165
170	+ 0.253 577	+ 0.252 944	+ 0.252 313	+ 0.251 686	+ 0.251 061	+ 0.250 441	+ 0.249 822	+ 0.249 207	+ 0.248 594	+ 0.247 985	170
175	+ 0.253 589	+ 0.252 956	+ 0.252 325	+ 0.251 698	+ 0.251 073	+ 0.250 452	+ 0.249 833	+ 0.249 218	+ 0.248 605	+ 0.247 996	175
180	+ 0.253 593	+ 0.252 960	+ 0.252 329	+ 0.251 702	+ 0.251 077	+ 0.250 456	+ 0.249 837	+ 0.249 222	+ 0.248 609	+ 0.248 000	180
ψ / z	3.91	3.92	3.93	3.94	3.95	3.96	3.97	3.98	3.99	4.00	z \ ψ

APPLICATIONS OF LEUSCHNER'S SHORT METHODS OF DETERMINING ORBITS.

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INTRODUCTION.

The purpose of this Part is to show in complete detail the computation necessary for determining orbits of all kinds. Among other things there will be shown (1) when parallax should be taken into consideration and how, in such cases, this is accomplished; (2) how to determine, in the course of the computation, whether a parabola is in the range of possible solutions; (3) the ease and rapidity with which transference may be made from a parabola to a general orbit; (4) the small amount of labor involved in making successive approximations in order to remove residuals completely; (5) how arbitrary methods of "cooking" to shorten the computations may sometimes be employed, due principally to the perfect perspicuity of LEUSCHNER'S methods, which enables the computer to see the meaning of every step in the computation, nothing being hidden in abstruse analytical developments.

In order to study the efficiency of LEUSCHNER'S methods as compared with others, all the available orbits of comets have heen collected and arranged in the order of the length of arc from which they were computed. In most of the cases the quantities tabulated are ΔT, $\Delta \omega$, etc., these being the differences between the elements of the particular orbit and the elements from the longest arc (considered to be the best). In a few cases these are not given, but the elements themselves are tabulated for reasons stated under each case.

The orbits taken are all those computed here beginning with that of Comet ***a***, 1905 (GIACOBINI), as this is the first case of an orbit computed here by the *Short* ***Method*** in its present form as set forth in Part 7.

The data for orbits computed in the Berkeley Astronomical Department are given in bolder type.

COMET *a* 1905 (GIACOBINI).

The orbit of reference in this case is that of WEDEMEYER who used a 33 day arc. His elements are given in *A. N.* **168** 243.

No.	ΔT	$\Delta\omega$	$\Delta\Omega$	Δi	$\Delta \log q$	Arc	Computer.	Reference.
I	+ 0.49	+ 1° 0′	+ 3 15′	− 2° 44′	− 0.0137	2 days	MORGAN, LAMSON	*A. N.* **168** 14
II	+ 0.89	+ 1 4	+ 1 15	− 1 23	− 0.0050	4 days	STRÖMGREN	*A. N.* **168** 14
III	− 0.07	− 0 5	+ 0 16	0 10	0.0010	4 days	GIACOBINI	*A. N.* **168** 27
IV	**+ 0.33**	**+ 0 24**	**+ 0 38**	**− 0 37**	**− 0.0025**	**4 days**	**Crawford, Maddrill**	*L. O. B.* **73**
V	+ 0.30	+ 0 20	+ 0 32	0 28	0.0021	5 days	MAUBANT	*A. N.* **168** 27
VI	+ 0.01	+ 0 1	+ 0 16	0 9	0.0011	6 days	LAMSON	*A. N.* **168** 63
VII	+ 0.09	+ 0 6	+ 0 21	0 16	0.0014	14 days	EBELL *et al.*	*A. N.* **168** 46

There are two other sets of elements from longer arcs (27 days and 31 days) which are omitted.

There are three orbits from 4 day arcs. Ours (IV) is considerably better than No. II, but not so good as No. III. The reason for the latter is to be found in the fact that GIACOBINI had equal intervals (each two days) while we had intervals of one day and three days, respectively. Our orbit, however, is practically the same as No. V in which a 5 day arc was used. Comparing Nos. IV and V we have more nearly similar conditions (unequal intervals), and it is seen that our orbit is of the same order of accuracy as that in which an arc of one more day is used.

COMET *b* 1905 (SCHAER).

No.	T (B. M. T.)	ω	Ω	i	$\log q$	φ	Arc	Computer.	Reference.
I	1905 Oct. 18.93	120° 6′	219° 50′	145° 32′	9.9997		2 days	MORGAN	*A. N.* **169** 413
II	" 27.49	135 39	223 45	138 55	0.0263		2 days	EBELL	*A. N.* **169** 415
III	" 25.71	132 35	222 55	14[illegible] 37	0.0219		5 days	EBELL	*A. N.* **170** 13
IV	" 25.77	132 40	222 56	140 36	0.0220		5 days	LAMSON	*A. N.* **170** 112
V	**" 25.06**	**132 7**	**223 4**	**140 27**	**0.0209**	**74° 41′**	**7 days**	**Crawford, Champreux**	*L. O. B.* **86**
VI	" 25.80	132 43	222 56	140 35	0.0221		25 days	WEDEMEYER	*A. N.* **170** 97

In this case we had no parabola. For this reason the differences in the elements are not given. The elements themselves are exhibited, however, in order that it may be noted that the longer the arc used the nearer are the various parabolic elements approaching our elliptical elements.

Orbit No. VI is based upon observations of November 18, December 1, and December 13, and does not represent two observations within this period (November 25 and November 28). The residual for the middle place is 8″.

Assuming that the orbit is ultimately a parabola it would seem as though the parabolas of EBELL and LAMSON from 5 day arcs were better than our orbit from a

7 day arc. Comparison with the last available observation, however, does not bear this out. Such an observation by E. SMITH at the Lick Observatory on December 29 is represented practically exactly by the elements of WEDEMEYER. Our orbit, however, is closer to this position than the parabola No. III of EBELL. The residuals being:

		CR. and CH.	EBELL.
O - C	$\Delta\alpha$	$+4^s$	-3^s
	$\Delta\delta$	$+0'.3$	$-2'.5$

COMET *c* 1905 (GIACOBINI).

The orbit of reference in this case is one by SCHOENBERG and BUSS. They used a 38 day arc. Their elements have been taken from the *Astronomischer Jahresbericht* for the year 1906.

No.	Δ T	$\Delta\,\omega$	$\Delta\,\Omega$	$\Delta\,i$	Δ log q	Arc.	Computer.	Reference.
I	$+6^d.16$	$-14°\ 44'.4$	$-1°\ 15'.3$	$-0°\ 43'.1$	+ 0.3670	2 days	MORGAN	*A. N.* 170 27
II	-9.26	+27 48.0	+ 2 23.1	+ 0 55.3	- 0.3930	2 days	GIACOBINI	*A. N.* 170 68
III	-3.50	+ 8 43.4	+ 0 56.1	+ 0 22.5	- 0.1530	3 days	THIELE	*A. N.* 170 82
IV	-2.36	+ 5 46.2	+ 0 29.3	+ 0 13.4	- 0.1040	3 days	Crawford Champreux	*L. O. B.* 87
V	+4.15	- 9 59.0	- 0 59.7	- 0 29.4	+ 0.2285	4 days	STRÖMGREN	*A. N.* 170 67
VI	+1.12	- 2 41.8	- 0 2.0	- 0 3.7	+ 0.0526	5 days	MAUBANT	*A. N.* 170 84
VII	-0.27	+ 0 50.0	+ 0 10.0	+ 0 2.5	- 0.0155	10 days	STRÖMGREN	*A. N.* 170 100
VIII	-0.06	+ 0 10.2	+ 0 4.1	+ 0 1.0	- 0.0034	11 days	Crawford	*L. O. B.* 88

Our orbit (IV) from a 3 day arc is considerably better than that of THIELE (III) from a like arc which in turn is better than that of STRÖMGREN (V) from a 4 day arc. Our orbit from an 11 day arc (VIII) is decidedly better than that of STRÖMGREN (VII) from a 10 day arc.

COMET *a* 1906 (BROOKS).

The orbit of reference in this case is that of EBELL from a 14 day arc. His elements are given in *A. N.* 170 323.

No.	Δ T	$\Delta\,\omega$	$\Delta\,\Omega$	$\Delta\,i$	Δ log q	Arc.	Computer.	Reference.
I	$+2^d.71$	$+3°\ 21'$	$+0°\ 55'$	$-0°\ 21'$	+ 0.0043	2 days	Crawford Champreux	*L. O. B.* 90
II	-9.38	-11 14	- 3 15	+ 1 28	- 0.0117	2 days	MAUBANT	*A. N.* 170 308
III	-2.31	- 2 53	+ 0 54	- 0 23	- 0.0032	2 days	LAMSON	*Pop. Ast.* 14 172
IV	+2.20	+ 2 43	+ 0 43	- 0 18	+ 0.0040	4 days	EBELL	*A. N.* 170 276
V	+0.20	+ 0 14	+ 0 3	- 0 1	+ 0.0004	5 days	LAMSON	*A. J.* 25 60
VI	+0.20	+ 0 15	+ 0 4	+ 0 1	+ 0.0003	6 days	MORGAN	*Pop. Ast.* 14 170
VII	-0.11	- 0 8	- 0 2.3	+ 0 1.6	- 0.00029	10 days	Crawford Champreux	*L. O. B.* 91
VIII	-0.14	- 0 11	- 0 3.2	+ 0 1.8	- 0.00035	12 days	MORGAN	*A. J.* 25 75

Our orbit (I), given above as from a 2 day arc was computed from observations actually covering *only 1.76 days*. It is considerably better than that of MAUBANT (II), and of the same order but not quite so good as that of LAMSON (III), both of which are based upon a 2 day arc. It is further of practically the same order as that of EBELL (IV), who used an arc of four days. Our orbit (VII) from a 10 day arc is a little better than that of MORGAN (VIII) from a 12 day arc.

COMET *b* 1906 (KOPFF).

The orbit of reference is that of EBELL from a 51 day arc. His elements are given in *A. N.* 171 111.

No.	ΔT	$\Delta\omega$	$\Delta\Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	$+67^d.5$	15'.0	− 12'.8	− 3° 6'	−0.404	2 days	Crawford Champreux	*L. O. B.* 92
II	+77.5	− 20 .3	− 13 .8	− 3 21	−0.487	3 days	EBELL	*A. N.* 170 376
III	+17.8	+ 1 .9	1 2	0 33	0.031	14 days	EBELL	*A. N.* 170 389
IV	+ 2.2	+ 0 .4	− 0 .06	− 0 2	0.0015	22 days	Crawford Champreux	*L. O. B.* 97

This comet had a very slow motion, and, as noted in *L. O. B.* 97 the computations indicated a wide range of possible solutions. Our orbit (I) from a 2 day arc is closer than that of EBELL (II) from a 3 day arc.

COMET *e* 1906 (KOPFF).

No.	T (B. M. T.)	ω	Ω	i	$\log q$	φ	Arc.	Computer.	Reference.
I	1907 Apr. 12.3	221° 38'	230° 2'	12° 44'	0.0485		2 days	Crawford	*L. O. B.* 99
II	1906 Dec 7.3	243 13	179 19	15 18	9.9143		2 days	MORGAN	*A. N.* 172 224
III	1906 May 14.69	353 32	279 26	9 46	0.1457		4 days	EBELL	*A. N.* 172 223
IV	1905 Oct. 16.05	292 25	297 33	17 28	0.0842		4 days	EBELL	*A. N.* 172 223
V	1906 May 3.09	19 33.7	263 48.7	8 42.1	0.22875	31° 18'.2	20 days	EBELL	*A. N.* 172 303
VI	1906 May 2.125	19 28.7	263 45.4	8 44.2	0.23011	31 21.6	22 days	Crawford Champreux	*L. O. B.* 100

This comet presents a peculiar case. As indicated in *L. O. B.* 99 a wide range of solutions was expected. The orbit was ultimately shown to be a short period ($6\frac{2}{3}$ years) ellipse, announcement of which was first made in *L. O. B.* 100. The geocentric motion at the time of discovery was very slow. With the exception of T and ω in which there is a great mutual uncertainty our elements (I) from a 2 day arc are closer than those of MORGAN (II) from a like arc.

Elements III and IV by EBELL are two solutions from the same observations.

COMET *a* 1907 (GIACOBINI).

The orbit of reference is that of TRINGALI who used a 31 day arc. His elements are given in *A. N.* 176 73.

No.	Δ T	$\Delta\omega$	$\Delta\Omega$	Δi	Δ log q	Arc.	Computer.	Reference.
I	$-$ 4.35 d	$-$ 2° 28′	$-$ 0° 30′	+ 0° 19′	+0.0003	2 days	EBELL	*A. N.* 174 143
II	+10.81	+ 5 55	+ 1 16	$-$ 0 48	+0.0008	2 days	LAMSON	*Pop. Ast.* 15 248
III	+ 6.52	+ 3 38	+ 0 45	$-$ 0 29	+0.0001	3 days	DUNCAN, WILLIAMS	*A. N.* 174 159
IV	+ 6.49	+ 3 48	+ 0 53	$-$ 0 34	+0.0002	3 days	Einarsson, Glancy	*L. O. B.* 111
V	+ 3.48	+ 1 57	+ 0 21	$-$ 0 14	0.0000	4 days	GIACOBINI	*A. N.* 174 173
VI	$-$ 2.82	$-$ 1 36	$-$ 0 19	+ 0 14	0.0000	5 days	BRÜCK	*C. R.* 144 613
VII	$-$ 6.89	$-$ 3 51	$-$ 0 44	+ 0 32	0.0000	7 days	EBELL	*A. N.* 174 207
VIII	+ 1.47	+ 0 50	+ 0 10	+ 0 7	+0.0001	8 days	LAMSON	*A. N.* 174 335
IX	$-$ 0.10	$-$ 0 4	$-$ 0 1	0 0	0.0000	28 days	Einarsson, Glancy, Joy	*L. O. B.* 113

The data here present a peculiar state of affairs. EBELL'S orbit (I) from a 2 day arc is very good and considerably better than his later orbit (VII) from a 7 day arc. Our orbit (IV) from a 3 day arc is practically the same as that of DUNCAN and WILLIAMS (III) from a similar arc. It may be remarked that our orbit could have been improved by removing a residual of 0′.2. As a second orbit was contemplated it was not considered advisable at the time to make the improvement to the preliminary orbit.

COMET *d* 1907 (DANIEL).

The orbit of reference is that of KRITZINGER from a 74 day arc. His elements are given in *A. N.* 176 15.

No.	Δ T	$\Delta\omega$	$\Delta\Omega$	Δi	Δ log q	Arc.	Computer.	Reference.
I	+ 1.97 d	+52° 23′	$-$ 0° 39′	+ 2° 43′	$-$0.4047	3 days	STRÖMGREN	A. N. 175 155
II	+ 0.18	+ 0 25	0 0	$-$ 0 11	+0.0052	6 days	Crawford, Einarsson, Glancy	*L. O. B.* 119
III	$-$ 5.43	+14 59	$-$ 0 22	+ 1 4	$-$0.1574	8 days	STRÖMGREN	*A. N.* 175 191
IV	$-$ 0.29	+ 0 32	0 0	+ 0 2	$-$0.0062	19 days	KRITZINGER	*A. N.* 175 259
V	$-$ 0.04	+ 0 7	+ 0 2	+ 0 1	$-$0.0016	27 days	DYBECK	*A. N.* 175 307

Our orbit (II) from a 6 day arc is considerably better than that of STRÖMGREN (III) from an 8 day arc and slightly better than that of KRITZINGER (IV) from a 19 day arc. No improvement is found until DYBECK'S orbit (V) from a 27 day arc. Another orbit, not given here, by MILLOSEVICH from a 67 day arc is practically the same as the orbit of reference.

COMET *c* 1907 (MELLISH).

The orbit of reference is one by KOBOLD who used a 42 day arc. His elements are given in *A. N.* 177 240.

No.	ΔT	$\Delta\omega$	$\Delta\Omega$	Δi	$\Delta \log q$	Arc	Computer	Reference.
I	+ 1.89	+ 2° 40′	− 0° 57′	+ 0° 45′	+0.0044	2 days	LAMSON	*A. N.* 176 127
II	1.77	2 24	+ 0 59	0 34	0.0020	2 days	Crawford Glancy Morgan	*L. O. B.* 121
III	− 2.28	− 3 22	+ 0 38	0 17	0.0058	2 days	WILSON	*Pop. Ast.* 15 570
IV	+ 0.65	+ 0 55	− 0 19	+ 0 14	+0.0013	3 days	EBELL	*A. N.* 176 147
V	− 0.52	− 0 44	+ 0 13	− 0 9	0.0011	6 days	LAMSON	*Pop. Ast.* 15 632
VI	− 0.107	− 0 10.0	+ 0 1.5	0 3.1	0.00051	15 days	Crawford Glancy	*L. O. B.* 124
VII	− 0.009	− 0 0.6	0 0.0	+ 0 0.3	0.00001	18 days	EBELL	*A. N.* 176 195

Of the three orbits from 2 day arcs ours (II) is better than those of LAMSON (I) and WILSON (III). EBELL'S (IV) from a 3 day arc is a decided improvement. Our orbit (VI) from a 15 day arc is not so close as that of EBELL (VII) from an 18 day arc.

COMET *c* 1908 (MOREHOUSE).

The orbit of reference is one by KOBOLD who used an 83 day arc. His elements are given in *A. N.* 179 273.

No.	ΔT	$\Delta\omega$	$\Delta\Omega$	Δi	$\Delta \log q$	Arc.	Computer.	Reference.
I	+ 1.51	− 2° 53′	1° 53′	0° 26′	+0.0112	2 days	KOBOLD	*A. N.* 179 15
II	− 10.92	+19 34	+12 49	+ 4 15	0.0921	2 days	Einarsson Meyer	*L. O. B.* 138
III	− 0.023	+ 0 6.7	+ 0 5.1	+ 0 0.6	− 0.00036	15 days	Einarsson Meyer	*L. O. B.* 139
IV	− 0.015	+ 0 14.8	+ 0 8.1	+ 0 0.8	− 0.00103	15 days	CHACON	*Mem. [illegible] Mex.* 25 292
V	+ 0.011	− 0 1.9	0 2.1	0 0.2	+ 0.00014	21 days	KOBOLD	*A. N.* 179 47
VI	− 0.078	− 0 15.2	− 0 6.6	+ 0 1.0	+ 0.00086	26 days	GABBA	*Lomb. Ist. Rend.* (2) 42

Our preliminary orbit (II) from a 2 day arc is decidedly in error. After it was issued an error was discovered in one of the observations upon which it was based. Our orbit (III) from a 15 day arc is better than that of CHACÓN (IV) from a like arc, and also better than that of GABBA (VI) from a 26 day arc. It is not so good as KOBOLD'S (V) from a 21 day arc.

COMET *a* 1909 (DANIEL).

No.	T (B. M. T.)	ω	Ω	i	$\log q$	Arc	Computer.	Reference.
I	1909 June 5.35	5° 4′	305 21′	51° 32	9.9252	2 days	KOBOLD	*A. N.* 181 211
II	1909 June 5.21	4 59	306 19	52 26	9.9275	5 days	Crawford	*L. O. B.* 159
III	1909 June 5.16	4 56	306 40	52 39	9.9281	5 days	BOSS	*A. N.* 181 363
IV	1909 June 5.30	5 1	305 38	52 4	9.9258	8 days	KOBOLD	*A. N.* 181 283

In this case the elements themselves are given as there was no orbit from an arc long enough to be a standard for reference. Of these four the one from the longest arc (8 days) is that of KOBOLD (IV). His first orbit (I) is slightly nearer his second orbit than is our orbit (II) from a 5 day arc; but it should be noted that as KOBOLD'S arc is lengthened his elements *proceed in the direction from his first set toward our elements.* An orbit from a still longer arc might then well come still further toward the elements (II), so that, in this case, the data are rather insufficient from which to make a comparison or draw a conclusion. Our orbit could have been improved by the removal of some rather large residuals, but they were left with the idea that a second orbit would be computed. Owing to pressure of time this second orbit was not determined.

COMET *e* 1909 (DANIEL).

No.	T (B. M. T.)	ω	Ω	i	log q	e	Arc.	Computer.	Reference.
I	1909 Dec. 5.60	8° 16′	73° 33′	26° 57′	0.1967	1.0000	2 days	EBELL	*A. N.* 183 127
II	1909 Dec. 1.78	5 31	73 31	26 50	0.1944	1.0000	3 days	RUSSELL	*A. J.* 26 46
III	1909 Nov. 27.67	2 31	73 27	26 36	0.1911	1.0000	5 days	EBELL	*A. N.* 183 175
IV	1909 Nov. 29.80	10 52	68 58	19 44	0.1614	1.0000	7 days	KRASSOWSKI	*A. N.* 183 191
V	1909 Nov. 28.23	3 2	71 10	19 55	0.1433	0.6252	7 days	Einarsson Young	*L. O. B.* 179
VI	1909 Nov. 28.91	3 37	70 58	19 23	0.1402	0.5995	37 days	EBELL	*A. N.* 183 263
VII	1909 Nov. 28.76	3 29	71 0	19 27	0.1404	0.6025	86 days	Einarsson Young	*L. O. B.* 179

The elements themselves are given in this case because the orbit is a short period ellipse. The periods of the orbits (V), (VI) and (VII) are 7.ʸ15, 6.ʸ40 and 6.ʸ48 respectively. Our ellipse (V) is sufficiently close to establish the character of the orbit from a 7 day arc.

COMET *b* 1910 (METCALF).

The orbit of reference for this comet is the one by KOBOLD, which is based on an arc extending from August 11 to August 25 and then corrected by means of an observation taken September 18th. These elements are given in *A. N.* 186 80.

No.	Δ T	$\Delta\ \omega$	$\Delta\ \Omega$	$\Delta\ i$	Δ log q	Arc.	Computer.	Reference.
I	$-\ 5\overset{d}{.}73$	− 2° 55′	+ 0° 15′	+ 0° 11′	+0.0019	2 days	Young	*Pop. Ast.* 18 9
II	+66.25	+72 9	−17 4	+ 1 35	+0.2687	2 days	KOBOLD	*A. N.* 185 296
III	+17.31	+ 8 46	− 0 48	− 0 23	+0.0012	4 days	KOBOLD	*A. N.* 185 308
IV	+21.28	+10 55	− 1 10	− 0 22	+0.0069	14 days	KOBOLD	*A. N.* 185 408

Our orbit (I) from a 2 day arc is very much closer to the orbit of reference **than** any of KOBOLD'S orbits which were computed from arcs of 2, 4, and 14

days, respectively. It should be noted that KOBOLD attained an improvement over our orbit from a 2 day arc only by extending his arc to 38 days. As the arc is lengthened by KOBOLD his orbits (II), (III), and that of reference successively approach ours from a 2 day arc. Orbit (IV), however, may be affected by some error, probably of observation.

COMET *c* 1910 (CERULLI-FAYE).

No.	T (B. M. T.)	ω	Ω	i	log q	e	Arc	Computer.	Reference.
I	1910 Sept 15.06	168° 33′	211° 39′	15° 36′	0.0512	1.0000	2 days	EBELL	*A. N.* 186 209
II	1910 Nov. 19.52	200 17	212 58	18 17	0.3416	1.0000	4 days	Meyer Levy	*L. O. B.* 186
III	1910 Nov. 12.45	206 20	205 29	10 14	0.2175	0.5459	4 days	Meyer Levy	*L. O. B.* 186
IV	1910 Nov. 2.31	199 51	206 7	10 30	0.2178	0.5608	12 days	FAYET	*A. N.* 186 351
V	1910 Oct. 30.08	197 55	205 39	9 48	0.2009	0.5169	14 days	PRAGER RISTENPART	*A. N.* 186 370
VI	1910 Nov. 1.50	199 17	206 14	10 36	0.2189	0.5656	33 days	Meyer Levy	*L. O. B.* 187

The elements themselves are given in this case because the orbit is a short period ellipse. The periods of the orbits (III), (IV), (V) and (VI) are $6^{\mathrm{y}}.93$, $7^{\mathrm{y}}.29$, $5^{\mathrm{y}}.96$ and $7^{\mathrm{y}}.44$ respectively. Our orbit (III) from a 4 day arc is better than (V) from a 14 day arc.

It was from orbit (II) that Professor LEUSCHNER announced the probable identity of the comet discovered by CERULLI with FAYE'S comet. The elements (III) from the same observations made the identification certain.

In every case of orbits computed here the geocentric parallax for the middle observation has been eliminated.

In the first three cases (the comets for 1905) differential corrections have been used to improve the first solutions. In all other cases for short arcs the orbits represent a direct solution (no differential correction) for first orbits.

CONCLUSION.

It is to be noted from these data that, in general, orbits computed by LEUSCHNER'S *Methods are better than other orbits computed from the same arc, and frequently better even than orbits from longer arcs, although in no case of unequal intervals was the epoch chosen at the mean date as might have been done, with little additional labor, for the purpose of still further increasing the accuracy of the direct solution.*

In view of this, and also of the unquestionable brevity of the computations as compared with other methods, the value of LEUSCHNER'S *Methods, as established theoretically by him in Part 7, is fully substantiated by actual experience.*

EXAMPLES A. DIRECT SOLUTIONS.

NOTE.—In the computations given in the examples the formulæ referred to by numbers are those of the *Synopsis of Formulæ* in Part 7. In the discussions of the examples the formulæ and pages referred to are those of Part 7.

In the arguments log I means the logarithm of the first term in a formulæ composed of several terms. Log II means the logarithm of the second term of such an expression, etc. SUM means the logarithm of the sum of two terms of an expression. Similarly DIFF means the logarithm of the difference. ADD and SUB refer to the numbers taken from an addition and subtraction table. Log [ ] or { } means the logarithm of the expression contained within the square brackets or braces.

EXAMPLE NO. 1.

DIRECT SOLUTION OF A PARABOLIC ORBIT.

As an example of the details of the computation of an ordinary preliminary orbit, the complete calculation of the orbit of Comet *a* 1909 (Daniel) is given here. The computation was made by R. T. CRAWFORD, and has since been done in duplicate by P. W. MERRILL, Fellow in the Lick Observatory.

The computation is based on the following observations:

1909 Gr. M. T.	α (app.)	δ (app.)	Observer.
I June 16.5306	$1^h\ 41^m\ 54^s.1$	$+29°\ 58'\ 18''$	JAVELLE—Nice
II 18.9809	48 49 .5	+33 26 15	CAMPBELL—Lick
III June 21.9659	1 57 51 .0	+37 25 9	ALBRECHT—Lick

A Ia

Reduced to the beginning of the year 1909.0, including the aberration terms, these are

$\alpha_{,}$	$25°\ 28'\ 38''$	$\delta_{,}$	$+29°\ 58'\ 25''$
$\alpha_{,,}$	27 12 29	$\delta_{,,}$	+33 26 22
$\alpha_{,,,}$	29 27 51	$\delta_{,,,}$	+37 25 17

Using the data of the *American Ephemeris and Nautical Almanac*, the interpolation for the Sun's coördinates, referred to 1909.0, gives for the three dates of observation:

$X_{,}$	+0.085434	$Y_{,}$	+0.928875	$Z_{,}$	+0.402945
$X_{,,}$	+0.044042	$Y_{,,}$	+0.931466	$Z_{,,}$	+0.404071
$X_{,,,}$	−0.006472	$Y_{,,,}$	+0.932482	$Z_{,,,}$	+0.404513

Velocities of the Solar Coördinates at the Instant and Middle Date ($w = 1$).

1909	X	f^{I}	f^{II}	f^{III}
June 17.5	+0.0690722			
		−168981		
18.5	521741		−145	
		169126		+53
19.5	352615		−92	
		−169218		
20.5	+0.0183397			
	Y			
June 17.5	+0.9300897			
		+9930		
18.5	9310827		−2640	
		7290		+ 1
19.5	9318117		−2639	
		+4651		
20.5	+0.9322768			

1909	Z	f^{I}		f^{III}
June 17.5	+0.4034724			
		+ 4316		
18.5	4039040		−1147	
		3169		+ 1
19.5	4042209		−1146	
		+ 2023		
20.5	+0.4044232			

Date: June 18.9809	$f(l)$	X	Y	Z
	$f^{\mathrm{I}}(a + \frac{1}{2}w)$	−0.0169126	+0.0007290	+0.0003169
$m = -0.0191$	$m\, f^{\mathrm{II}}(a + \frac{1}{2}w)$	+ 2	+ 50	+ 22
	$M_r^3(m)\, f^{\mathrm{III}}(a + \frac{1}{2}w)$	− 2	0	0
$M_r^3(m) = -0.041$	$k \int f(l)\, dl$	−0.0169126	+0.0007340	+0.0003191
	$\log k \int f(l)\, dl$	8.22821n	6.86570	6.50393
	$\log X_0'$; Y_0'; Z_0'	9.99263n	8.63012	8.26835

Correction of the Solar Coördinates of the Middle Date for the Partial Elimination of the Geocentric Parallax.

The parallax factors for the middle observation were taken from the tables of Volume I, *Publications of the Lick Observatory*. The logarithms of these, expressed in circular measure, are:

$\log p_\alpha \rho$	5.5810n
$\log p_\delta \rho$	5.1699

Then

$\sin \alpha_{\prime\prime}$	9.6601
$\cos \alpha_{\prime\prime}$	9.9491
$\sin \delta_{\prime\prime}$	9.7412
$\cos \delta_{\prime\prime}$	9.9214
log I	5.1625n
log II	4.8602
$X_{\prime\prime}$	+0.044042
I	− 14
II	+ 7
$(X)_{\prime\prime}$	+0.044035
log I	5.4515
log II	4.5712
$Y_{\prime\prime}$	+0.931466
I	+ 28
II	+ 4
$(Y)_{\prime\prime}$	+0.931498
$\log \Delta Z_{\prime\prime}$	+5.0913n
$Z_{\prime\prime}$	+0.404071
$\Delta Z_{\prime\prime}$	− 12
$(Z)_{\prime\prime}$	+0.404059

$R \cos D \cos A$	8.64380
$R \cos D \sin A$	9.96918
$\tan A$	1.32538
A	87.°2934
$A - \alpha_{\prime\prime}$	60.0853
$\sin A$	9.99951
$\cos A$	8.67414
$\log S = R \cos D$	9.96966
$R \sin D$	9.60644
$\tan D$	9.63678
$\sin D$	9.59942
$\cos D$	9.96264
$\log R$	0.00702
$\log (t_{\prime\prime\prime} - t_{\prime\prime})$	0.474944
$\log (t_{\prime\prime} - t_{\prime})$	0.389219
$\log (t_{\prime\prime\prime} - t_{\prime})$	0.735224
$\log \theta_{\prime}$	8.7105
$\log \theta_{\prime\prime\prime}$	8.6248

A IIa.

p	α	δ
$p_{\prime\prime} - p_{\prime}$	$+ \quad 6231''$	$+ \quad 12467''$
$p_{\prime\prime\prime} - p_{\prime\prime}$	$+ \quad 8122$	$+ \quad 14335$
$\log (p_{\prime\prime} - p_{\prime})$	3.794558	4.096110
$\log (p_{\prime\prime\prime} - p_{\prime\prime})$	3.909663	4.156398
$\log p_{\prime\prime\prime}'$	3.405339	3.706891
$\log p_{\prime}'$	3.434719	3.681454
sub	8.845043	8.780467
$\log (p_{\prime}' - p_{\prime\prime\prime}')$	2.250382	2.461921n
$\log \frac{2}{k^2} \sin 1''$	8.515442	
colog $(t_{\prime\prime\prime} - t_{\prime})$	9.264776	
$\log p_0''$	0.030600	0.242139n
$\log (t_{\prime\prime\prime} - t_{\prime\prime}) p_{\prime\prime\prime}'$	3.880283	4.181835
$\log (t_{\prime\prime} - t_{\prime}) p_{\prime}'$	3.823938	4.070673
add	0.273771	0.248996
sum	4.154054	4.430831
$\log \frac{\sin 1''}{k}$	6.449993	
colog $(t_{\prime\prime\prime} - t_{\prime})$	9.264776	
$\log p_0'$	9.868823	0.145600

$\sec^2 \delta_{\prime\prime}$	0.157180
$\log (\tan \delta)_0'$	0.302780
$\log \delta_0'$	0.291200
$\tan \delta_{\prime\prime}$	9.819785
log I	0.412015
log II	0.242139n
add	9.680050
log (I + II)	9.922189
$\log (\tan \delta)_0''$	0.079369

A IIIa

$\log \alpha_0'^2$	9.73765	$\log \alpha_0''$	0.03060	$\sin \alpha_{\prime\prime}$	9.66013
$\tan \delta_{\prime\prime}$	9.81978	$\log n\Gamma$	1.11455n	$\cos \alpha_{\prime\prime}$	9.94907
$\log \alpha_0'^2 \tan \delta_{\prime\prime}$	9.55743	add	9.96265		
$\log (\tan \delta)_0''$	0.07937	log numer.	1.07720n	$\log {}^{\lambda}/_{\kappa} \cos \alpha_{\prime\prime}$	9.52105n
add	0.11416	$\log (-2\Phi)$	1.50522	$\log \alpha_0' \sin \alpha_{\prime\prime}$	9.52895
$\log n$	0.19353	$\log {}^{\lambda}/_{\kappa}$	9.57198n	sub	0.29710
				$\log a_x$	9.82605n
$\cos (A - \alpha_{\prime\prime})$	9.69785	$\sin \delta_{\prime\prime}$	9.74120		
$\tan \delta_{\prime\prime} \cos (A - \alpha_{\prime\prime})$	9.51763	$\sin D$	9.59942	$\log {}^{\lambda}/_{\kappa} \sin \alpha_{\prime\prime}$	9.23211n
$\tan D$	9.63678	log I	9.34062	$\log \alpha_0' \cos \alpha_{\prime\prime}$	9.81789
sub	9.49925	$\cos \delta_{\prime\prime}$	9.92141	add	9.86949
$\log C_1$	9.01688n	$\cos D$	9.96264	$\log a_y$	9.68738
$\log C_2$	9.93790	$\cos (A - \alpha_{\prime\prime})$	9.69785	$\log {}^{\lambda}/_{\kappa} \tan \delta_{\prime\prime}$	9.39176n
$\log \Gamma$	0.92102n	log II	9.58190	$\log (\tan \delta)_0'$	0.30278
		add	0.19694	add	9.94313
$\log \alpha_0'$	9.86882	$\log c$	9.77884	$\log a_z$	0.24591
$\log \Gamma (\tan \delta)_0'$	1.22380n				
add	9.98039	ψ	53°.062		
$\log \Phi$	1.20419n	$\log s$	9.90270		

A IVa

$\log a_x^2$	9.65210	$\log X_0'^2$	9.98526	$\log a_x X_0'$	9.81868
$\log a_y^2$	9.37476	$\log Y_0'^2$	7.26024	$\log a_y Y_0'$	8.31750
add	0.18413	add	0.00082	add	0.01348
sum	9.83623	sum	9.98608	sum	9.83216
$\log a_z^2$	0.49182	$\log Z_0'^2$	6.53670	$\log a_z Z_0'$	8.51426
add	0.08672	add	0.00015	add	0.02040
log [ ]	0.57854	$\log G_s^2$	9.98623	log [ ]	9.85256
$\cos^2 \delta_{\prime\prime}$	9.84282			$\cos \delta_{\prime\prime}$	9.92141
$\log a^2$	0.42136			$\log b$	9.77397

A IVa—Continued.

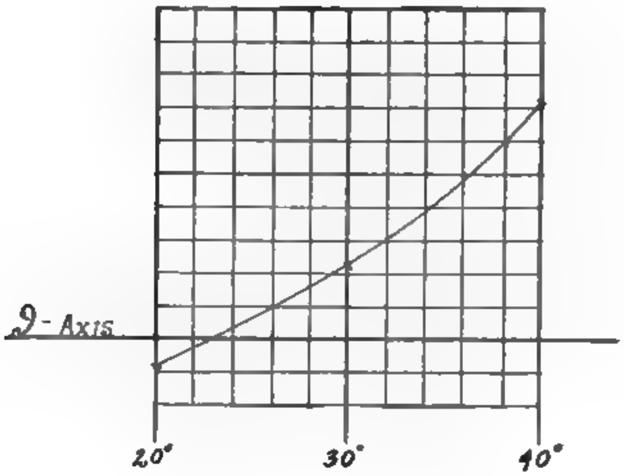
log $a^2 G_1^2$	0.40759
log b^2	9.54794
sub	9.93543
diff	0.34302
log $a^4 R^2$	0.85676
log q'^2	9.48626
q'^2	+0.30638
log $a^2 R$	0.42838
log p'	9.34559

log 2	0.30103
colog a^2	9.57864
colog R^2	9.97894
log k	9.85861
log c	9.77884
log p'	9.34559
sub	9.80019
log c'	9.57903
c'	+0.37934

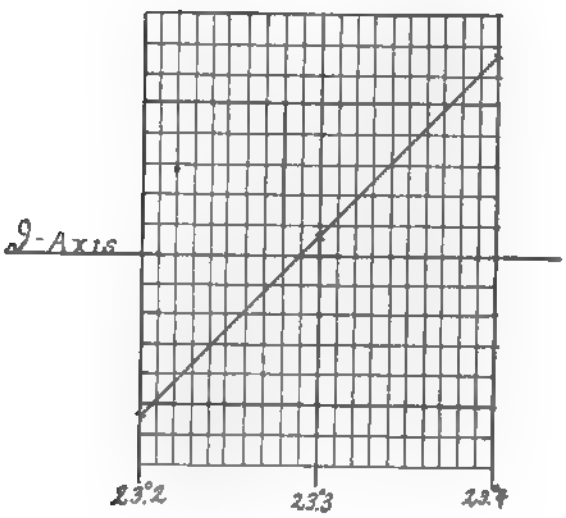
log s	9.9027
log a^2	0.4214
log q'^2	9.4863
colog 2	9.6990
log ($m' = \cos \vartheta_m$)	9.5094
ϑ_m	71°
log k/s	9.95591
k/s	+0.90346
s	+0.79928

Graphical Determination of z_i.

ϑ	z'^2	y	$f(\vartheta)$
70°	6.7	0.0	+ 6.7
40	1.10	0.39	+ 0.71
30	0.71	0.48	+ 0.23
20	0.45	0.54	− 0.09



ϑ	z'^2	y	$f(\vartheta)$
23.°4	+0.5261	+0.5228	+0.0033
23. 3	+0.5237	+0.5234	+0.0003
23. 2	+0.5213	+0.5240	−0.0027



ϑ	23°.290
log tan ϑ	9.63393
log s tan ϑ	9.53663
log c	9.77884
add	0.19659
log z_i	9.97543

Note.—In this case a second figure is not really necessary as a simple inspection of the numbers in the column (ϑ) shows that $f(\vartheta) = 0$ for $\vartheta = 23.°290$.

Final Solution for z.

i	1	2
$\log z_i$	9.97543	9.97548
sub	9.88394	9.88396
$\log (z_i - p')$	9.85937	9.85944
$\log$ I	9.71874	9.71888
sub		9.75792
$\log (z_i - c)$	9.53663	9.53676
$\log (z_i - c)^2$	9.07326	9.07352
add	0.07383 −	0.07387 −
$\log \mu$	9.87923	9.87927
$\log \mu^{½}$	9.93961	9.93963
$\log (\text{III} = h\,\mu^{-½})$	9.91900	9.91898
I	+0.52329	+0.52346
− III	− 0.82985	− 0.82982
M_i	− 0.00018	+0.00002 check

i	1
$\log h\,\mu^{-½}$	9.9190
$\log (z_i - c)$	9.5366
$\log \mu^{-1}$	0.1208
$\log$ II	9.5764
$\log$ I	0.1604
add	0.1006
log denom.	0.2610
$\log (-M_i)$	6.2553
$\log \partial z$	5.9943
add	0.00005
$\log z_{i+1}$	9.97548

It is to be noted that the graphical solution gave the value of $\log z$ correct to within one half a unit of the fourth decimal place and that a single differential correction gave the final value to 5 decimals.

A Va

$\log R$	0.00702
$\log z$	9.97548
$\log \rho_0$	9.98250
$\log \alpha\rho_0$	7.7438n
$\alpha\rho_0$	− 0.0055
t_{II} (obs.)	18.9809
t_{II}	18.9754
$\cos \delta_{II}$	9.92141
$\log \sigma_0$	9.90391
$\cos \alpha_{II}$	9.94907
$\log \sigma_0 \cos \alpha_{II}$	9.85298
$\log (X)_{II}$	8.64380
sub	9.97230
$\log x_0$	9.82528
$\sin \alpha_{II}$	9.66013
$\log \sigma_0 \sin \alpha_{II}$	9.56404
$\log (Y)_{II}$	9.96918
sub	0.18803
$\log y_0$	9.75207n
$\tan \delta_{II}$	9.81978
$\log \sigma_0 \tan \delta_{II}$	9.72369
$\log (Z)_{II}$	9.60644
sub	9.49126
$\log z_0$	9.09770

$\log \lambda/\kappa\, \theta_{III}$	8.1968n
$\log (1 - \lambda/\kappa\, \theta_{III})$	0.0068
$\sec \delta_I$	0.0624
$\log \rho_I$	9.9731
$\log \alpha\rho_I$	7.7344n
$\alpha\rho_I$	− 0.0054
t_I (obs.)	16.5306
t_I	16.5252
$\log (t_{II} - t_I)$	0.38920
$\log \theta_{III}$	8.62478
$\log a_x \sigma_0$	9.72996n
$\log X_0'$	9.99263n
sub	9.91956
$\log x_0'$	9.64952
$\log a_y \sigma_0$	9.59129
$\log Y_0'$	8.63012
sub	9.94970
$\log y_0'$	9.54099
$\log a_z \sigma_0$	0.14982
$\log Z_0'$	8.26835
sub	9.99425
$\log z_0'$	0.14407
$\log x_0'^2$	9.29904
$\log y_0'^2$	9.08198
add	0.20592
sum	9.50496
$\log z_0'^2$	0.28814
add	0.06623
$\log G_0^2$	0.35437
Check	
$\log {}^2/r_0$	0.35437

$\log \lambda/\kappa\, \theta_I$	8.2825n
$\log (1 + \lambda/\kappa\, \theta_{II})$	0.9916
$\sec \delta_{III}$	0.1001
$\log \rho_{III}$	9.9956
$\log \alpha\rho_{III}$	7.7569n
$\alpha\rho_{III}$	− 0.0057
t_{III} (obs.)	21.9659
t_{III}	21.9602
$\log (t_{III} - t_{II})$	0.47491
$\log \theta_I$	8.71049
$\log x_0^2$	9.65056
$\log y_0^2$	9.50414
add	0.23396
sum	9.88452
$\log z_0^2$	8.19540
add	0.00880
$\log r_0^2$	8.89332
$\log r_0$	9.94666
Check	
$\log R^2$	0.01404
$\log \mu$	9.87927
$\log r_0^2$	9.89331
$\log x_0 x_0'$	9.47480
$\log y_0 y_0'$	9.29306n
add	9.71571
sum	9.00877
$\log z_0 z_0'$	9.24177
add	0.19997
$\log r_0 r_0'$	9.44174
$\log r_0'$	9.49508

A VIa

Representation of the First and Third Places.

	I	III
$\log \theta$	8.62478	8.71049
$\log \theta^2$	7.24956	7.42098
$\log \theta^3$	5.8743	6.1315
$\log \theta^4$	4.499	4.842
$\log r_0^3$	9.83998	
$\log f_2$	9.85899n	
$\log r_0'$	9.49508	
$\log 2\, r_0^4$	0.08767	
$\log f_3$	9.4074	
$\log (r_0 r_0')^2$	8.8835	
$\text{colog}\, r_0$	0.0533	
$\log \frac{15}{4}$	0.5740	
$\log$ II	9.5108	
$\log$ (1 − II)	9.8298	
$\log 6\, r_0^6$	0.4581	
$\log f_4$	9.372	
$\log g_3$	9.3819n	
$\log 4\, r_0^4$	0.3887	
$\log g_4$	9.106	
$\log \theta^2 f_2$	7.10855n	7.27997n
$\log \theta^3 f_3$	5.2817	5.5389
$\log \theta^4 f_4$	3.871	4.214
$\log \theta^3 g_3$	5.2562n	5.5134n
$\log \theta^4 g_4$	3.605	3.948
	1.000000	1.000000
$\theta^2 f_2$	− 1284	− 1905
$\mp \theta^3 f_3$	− 19	+ 35
$\theta^4 f_4$	+ 1	+ 2
$f_{I},\ f_{III}$	+0.998698	+0.998132
$\log f_{I};\ \log f_{III}$	9.99943	9.99919
$\mp \theta$	−0.042148	+0.051344
$\mp \theta^3 g_3$	+ 18	− 33
$\theta^4 g_4$	0	+ 1
$g_{I};\ g_{III}$	−0.042130	+0.051312
$\log g_{I};\ \log g_{III}$	8.62459n	8.71022

	I	III
$\log f x_0$	9.82471	9.82447
$\log f y_0$	9.75150n	9.75126n
$\log f z_0$	9.09713	9.09689
$\log g x_0'$	8.27411n	8.35974
$\log g y_0'$	8.16558n	8.25121
$\log g z_0'$	8.76866n	8.85429
X	+0.085434	−0.006472
$f x_0$	+0.667900	+0.667519
$g x_0'$	−0.018798	+0.022895
ξ	+0.734536	+0.683942
Y	+0.928875	+0.932482
$f y_0$	−0.564288	+0.563975
$g y_0'$	−0.014641	+0.017832
η	+0.349946	+0.386339
Z	+0.402945	+0.404513
$f z_0$	+0.125063	+0.124994
$g z_0'$	−0.058703	+0.071497
ζ	+0.469305	+0.601004
$\rho \cos \delta \cos \alpha$	9.86601	9.83502
$\rho \cos \delta \sin \alpha$	9.54400	9.58697
$\tan \alpha$	9.67799	9.75195
α	25°.4740	29°.4611
α (Obs.)	25.4750	29.4617
(O − C) $\Delta \alpha$	+0.0010	+0.0006
(O − C) $\Delta \alpha \cos \delta$	+0.0009	+0.0005
$\sin \alpha$	9.63357	9.69182
$\cos \alpha$	9.95558	9.93987
$\rho \cos \delta$	9.91043	9.89515
$\rho \sin \delta$	9.67145	9.77887
$\tan \delta$	9.76102	9.88372
δ	+29°.9761	+37°.4200
δ (Obs.)	+29 .9753	+37 .4222
(O − C) $\Delta \delta$	− 0 .0008	+ 0 .0022
$\sin \delta$	9.69866	9.78366
$\cos \delta$	9.93764	9.89993
$\log \rho$	9.97279	9.99522
$\log p_\alpha \rho$	0.864n	0.932n
$\log p_\delta \rho$	0.748	0.493
$\log p_\alpha$	0.891n	0.937n
$\log p_\delta$	0.775	0.498
p_α	− 8"	− 9"
p_δ	+ 6	+ 3

A VIIa

Constants for the Equator 1909.0, and Elements.

$d \quad w \quad D'$	$a \quad x \quad A'$	$b \quad y \quad B'$	$c \quad z \quad C'$
$\log r_0 w_0'$	9.59618	9.48765	0.09073
$\log w_0 r_0'$	9.32036	9.24715n	8.59278
sub	9.94802	0.19722	9.98598
$\log [r_0 w' - w_0 r_0']$	9.26838	9.68487	0.07671
$\sin d \cos (D' + v_0)$	9.15413	9.57062	9.96246
$\sin d \sin (D' + v_0)$	9.87862	9.80541n	9.15104
$\tan (D' + v_0)$	0.72449	0.23479n	9.18858
$D' + v_0$	79°.3202	300°.2159	8°.7758
$D' = A'; B'; C'$	55 .3191	276 .2148	344 .7747
$\sin (D' + v_0)$	9.99241	9.93658n	9.18346
$\cos (D' + v_0)$	9.26792	9.70180	9.99488
$\sin d = \sin a; \sin b; \sin c$	9.88621	9.86882	9.96758

$\log r_0'$	9.49508
$\log G$	0.17718
$\sin \frac{1}{2} v_0$	9.31790
$\frac{1}{2} v_0$	12°.0005[6]
v_0	24 .0011
$\log M$	1 .24892
$\log 2 r_0$	0 .24769
$\log (r_0 r_0')^2$	8 .88348
sub	9 .98081
$\log p$	0 .22850
$\log q$	9 .92747
$\log q^{\frac{3}{2}}$	9.89120
$\log M q^{\frac{3}{2}}$	1.14012
$M q^{\frac{3}{2}}$	13.8077
t_0	June 18.9754
T	June 5.1677 Gr. $M.$ $T.$

$\tan \varepsilon$	9.63728
log I	9.96246
log II	9.20790
sub	9.91594
$\sec \varepsilon \sin i \cos u$	9.87840
log I	9.15104
log II	9.44269n
sub	0.17924
$\sec \varepsilon \sin i \sin u$	9.62193
$\tan u$	9.74353
u	28°.9878
ω	4°.9867
$\sin u$	9.68540
$\cos u$	9.94187
$\sec \varepsilon \sin i$	9.93653
$\cos \varepsilon$	9.96256
$\sin i$	9.89909

Check

$\sin b \sin c$	9.83640
$\sin (C - B)$	9.96886
$\operatorname{cosec} a$	0.11379
$\sec A$	0.19495
$\tan i$	0.11400

A	50°.3324
B	271 .2281
C	339 .7880
$\sin B$	9.99990n
$\sin b$	9.86882
$\sec \varepsilon$	0.03744
$\sin \Omega$	9.90616n
$\sin a$	9.88621
$\sin A$	9.88635
$\cos \Omega$	9.77256
Ω	306°.3220
$(-\sin a)$	9.88621n
$\cos A$	9.80505
$\operatorname{cosec} \Omega$	0.09384n
$\cos i$	9.78510
i	52°.4340

Check

$\sin c$	9.96758
$\sin C$	9.53844n
$\operatorname{cosec} \varepsilon$	0.40016
$\sin \Omega$	9.90618n

COLLECTION OF RESULTS.

Elements.

T	1909 June 5.1677 Gr. M. T.	
ω	4°.9867	1909.0
Ω	306 .3220	1909.0
i	52 .4340	1909.0
$\log q$	9 .92747	

Constants for the Equator 1909.0.

$x = r\,[9.88621] \sin (55°.3191 + v)$

$y = r\,[9.86882] \sin (276 .2148 + v)$

$z = r\,[9.96758] \sin (344 .7747 + v)$

SUPPLEMENTARY DISCUSSION.

This supplementary discussion is given in full as a sample of the analysis that should be made in connection with every preliminary orbit. The experienced computer can arrive at the conclusions very quickly without going into such elaborate detail as is given here for the sake of making the process clear.

First of all we should investigate the number of mathematical solutions through the use of the criteria given on page 288, so as to avoid the adoption of a fictitious solution in case of multiple solutions. Since p' and c are positive, three solutions are possible only if $\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$ is negative. For this expression to be negative p must be negative, *i. e.* $9\,(2s^2 + q'^2) < 7\,c'^2$. But a glance at the numerical values of s, q'^2, and c' shows that $9\,(2s^2 + q'^2) > 7\,c'^2$. Hence only a single solution is possible.

It is further essential to investigate whether the parabolic hypothesis is admissible within the uncertainty of the computation or whether it should be rejected. For this we have to compute only approximate values of N, κ and $\frac{1}{m}$ according to A IIIa, A IVa and determine the degree of accuracy to which $\frac{1}{m}$ is known.

In this let us place a dot over the last definitely known figure. If we assume that the error of observation is a matter of only a few seconds of arc, then the differences in α and δ are known to 3 and 4 places respectively. The corresponding differences and intervals are:

$t_{\prime\prime} - t_{\prime}$	$= 2.4503$	$\alpha_{\prime\prime} - \alpha_{\prime}$	$62\dot{3}1''$	$\delta_{\prime\prime} - \delta_{\prime}$	$12467''$
$t_{\prime\prime\prime} - t_{\prime\prime}$	2.9850	$\alpha_{\prime\prime\prime} - \alpha_{\prime\prime}$	$81\dot{2}2$	$\delta_{\prime\prime\prime} - \delta_{\prime\prime}$	$143\dot{3}5$

The intervals (not exactly equal) are in the neighborhood of 3 days. In order to get a better idea of the run of the differences they can be reduced by proportion to a three-day basis. On this basis, assigning the differences to the instants of the middle of the intervals, we would have

	f^{I}_{α}	Diff	f^{II}_{α}	f^{I}_{δ}	Diff.	f^{II}_{δ}
June 17.7557	$76\dot{2}9''$	$(5\dot{3}4'')$	$5\dot{8}9''$	$152\dot{7}6''$	$(8\dot{7}0'')$	$9\dot{6}0''$
June 20.4734	$81\dot{6}3$			$144\dot{0}6$		

The numbers in parentheses are the differences between the f^{I}_{α} and f^{I}_{δ} respectively which correspond to a time interval of 2.7177 days. These reduced to correspond to a three-day interval are the quantities shown as f^{II}_{α} and f^{II}_{δ} respectively. Since the principal or governing term in α'' and δ'' is the corresponding f^{II} it is seen that the accelerations are known only to two places. But now we should consider the error introduced by neglecting f^{III}. In α, f^{II} is somewhat smaller than $\frac{1}{10}\,f^{\text{I}}$. If now we assume the differences to decrease at the same rate we would have as an approximation $f^{\text{III}}_{\alpha} = 59''$, $f^{\text{III}}_{\delta} = 96''$.

According to (2) page 267, or also according to the discussion, pages 274 and 275, the error produced by neglecting the third differences is $\frac{\theta_{,}\theta_{,,,}}{6}\alpha'''$ and $(\theta_{,,,}-\theta_{,})\frac{\alpha'''}{3}$ in the velocities and accelerations respectively. Or, since $\alpha''' = \frac{f^{\text{III}}}{\theta^3}$ approximately, and further $\theta_{,,,} - \theta_{,} = 0.5\,k$ approximately, and since (on our three-day basis) we can place $\theta_{,} = \theta_{,,,} = \theta = 3\,k$, the error created in the velocities is approximately $\frac{f^{\text{III}}}{6\,\theta}$, and in the accelerations $\frac{0.5\,k}{3\,\theta^3}f^{\text{III}}$ or $\frac{0.5}{9\,\theta^2}f^{\text{III}}$. For the velocities, therefore, $\frac{f^{\text{III}}}{6}$ should be compared with f^{I}, and for the accelerations $\frac{0.5}{9}f^{\text{III}}$ with f^{I}. For $\frac{f^{\text{III}}}{6}$ we have $\frac{59''}{6} = 10''$ in α, and $\frac{96''}{6} = 16''$ in δ. So that, through neglecting the third differences, the uncertainty gets slightly into the tens of seconds and consequently the uncertainty of the velocities in α and δ is in the third and fourth places respectively. For the accelerations, $\frac{0.5}{9}f^{\text{III}}$ becomes $\frac{59''}{18}$ in α and $\frac{96''}{18}$ in δ, or $3''+$ and $5''+$ respectively. It is seen, therefore, that the neglect of the third differences introduces no greater uncertainty into the accelerations than was assumed for them arising from errors of observation. These two errors, however, might be cumulative so that their combination might affect the last two places, and since in f^{II} only three places are available we can say that we know *absolutely* only one place.

Collecting these results then it is seen that we have *certain*

in α' 2 places $\quad \delta'$ 3 places $\quad \alpha''$ 1 place $\quad \delta''$ 1 place.

It should be noted that the conditions are somewhat more favorable for δ'' than for α''. From the computation for $(\tan\delta)'$ and $(\tan\delta)''$ it follows that these quantities are of the same order of certainty as δ' and δ'' respectively.

After the investigation into the accuracy of the velocities and accelerations the subject of parallax should be studied. In this case there is no necessity for the *complete* elimination of the parallax, since the correction for parallax can scarcely affect more than the already uncertain last two places. As should always be done, the partial elimination of the geocentric parallax was applied as a correction to the middle solar place merely to give the correct residuals for the first and third places, in that, through this elimination, the geocentric parallax in the α and δ of the middle date is taken into account.

There remains the investigation into the character of the orbit. Through an approximate computation we find for the three terms in the quantity N, according to AIIIa, the following:

$$N = 0.267 - 2.155 + 0.888 = -1.000$$

Since α'' is the least certain, so is the second term, the conclusive one for the sufficiency of N. Therefore, the uncertainty of N is one unit of the first decimal place.

Further, according to A IIIa, log $\varkappa = 0.191$ and, according to A IVa, $\frac{1}{m} = -0.57$. $\varkappa$ and $\frac{1}{m}$ are of the order of accuracy of N. If we place the uncertainty of N at about 0.2, *i. e.* $\frac{1}{5}$ of its value, then the uncertainty of $\frac{1}{m}$ will be a little more than 0.1. If we interpolate the general solution z_g from the Table at the end of Part 7 with the arguments $\psi = 53^\circ.062$ and a somewhat closer value of $\frac{1}{m}$, viz., $\frac{1}{m} = -0.5739$, we find $z_g = 0.936$. From the appropriate part of the Table in Part 7 we find a variation of 0.01 in z corresponds to a variation of 0.03 in $\frac{1}{m}$. The difference of 0.009 between $z = 0.945$ and z_g corresponds, therefore, to an uncertainty of 0.03 in $\frac{1}{m}$. This is so much less than the determined uncertainty of more than 0.1 in $\frac{1}{m}$ that we are not to expect a general orbit from the observational material at hand.

One might draw the conclusion from the fair agreement between the parabolic and the general values of z, that, in practice, it would be better to take the first appoximation for z from the Table than to use the graphical method which was employed in the example. But this is not the case, since, generally, in using the tabular value several applications of the differential formulæ AIVa would be required to get the final value of z.

In conclusion it is to be noted that the greatest residual (O — C) is about 8″, and that on account of the difficulty of observing this comet accurately, it is not advisable to attempt an improvement of the orbit through a differential correction to remove the residuals.

EXAMPLE NO. 2.

DETERMINATION OF α', δ'; α'', δ'' FROM FIVE OBSERVATIONS.

It sometimes happens that the computer has reason to believe, at the start, that three observations can not give velocities and accelerations in α and δ accurately enough, and that, therefore, five or more observations should be used according to the scheme outlined in the footnote, page 314. To illustrate the workings of these formulæ the case of Comet e 1906 (KOPFF) is selected. The computations were made by R. T. CRAWFORD and A. J. CHAMPREUX.

A preliminary orbit of this comet was available, but because of the very slow motion of the object large changes in the orbit were expected. It was deemed advisable, therefore, not to attempt to derive a second orbit based upon the preliminary, but to make another *direct solution* starting with more accurate geocentric velocities and accelerations as determined from five observations.

The five observations used were made by FATH at the Lick Observatory. They are:

	Gr. M. T. 1906	α (1906.0)	δ (1906.0)
I	Aug. 24.7063	34[illegible]° 59′ [illegible]″.6	+10° 18′ 6″.2
II	Aug. 29.9817	341 [illegible] [illegible].2	10 0 49 .3
III	Sept. 5.6771	339 43 57 .0	9 28 40 .3
IV	Sept. 8.7203	339 11 54 .0	9 11 13 .0
V	Sept. 15.7371	338 6 4 .4	+ 8 25 45 .4

Using the first, third, and fifth observations, approximate values of the velocities and accelerations were found, according to A IIa (unit of time being one Mean Solar Day). They are:

log α_0'	7.4829731n	log α_0''	5.621814
log δ_0'	7.186673n	log δ_0''	5.749006n

The more accurate determination depending upon the five observations, according to the formulæ, page 314, proceeds as follows:

i	1	2	4	5
t_i	24.7063	29.9817	39.7203	46.7371
$t_i - t_0$	− 11.9708	− 6.6954	+ 3.0432	+ 10.0600
log $(t_i - t_0)$	1.0781231n	0.8257771n	0.483330	1.002598
log $(t_i - t_0)^2$	2.156246	1.651554	0.966660	2.005196
log $(t_i - t_0)^3$	3.234369n	2.477331n	1.449990	3.007794
log a_i	1.854158n	1.097120n	0.069779	1.627583
log b_i	1.378095	0.873403	0.188509	1.227045
log c_i	0.777093n	0.524747n	0.182300	0.701568

i	2	4
log 206265″	5.314425	5.314425
log $2\,c_i\,\alpha_0'$	3.623175	3.280728n
log $3\,b_i\,\alpha_0''$	2.286763	1.601869
α_i	341° 0′ 0″.2	339° 11′ 54″.0
$-\alpha_0$	− 339 43 57 .0	
$-2\,c_i\,\alpha_0'$	− 1 9 59 .3	+ 0 31 48 .7
$-3\,b_i\,\alpha_0''$	− 0 3 13 .5	− 0 0 40 .0
sum	+ 0 2 50 .4	− 0 0 54 .3
log sum	2.231470	1.734800n
log $n_{i,\alpha}$	1.405693n	1.251470n

i	2	4
log $2\,c_i\,\delta_0'$	3.326875	2.984428n
log $3\,b_i\,\delta_0''$	2.413955n	1.729061n
δ_i	+10° 0′ 49″.3	+9° 11′ 13″.0
$-\delta_0$	− 9 28 40 .3	
$-2\,c_i\,\delta_0'$	− 0 35 22 .6	+0 16 4.8
$-3\,b_i\,\delta_0''$	+ 0 4 19 .4	+0 0 53.6
sum	+ 0 1 5 .8	−0 0 28.9
log sum	1.818226	1.460898n
log $n_{i,\delta}$	0.992449n	0.977568n

p_i	b_i	c_i	1_i
$\log p_2$	0.873403	0.524747n	0.000000
$\log \frac{a_2}{a_1} \cdot p_1$	0.621057	0.020055n	9.242962
sub	9.896477	9.837064	
$\log [p_2 \cdot 1]$	0.517534	0.361811n	9.916470
$\log p_4$	0.188509	0.182300	0.000000
$\log \frac{a_4}{a_1} \cdot p_1$	9.593716n	8.992714	8.215621n
sub	0.098373	9.970985	
$\log [p_4 \cdot 1]$	0.286882	0.153285	0.007077
$\log p_5$	1.227045	0.701568	0.000000
$\log \frac{a_5}{a_1} \cdot p_1$	1.151520n	0.550518	9.773425n
sub	0.264907	9.619048	
$\log [p_5 \cdot 1]$	1.491952	0.169566	0.202354

p_i	c_i	1_i
$\log [p_2 \cdot 1]$	0.361811n	9.916470
log II	9.195148	9.227936
sub	0.028624	9.900441
$\log [p_2 \cdot 2]$	0.390435n	9.816911
$\log [p_4 \cdot 1]$	0.153285	0.007077
log II	8.964496	8.997284
sub	9.970929	9.955317
$\log [p_4 \cdot 2]$	0.124214	9.962394

γ	α	δ
$\log [1_4 \cdot 2]\, n_{2,\gamma}$	1.368087n	0.954843n
$\log [1_2 \cdot 2]\, n_{4,\gamma}$	1.068381n	0.794479n
sub	9.997348	9.649968
log numer.	1.065729n	0.444447n
$\log [c_2 \cdot 2][1_4 \cdot 2]$	0.352829n	
$\log [c_4 \cdot 2][1_2 \cdot 2]$	9.941125	
sub	0.142240	
log denom.	0.495069n	0.495069n
log sin 1″	4.685575	4.685575
$\log y$	5.256235	4.634953

γ	α	δ
$\log [c_2 \cdot 2]\, n_{4,\gamma}$	1.641905	1.368003
$\log [c_4 \cdot 2]\, n_{2,\gamma}$	1.529907n	1.116663n
sub	0.248631	0.193294
log numer.	1.890536	1.561297
log denom.	0.495069n	0.495069n
log sin 1″	4.685575	4.685575
$\log z_\gamma$	6.081042n	5.751803n

γ	α	δ
$\log \gamma_0'$	7.482973n	7.186673n
$\log z_\gamma$	6.081042n	5.751803n
add	0.016881	0.015670
colog k	1.764419	1.764419
$\log \gamma'$	9.264273n	8.966762n

γ	α	δ
$\log \gamma_0''$	5.621814	5.749006n
$\log y_\gamma$	5.256235	4.634953
add	0.155622	9.965247
colog k^2	3.528838	3.528838
$\log \gamma''$	9.306274	9.243091n

With these values of the velocities and accelerations in α and δ the computation then proceeds exactly as is shown in Example No. 1.

EXAMPLE NO. 3.

DIRECT SOLUTION OF A CIRCULAR ORBIT.

To illustrate the computation for the direct solution of a circular orbit, three observations of (158) *Koronis* have been used. The computation is by R. T. CRAWFORD.

The observations were taken from *A. N.* 4127, page 373. All of them were made at the Naval Observatory, Washington; the first and third by HAMMOND, the second by FREDRICKSON.

A Ib

Reduced to the beginning of the year 1906.0, including the aberration terms, they are

1906 Gr. M. T.		α (1906.0)	δ (1906.0)
I	April 23.63642	180° 12′ 14″.6	−1° 38′ 53″.4
II	April 25.60822	179 58 17 .0	−1 32 7 .8
III	April 27.56370	179 45 39 .6	−1 25 55 .3

The solar coördinates, reduced to 1906.0, are

$X_{\prime}$	+0.843341	$(X)_{\prime\prime}$	+0.824920	$X_{\prime\prime\prime}$	+0.805728
$Y_{\prime}$	+0.503136	$(Y)_{\prime\prime}$	+0.529043	$Y_{\prime\prime\prime}$	+0.554147
$Z_{\prime}$	+0.218260	$(Z)_{\prime\prime}$	+0.229471	$Z_{\prime\prime\prime}$	+0.240390

NOTE.—The solar coördinates for the middle date have been corrected so as to eliminate the geocentric parallax for that place.

Further

$\log X_0'$	9.745839n	$\log R$	0.002813	$\sin D$	9.357915
$\log Y_0'$	9.878045	$\log S$	9.991222	$\cos D$	9.988409
$\log Z_0'$	9.515359				
		$\log (t_{\prime\prime} - t_{\prime})$	0.294863	$\log (t_{\prime\prime\prime} - t_{\prime\prime})$	0.291254
		$\log \theta_{\prime\prime\prime}$	8.5304	$\log \theta_{\prime}$	8.5268

A IIb

p	α	δ
$p_{\prime\prime} - p_{\prime}$	−837″.6	+405″.6
$p_{\prime\prime\prime} - p_{\prime\prime}$	−757 .4	+372 .5
$\log (p_{\prime\prime} - p_{\prime})$	2.923037n	2.608098
$\log (p_{\prime\prime\prime} - p_{\prime\prime})$	2.879325n	2.571126
$\log p_{\prime\prime\prime}'$	2.628174n	2.313235
$\log p_{\prime}'$	2.588071n	2.279872
sub	8.98560	8.90227
$\log (p_{\prime}' - p_{\prime\prime\prime}')$	1.57367	1.18214n
$\log \frac{2}{k^2} \sin 1''$	8.51544	
colog $(t_{\prime\prime\prime} - t_{\prime})$	9.40591	
$\log p_0''$	9.49502	9.10349n
$\log (t_{\prime\prime\prime} - t_{\prime\prime}) p_{\prime\prime\prime}'$	2.919428n	2.604489
$\log (t_{\prime\prime} - t_{\prime}) p_{\prime}'$	2.882934n	2.574735
add	0.283166	0.286408
sum	3.202594n	2.890897
$\log \frac{\sin 1''}{k}$	6.449993	
colog $(t_{\prime\prime\prime} - t_{\prime})$	9.405908	
$\log p_0'$	9.058495n	8.746798

$\sec^2 \delta_{\prime\prime}$	0.00031
$\log (\tan \delta)_0'$	8.74711
$\log \delta_0'^2$	7.49360
$\tan \delta_{\prime\prime}$	8.42823n
log I	6.22286n
log II	9.10349n
add	0.00057
log (I + II)	9.10406n
$\log (\tan \delta)_0''$	9.10437n

A IIIb

$\log \alpha_0'^2$	8.11699
$\tan \delta_u$	8.42823n
$\log \alpha_0'^2 \tan \delta_u$	6.54522n
$\log (\tan \delta)_0''$	9.10437n
add	0.00119
$\log n$	9.10556n
$\cos (A - \alpha_u)$	9.92505n
$\tan \delta_u \cos (A - \alpha_u)$	8.35328
$\tan D$	9.36951
sub	9.95600
$\log C_1$	9.32551n
$\log C_2$	9.73261n
$\log \Gamma$	0.40710
$\log \alpha_0'$	9.05850n
$\log \Gamma (\tan \delta)_0'$	9.15421
add	9.39190
$\log \Phi$	8.45040

$\log \alpha_0''$	9 49502
$\log n \Gamma$	9.51266n
add	8.61750
log numer.	8.11252n
$\log (-2 \Phi)$	8.75143n
$\log \frac{\lambda}{\varkappa}$	9.36109
$\sin \delta_u$	8.42808n
$\sin D$	9.35791
log I	7.78599n
$\cos \delta_u$	9.99984
$\cos D$	9.98841
$\cos (A - \alpha_u)$	9.92505n
log II	9.91330n
add	0.00323
$\log c$	9.91653n
ψ	145°.6033
$\log s$	9.75198

$\sin \alpha_u$	6.69841
$\cos \alpha_u$	0.00000n
$\log \frac{\lambda}{\varkappa} \cos \alpha_u$	9.36109n
$\log \alpha_0' \sin \alpha_u$	5.75691n
sub	9.99989
$\log a_x$	9.36098n
$\log \frac{\lambda}{\varkappa} \sin \alpha_u$	6.05950
$\log \alpha_0' \cos \alpha_u$	9.05850
add	0.00043
$\log a_y$	9.05893
$\log \frac{\lambda}{\varkappa} \tan \delta_u$	7.78932n
$\log (\tan \delta)_0'$	8.74711
add	9.94928
$\log a_z$	8.69639

At this point we may proceed in two different ways. We may compute either a conditioned ellipse and apply the criterion $r_0' = 0$, or a general orbit and apply the criteria $a = r_0$ and $r_0' = 0$.

Method of Conditioned Ellipse.

A IVb

$\log a_x^2$	8.72196
$\log a_y^2$	8.11786
add	0.09650
sum	8.81846
$\log a_z^2$	7.39278
add	0.01600
log [ ]	8.83446
$\cos^2 \delta_u$	9.99969
$\log a^2$	8.83415
$\log a_x X_0'$	9.10682
$\log a_y Y_0'$	8.93698
add	0.22436
sum	9.33118
$\log a_z Z_0'$	8.21175
add	0.03180
log [ ]	9.36298
$\cos \delta_u$	9.99984
$\log b$	9.36282

$\log X_0'^2$	9.49168
$\log Y_0'^2$	9.75609
add	0.18865
sum	9.94474
$\log Z_0'^2$	9.03072
add	0.04995
$\log G_0^2$	9.99469
$\log a^2 G_0^2$	8.82884
$\log b^2$	8.72564
sub	9.42852
diff	8.15416
$\log a^4 R^2$	7.67392
$\log q'^2$	0.48024
q'^2	+3.02164
$\log a^2 R$	8.83696
$\log p'$	0.52586

$\log a^2 R^3$	8.84259
$\log h$	1.15741
$\log c$	9.91653n
$\log p'$	0.52586
sub	0.09546
$\log c'$	0.62132n
c'	−4.1814
$\log s$	9.75198
$\log a^2$	8.83415
$\log q'^2$	0.48024
$\log (m' = \cos \vartheta_m)$	9.06637
ϑ_m	83°.31
$\log \frac{h}{s}$	1.40543
$\frac{h}{s}$	+25.435
s	+0.56491

Since $p' > 0$, $c < 0$, $\psi > 125° \; 23'$, and since the criterion for no solution expressed by means of the inequalities on page 306 is not fulfilled, the initial

velocities and accelerations can be represented by a *conditioned* ellipse ($r_0 = a$) with *one* solution.

The details for the solution of z, which are shown fully in Example No. 1, are omitted here. The solution gives $\log z = 0.27450$. To check this, the substitution of this value into the equation for z is here given.

$\log z$	0.27450	sub	0.15793	I	+2.1750
sub	9.89423	$\log (z - c)$	0.43243	II = q'^2	+3.0216
$\log (z - p')$	0.16873n	$\log (z - c)^2$	0.86486	− III	−5.1966
$\log$ I	0.33746	add	0.01852	I + II − III	0.0000
		$\log \mu$	0.88338	Check	
		$\log \mu^{\frac{1}{2}}$	0.44169		
		$\log (\text{III} = h\mu^{-\frac{1}{2}})$	0.71572		

A Vb

$\log R$	0.00281	$\log \frac{\lambda}{\varkappa} \theta_{\prime\prime\prime}$	7.8915	$\log \frac{\lambda}{\varkappa} \theta_{\prime}$	7.8879
$\log z$	0.27450				
$\log \rho_{\prime\prime}$	0.27731	$\log (1 - \frac{\lambda}{\varkappa} \theta_{\prime\prime\prime})$	9.9966	$\log (1 + \frac{\lambda}{\varkappa} \theta_{\prime})$	0.0033
		$\sec \delta_{\prime}$	0.0002	$\sec \delta_{\prime\prime\prime}$	0.0001
		$\log \rho_{\prime}$	0.2740	$\log \rho_{\prime\prime\prime}$	0.2806
$\log \alpha \rho_{\prime\prime}$	8.03859n	$\log \alpha \rho_{\prime}$	8.0353n	$\log \alpha \rho_{\prime\prime\prime}$	8.0419n
$\alpha \rho_{\prime\prime}$	−0.01093	$\alpha \rho_{\prime}$	−0.01085	$\alpha \rho_{\prime\prime\prime}$	−0.01101
$t_{\prime\prime}$ (obs)	25.60822	$t_{\prime}$ (obs)	23.63642	$t_{\prime\prime\prime}$ (obs)	27.56370
$t_{\prime\prime}$	25.59729	$t_{\prime}$	23.62557	$t_{\prime\prime\prime}$	27.55269
		$\log (t_{\prime\prime} - t_{\prime})$	0.29485	$\log (t_{\prime\prime\prime} - t_{\prime\prime})$	0.29124
$\cos \delta_{\prime\prime}$	9.99984	$\log \theta_{\prime\prime\prime}$	8.53043	$\log \theta_{\prime}$	8.52682
$\log \sigma_{\prime\prime}$	0.27715				
$\log \sigma_{\prime\prime} \cos \alpha_{\prime\prime}$	0.27715n				
$\log (X)_{\prime\prime}$	9.91461	$\log a_x \sigma_{\prime\prime}$	9.63813n	$\log x_0 x_0'$	9.52182n
sub	0.15709	$\log X_0'$	9.74584n	$\log y_0 y_0'$	9.45378
$\log x_0$	0.43424n	sub	9.44945	add	9.22943
		$\log x_0'$	9.08758	sum	8.68321n
$\log \sigma_{\prime\prime} \sin \alpha_{\prime\prime}$	6.97556			$\log z_0 z_0'$	8.81582
$\log (Y)_{\prime\prime}$	9.72349	$\log a_y \sigma_{\prime\prime}$	9.33608	add	9.55278
sub	9.99923	$\log Y_0'$	9.87804	$\log r_0 r_0'$	8.23599
$\log y_0$	9.72272n	sub	9.85302	$\log r_0'$	7.79149
		$\log y_0'$	9.73106n		
$\log \sigma_{\prime\prime} \tan \delta_{\prime\prime}$	8.70538n				
$\log (Z)_{\prime\prime}$	9.36073	$\log a_z \sigma_{\prime\prime}$	8.97354		
sub	0.08676	$\log Z_0'$	9.51536		
$\log z_0$	9.44749n	sub	9.85297		
		$\log z_0'$	9.36833n		
$\log x_0^2$	0.86848				
$\log y_0^2$	9.44544	$\log x_0'^2$	8.17516		
add	0.01610	$\log y_0'^2$	9.46212		
sum	0.88458	add	0.02187		
$\log z_0^2$	8.89498	sum	9.48399		
add	0.00442	$\log z_0'^2$	8.73666		
$\log r_0^2$	0.88900	add	0.07149		
$\log r_0$	0.44450	$\log G_0^2$	9.55548		
		Check			
Check		$\log \frac{1}{r_0}$	9.55550		
$\log R^2$	0.00563				
$\log \mu$	0.88338				
$\log r_0^2$	0.88901				

r_0' is not zero since we have solved for a conditioned ellipse as pointed out on pages 305–306. Following the argument of page 306 we must now compute $\frac{1}{m}$ and trace the uncertainty in G_0^2.

A IIIb

log $\alpha_0'^3$	7.17548n	log C_1	9.32551n	log R^4	0.01125
tan $\delta_{\prime\prime}$	8.42823n	log $C_1\ \alpha_0'$	8.38401	cos $\delta_{\prime\prime}$	9.99984
log I	5.60371			log R^4 cos $\delta_{\prime\prime}$	0.01109
		log C_2	9.73261n	log m	0.29768
log α_0''	9.49502	log C_2 (tan $\delta)_0'$	8.47972n	log $\frac{1}{m}$	9.70232
log (tan $\delta)_0'$	8.74711	add	9.39190		
log II	8.24213	sum	7.77591n	$\frac{1}{m}$	+0.503875
		log S	9.99122		
log α_0'	9.05850n	colog N	2.54164n		
log (tan $\delta)_0''$	9.10437n	log $\varkappa$	0.30877n	From Table (Part 7)	
log III	8.16287			$z = 1.89176$	
I	+0.0000402				
− II	−0.0174636				
III	+0.0145503				
N	−0.0028731				
log N	7.45836n				

From the differences of the α and δ it is seen that (assuming the seconds of arc to be without error) we know three figures in the velocities and two in the accelerations in both coördinates. In N the governing terms are the second and the third. The first of these contains α_0'' and the third $(\tan\delta)_0''$. In these then we know two figures, and consequently in N we know but one figure. The terms of the denominator of the expression (11) page 271 are proportional to the velocities so that we know three figures in each. But the computation shows that these terms have opposite signs, and are of such magnitude that we lose one figure in forming the denominator. It is then known to two figures, and the whole expression (11) is known to one figure. We therefore know but one figure in $\frac{1}{m}$. If we had assumed some slight errors of observation this one figure would be somewhat uncertain.

From the table it is seen that to a range of five units in the second decimal place of $\frac{1}{m}$ corresponds a range of two units in the first decimal of z. For $z = 1.9$ this uncertainty causes an uncertainty of six and one half units in the second decimal of log z. Inspecting the computation of x_0, y_0, z_0 and r_0 it is seen that this same uncertainty extends to log r_0. Log G_0^2 is therefore known definitely, at best, only to one decimal. As the characteristic of log r'^2 is $5-10$ it is seen that it cannot affect the definitely known portion of G_0^2 when subtracted to form $(G_0^2 - r_0'^2)$ in the expression $p_0 = r_0^2 (G_0^2 - r_0'^2)$. We can therefore call $r_0' = 0$ and write $p_0 = r_0 = a$ and obtain a *circular* orbit which will represent the three observations as well as the *conditioned ellipse* will.

A VIIIb

$\log x_0 y_0'$	0.16530	$\log x_0 \cos(\Omega)$	0.43387n		$\log a$	0.44450
$\log y_0 x_0'$	8.81030n	$\log y_0 \sin(\Omega)$	8.33671		$\log a^{3/2}$	0.66675
sub	0.01877	add	9.99652		$\log \mu''$	2.88326
$\sqrt{p} \cos(i)$	0.18407	$\log r_0 \cos(u)_0$	0.43039n		μ''	764".29
$\log y_0 z_0'$	9.09105	$\log z_0$	9.44749n			
$\log z_0 y_0'$	9.17855	$\sin(i)$	9.60356			
sub	9.34872	$\log r_0 \sin(u)_0$	9.84393n			
$\sqrt{p} \sin(i) \sin(\Omega)$	8.43977n	$\tan(u)_0$	9.41354			
		$(u)_0$	194° 31′ 42″			
$\log x_0 z_0'$	9.80257					
$\log z_0 x_0'$	8.53507n	$\sin(u)_0$	9.39943n			
sub	0.02284	$\cos(u)_0$	9.98589n			
$\sqrt{p} \sin(i) \cos(\Omega)$	9.82541	$\log r_0$	0.44450	Chec		
$\tan(\Omega)$	8.61436n					
(Ω)	357° 38′ 37″					
$\sin(\Omega)$	8.61399n					
$\cos(\Omega)$	9.99963					
$\sqrt{p} \sin(i)$	9.82578					
$\tan(i)$	9.64171					
(i)	23° 39′ 53″					

Constants for the Equator 1906.0.

$\sin(\Omega)$	8.61399n	$\cos(\Omega)$	9.99963		
$\cos(i)$	9.96185	$\cos(i)$	9.96185		
$\sin a \cos(A)$	8.57584	$\sin b \cos(B)$	9.96148		
$\sin a \sin(A)$	9.99963	$\sin b \sin(B)$	8.61399n		
$\tan(A)$	1.42379	$\tan(B)$	8.65251n		
(A)	87° 50′ 30″	(B)	−2° 34′ 21″		
$(u)_0$	194 31 42				
A'	282 22 12	B'	191 57 21	C'	194° 31′ 42″
$\sin(A)$	9.99969	$\sin(B)$			
$\cos(A)$		$\cos(B)$	9.99956		
$\sin a$	9.99994	$\sin b$	9.96192	$\sin c$	9.60356
$\log \alpha$	0.44444	$\log \beta$	0.40642	$\log \gamma$	0.04806

Representation of the First and Third Places.

	I	III
$\log(t - t_0)$	0.29485n	0.29124
$\log \mu(t - t_0)$	3.17811n	3.17450
$\mu \Delta t$	−0° 25′ 7″	+0° 24′ 55″
$A' + \mu \Delta t$	281 57 5	282 47 7
$B' + \mu \Delta t$	191 32 14	192 22 16
$C' + \mu \Delta t$	194 6 35	194 56 37
$\sin[A' + \mu \Delta t]$	9.99048n	9.98910n
$\log x$	0.43492n	0.43354n
$\sin[B' + \mu \Delta t]$	9.30104n	9.33091n
$\log y$	9.70746n	9.73733n
$\sin[C' + \mu \Delta t]$	9.38700n	9.41140n
$\log z$	9.43506n	9.45946n
x	−2.72219	−2.71356
X	+0.84334	+0.80573
ξ	−1.87885	−1.90783
y	−0.509867	−0.546175
Y	+0.503136	+0.554147
η	−0.006731	+0.007972
z	−0.272306	−0.288047
Z	+0.218260	+0.240390
ζ	−0.054046	−0.047657

	I	III
$\rho \cos \delta \cos \alpha$	0.27390n	0.28054n
$\rho \cos \delta \sin \alpha$	7.82808n	7.90157
$\tan \alpha$	7.55418	7.62103n
α	180° 12′ 19″	179° 45′ 38″
α (obs)	180 12 15	179 45 38
(O − C) $\Delta \alpha$	− 4	0
$\cos \alpha$	0.00000n	0.00000n
$\rho \cos \delta$	0.27390	0.28054
$\rho \sin \delta$	8.73277n	8.67812n
$\tan \delta$	8.45887n	8.39758n
δ	− 1° 38′ 52″	− 1° 25′ 51″
δ (obs)	− 1 38 50	− 1 25 52
(O − C) $\Delta \delta$	+ 2	− 1
$\cos \delta$	9.99982	9.99986
$\log \rho$	0.27408	0.28068
$\log p_{\alpha\rho}$	9.567	0.344n
$\log p_{\delta\rho}$	0.755	0.753
$\log p_{\alpha}$	9.293	0.063n
$\log p_{\delta}$	0.481	0.472
p_{α}	+ 0″.2	− 1″.2
p_{δ}	+ 3 .0	+ 3 .0

A VIb

Transformation to Ecliptical Elements.

sin (i)	9.60356
cos (Ω)	9.99963
log m sin M	9.60319
log m cos M	9.96185
tan M	9.64134
M	23° 38′ 49″
sin M	9.60325
cos M	9.96191
log m	9.99994
cos (i)	9.96185
log n cos N	9.96148
log n sin N	9.60356
tan N	9.64208
N	23° 40′ 58″
sin N	9.60387
cos N	9.96179
log n	9.99969

ε	23° 27′ 5″
$M - \varepsilon$	0 11 44
sin ($M - \varepsilon$)	7.53315
sin i cos Ω	7.53309
sin i sin Ω	8.21755n
tan Ω	0.68446n
Ω	281° 41′ 2″
sin Ω	9.99091n
sin i	8.22664
i	0° 57′ 56″
cos ($M - \varepsilon$)	0.00000
cos i	9.99994

sin ε	9.59985
sin i sin σ	8.21384n
$N - \varepsilon$	0° 13′ 53″
sin ($N - \varepsilon$)	7.60622
sin i sin σ	7.60591
tan σ	0.60793n
σ	283° 51′ 18″
$(u)_0$	194 31 42
u_0	270 40 24

COLLECTION OF RESULTS.

Elements.

Epoch = t_0 = 1906 April 25.5973 Gr. M. T.

u_0	270° 40′ 24″	Ecliptic and Equinox 1906.0
Ω	281 41 2	
i	0 57 56	
log a	0.44450	
μ	764″.29	

Constants for the Equator 1906.0.

$$x = [0.44444] \sin (282^\circ\ 22'\ 12'' + \mu\,\Delta t)$$
$$y = [0.40642] \sin (191\ 57\ 21 + \mu\,\Delta t)$$
$$z = [0.04806] \sin (194\ 31\ 42 + \mu\,\Delta t)$$

Method of General Solution.

The value of z for the general solution taken from the table is $z = 1.89176$ (*cf.* the computation under A IIIb, page 404). Checking this by the method of A IVc we find the final value of log z = 0.27687. With this z we find from A V

log x_0	0.43589n	log x_0'	9.07905	log a	0.44512
log y_0	9.72271n	log y_0'	9.73011n	log r_0'	7.89111
log z_0	9.44792n	log z_0'	9.36737n		
log r_0	0.44608	log G_0^2	9.55296		

The preceding discussion showed that the uncertainty in log r_0 amounts to six units of the second decimal place. In the general solution log r_0 differs from log a by only one unit in the third place. Therefore, we are justified in calling log a = log r_0. Further, as before, it is evident that r_0' can be put equal to zero, and the two criteria for the circle hold (*cf.* page 306).

With the values of the heliocentric coördinates and their velocities derived from the general solution for z the elements and constants for the Equator are computed exactly as before under A VIb. They are:

Elements.

Epoch = t_0 = 1906 April 25.5972 Gr. M. T.

u_0	270° 49′ 59″	Ecliptic and Equinox 1906.0
Ω	281 28 33	
i	0 58 3	
$\log a$	0.44608	
μ	760″.13	

Constants for the Equator 1906.0.

$$x = [0.44602] \sin (282^\circ\ 19'\ 35'' + \mu\,\Delta t)$$
$$y = [0.40801] \sin (191\ \ 54\ \ 40\ \ + \mu\,\Delta t)$$
$$z = [0.04959] \sin (194\ \ 29\ \ 27\ \ + \mu\,\Delta t)$$

This set of circular elements represents the first and third observations as follows:

		I	III
(O − C)	$\Delta\alpha$	−2″	+3″
	$\Delta\delta$	+1	−1

The second method is to be preferred since no part of the computation needs to be discarded if the criteria fail and a general solution becomes necessary.

EXAMPLE NO. 4.

DIRECT SOLUTION OF A COMET ORBIT WITHOUT HYPOTHESIS REGARDING THE ECCENTRICITY.

For an example of a direct solution of a comet orbit without hypothesis regarding the eccentricity the computation on the orbit of FAYE'S comet is given. This comet was picked up by CERULLI in November, 1910, and was announced as an unexpected comet. From the preliminary elements Professor LEUSCHNER identified it as FAYE'S comet. The identification was made certain after the derivation of elements without assumption regarding the eccentricity. The computations were made by W. F. MEYER and Miss SOPHIA LEVY.

The observations upon which the computation is based are:

	1910 Gr. M. T.	α (1910.0)	δ (1910.0)	Observer.
I	Nov. 9.3131	54° 38′ 11″.2	+8° 43′ 3″	MILLOSEVICH, Rome
II	Nov. 11.5801	54 35 57 .4	+8 9 0	EPPES, Washington
III	Nov. 13.8217	54 32 54 .3	+7 36 17	YOUNG, Lick

Solar Coördinates (1910.0)

$X_{\prime}$	−0.681211	$(X)_{\prime\prime}$	−0.651757	$X_{\prime\prime\prime}$	−0.621586
$Y_{\prime}$	−0.659264	$(Y)_{\prime\prime}$	−0.683218	$Y_{\prime\prime\prime}$	−0.705867
$Z_{\prime}$	−0.285987	$(Z)_{\prime\prime}$	−0.296406	$Z_{\prime\prime\prime}$	−0.306201

NOTE.—The solar coördinates for the middle date have been corrected so as to eliminate the geocentric parallax for that place.

Further

$\log X_0'$	9.885999	$\log R$	9.995487
$\log Y_0'$	9.778785n	$A-\alpha_{\prime\prime}$	171° 45′ 3″
$\log Z_0'$	9.416017n	$\log S$	9.975079
$\log G_0^2$	0.00884		

The details of the computation that are similar to those of Example No. 1 will be omitted here. The values of the principal qantities are given.

A IIc

$\log \alpha_0'$	8.29767n	$\log \delta_0'$	9.39859n	$\log (\tan \delta)_0'$	9.40741n
$\log \alpha_0''$	9.21670n	$\log \delta_0''$	9.26776	$\log (\tan \delta)_0''$	9.31676

A IIIc

$\log (\alpha_0')^3$	4.89301n	$\log C_{\prime}$	9.23599	$\log R^4$	9.98195
$\tan \delta_{\prime\prime}$	9.15598	$\log \alpha_0'$	8.29767n	$\cos \delta_{\prime\prime}$	9.99559
log I	4.04899n	$\log \frac{1}{2}$	7.53366n	colog κ	0.08666n
				$\log \frac{1}{m}$	0.06420
$\log (\tan \delta)_0'$	9.40741n	$\log C_2$	9.15679	$\frac{1}{m}$	+1.15932
$\log \alpha_0''$	9.21670n	$\log (\tan \delta)_0'$	9.40741n		
log II	8.62411	log II	8.56420n	$\log m$	9.93580
		add	0.03870		
$\log (\tan \delta)_0''$	9.31676	log sum	8.60290n		
$\log \alpha_0'$	8.29767n	colog N	1.33536n	ψ	167° 43′ 36″
log III	7.61443n	$\log S$	9.97508		
		$\log \kappa$	9.91334n		
I	−0.000001			$\log c$	9.989959
−II	−0.042083			$\log s^2$	8.655028
III	−0.004116				
N	−0.046200				
$\log N$	8.66464n				

With ψ and $\frac{1}{m}$ as arguments take a starting value of z from the table, Part 7. It is (calling it z_1)

$$\log z_1 \quad 9.830191$$

A IVc

Solution for z.

$\log z_1$	9.830191	$\log 3\,\nu_1$	9.74710n
$\log c$	9.989959n	log II	9.96551n
sub	0.228452	log I	0.44394
$\log (z_1 - c)$	0.218411	add	9.82456
$\log (z_1 - c)^2$	0.436822	log [ ]	0.26850
$\log s^2$	8.655028	$\log \nu_1$	9.26998n
add	0.007119	$\log \mu_1^2$	0.88788
$\log \mu_1$	0.443941	log 2	0.30103
		log denom	0.72739n
$\log m$	9.935800	$\log (z_2 - z_1)$	5.7782
sub	9.439791	add	0.000039
$\log \nu_1$	9.269982n	$\log z_2 = z$	9.830230
$\log \mu_1^3$	1.331823		
$\log \nu_1^2$	8.539964		
$\log \mu_1^3\,\nu_1^2$	9.871787		
$\log m^2$	9.871600		
sub	6.634		
$\log M_1$	6.5056		

A Vc

The computation from this point is exactly like that of Example No. 1, so the details will not be given. The principal quantities resulting from this value of z are:

$\log x_0$	0.01521	$\log r_0$	0.21747
$\log y_0$	0.08756	$\log G_0^2$	9.97171
$\log z_0$	9.59252		
$\log x_0'$	9.87604n	$\log \frac{1}{a}$	9.43966
$\log y_0'$	9.78011	$\log r_0'$	7.46052n
$\log z_0'$	8.96830		

The representation of the first and the third places gives

		I	III
(O − C)	$\Delta\alpha \cos\delta$	0″	+ 3″
	$\Delta\delta$	+ 2	− 3

Using the values of the heliocentric rectangular coördinates and their velocities given under A Vc the following elements result from A VIIIc:

T	1910 Nov. 12.4129 Gr. M. T.	
ω	206° 20′ 22″	1910.0
Ω	205 29 5	1910.0
i	10 14 11	1910.0
e	0.54590	
μ	512″.34	
$\log a$	0.56034	
Period	6.926 years.	

SUPPLEMENTARY DISCUSSION.

We will consider here the same points that were discussed under Example No. 1, viz., number of solutions, effect of neglecting third differences, parallax elimination, and character of orbit. The discussion will be condensed here; for the details of a full analysis see Example No. 1.

1. Computing the quantities p' and c under A IIIa we find $p' > 0$; $c < 0$. Then since $\psi > 125° \, 16'$ we might have but one parabolic solution. With $\frac{1}{m}$ and ψ as arguments the table gave but one *general* solution.

2. Reduced to a 2¼ day interval we have

f^{I}_α	f^{II}_α	f^{I}_δ	f^{II}_δ
133″		2028″	
	49″		58″
184		1970	

The intervals are practically equal (2.242 and 2.267 days respectively) so that no error is introduced into the accelerations of α and δ by neglecting third differences.

f^{II}_α is about $\frac{1}{3}\,f^{\text{I}}_\alpha$, and f^{II}_δ is about $\frac{1}{30}\,f^{\text{I}}_\delta$. Assuming f^{III} to bear the same ratio to f^{II} we have f^{III} to be approximately 16″ and 2″ in α and δ, respectively.

For the velocities, therefore, the errors would be $\frac{f^{\text{III}}}{6} = 3''$ in α and 1″ in δ. These are not greater than the errors of observation, so that we can conclude that neglecting the third differences produces no greater error or uncertainty than is caused by the accidental errors of observation.

3. The discussion of general parallax (geocentric and barycentric) is made clear by aid of the table of values of the parallax factors for the three observations. (In this the letter g distinguishes the geocentric data, and m the barycentric. A detailed computation of barycentric parallax factors is given in Example No. 6.)

	I	II	III		I	II	III
$p^g_\alpha\rho$	− 5″.82	− 5.49	− 0.97	$p^g_\delta\rho$	+ 5″.30	+ 4″.85	+ 4″.35
$p^m_\alpha\rho$	+ 5 .95	+ 5.97	+ 4.27	$p^m_\delta\rho$	+ 2 .28	+ 1 .61	+ 0 .38
$p_\alpha\rho$	+ 0 .13	+ 0.48	+ 3.30	$p_\delta\rho$	+ 7 .58	+ 6 .46	+ 4 .73

The greatest difference between two parallax factors is 2″.8, so that neglecting parallax can produce no greater error than neglecting third differences. Hence, in this example, as in Example No. 1, just the geocentric parallax was eliminated from the middle place.

4. To comply with the custom of publishing a parabola for a preliminary orbit, a parabola had been computed from these same observations, so that we have the parabolic z, viz., $\log z_p = 0.09037$.

The degree of accuracy of N is here again the degree of accuracy of α''_0, *i. e.*, of f^{II}_α the principal term upon which α''_0 depends. Since the error in this is the same as the error of observation, say about 5″ or $\frac{1}{10}$ part of its value, the uncertainty of N and also of $\frac{1}{m}$ is $\frac{1}{10}$ of its value. The value of $\frac{1}{m}$ is 1.159; hence its uncertainty is 0.12. In the Tables a range of 0.01 in z corresponds to a range of 0.0115 in $\frac{1}{m}$. Then $z - z_p = 0.555$ corresponds to a range of 0.638 in $\frac{1}{m}$. This is more than five times the uncertainty in $\frac{1}{m}$ (viz. 0.12). This then leads us to conclude that the parabola can not hold and that solution must be made without assumption regarding the eccentricity.

EXAMPLE NO. 5.

DIRECT SOLUTION OF THE ORBIT OF A MINOR PLANET WITHOUT HYPOTHESIS REGARDING THE ECCENTRICITY.

For a direct solution of the orbit of a minor planet, without hypothesis regarding the eccentricity, the object chosen is 1909 HC. The details that are similar to those of previous examples are not given. The computation by R. K. YOUNG and H. C. WILSON is based upon three observations made by Mr. WILSON at the Lick Observatory.

A Ic

	1910 Gr. M. T.	α (1910.0)	δ (1910.0)
I	Nov. 19.6507	3° 13′ 3″.2	+23° 26′ 40″.9
II	Nov. 26.7480	3 13 3 .0	+22 29 31 .3
III	Dec. 3.7632	3 29 28 .6	+21 38 48 .1

Attention is directed to the fact that, in this case, the right ascensions for the first two dates are practically the same, so that a large uncertainty might be expected due to the neglected third and higher differences.

Solar Coördinates.

$X_{,}$	−0.538875	$X_{,,}$	−0.430662	$X_{,,,}$	−0.317074
$Y_{,}$	−0.759593	$Y_{,,}$	−0.814331	$Y_{,,,}$	−0.856021
$Z_{,}$	−0.329503	$Z_{,,}$	−0.353249	$Z_{,,,}$	−0.371338

Owing to the irregularity of motion in α the parallax was entirely neglected.

Solar Velocities.

$\log X_0'$	9.961993
$\log Y_0'$	9.599906n
$\log Z_0'$	9.237278n

A IIc

$\log \alpha_0'$	8.29905	$\log \delta_0'$	9.11121n	$\log (\tan \delta)_0'$	9.17994n
$\log \alpha_0''$	9.51359	$\log \delta_0''$	9.05983	$\log (\tan \delta)_0''$	9.17794

A IIIc

$\log n$	9.17843	$\log \frac{\lambda}{\kappa}$	9.44318	$\log c$	9.76525n
$\log C_1$	9.22959	$\log \kappa$	0.36895n	$\log s$	9.91002
$\log C_2$	9.93266n	$\log N$	8.71917	ψ	125° 37′ 20″
$\log \Phi$	9.89418	$\frac{1}{m}$	+0.37433		

Entering the table (Part 7) with these values of ψ and $\frac{1}{m}$ the value of z was found for which $\log z = 0.41426$. Substituting this into the general equation for z, as shown in Example No. 4, the final value of z is found to give $\log z = 0.41425$, whence $\log \rho_0 = 0.40839$.

Using this value of ρ the middle date was corrected for aberration, and $\alpha_{,,}$ and $\delta_{,,}$ were corrected for parallax.

A Vc

$\log x_0$	0.44607	$\log x_0'$	9.42066n
$\log y_0$	9.97642	$\log y_0'$	9.68296
$\log z_0$	0.12481	$\log z_0'$	8.93672
$\log r_0$	0.51007	$\log r_0'$	8.70503n
$\log a$	0.51023	$\log G_0^2$	9.49009

It may be pointed out here that the application of the criteria for a circular orbit shows that these observations may be represented by a circular orbit as well as by a general orbit. Examination of the computation, exactly as in Example No. 3, shows that the uncertainty in r_0 amounts to 0.03. This is far greater than the difference between r_0 and a which is only 0.002. Further, $r_0'^2$ will not affect the definitely known part of G_0^2 in the expression $p = r_0\,(G_0^2 - r_0'^2)$. Hence we could put $p = a = r_0$ and proceed with a circular orbit.

However, continuing the computation for the general orbit, the representation of the first and third places is

		I	III
(O – C)	$\Delta\alpha \cos\delta$	$-3''$	$+1''$
	$\Delta\delta$	$+13$	-12

The intervals being equal, and the residuals in δ being practically equal with opposite signs, it is seen at once that they can be removed by an arbitrary change of z_0', so that no differential correction is necessary. Changing $\log z_0'$ from 8.93672 to 8.92958 removes the residuals in declination.

The elements were not computed from the values of the heliocentric coördinates and their velocities given above as the arc was extended and a new orbit computed. This second orbit was derived by differential correction on the basis of the series for f, g, ∂f, ∂g and is given as Example No. 9.

EXAMPLE NO. 6.

DIRECT SOLUTION AND COMPLETE ELIMINATION OF PARALLAX.

To illustrate the method of the *complete* elimination of the parallax in the case of a direct solution the orbit of Comet *e* 1909 (Daniel) is selected. The computations were performed by S. EINARSSON and Miss W. AITKEN.

I.

The observations, reduced to 1909.0, upon which this work is based are:

	1909 Gr. M. T.		α (1909.0)	δ (1909.0)	Observer.
I	December	11.74000	94° 23′ 43″.3	+38° 6′ 15″.2	Aitken
II	December	15.70788	94 29 59 .2	41 12 20 .6	Aitken
III	December	18.75589	94 30 46 .4	+43 25 59 .3	Aitken

The solar coördinates, reduced to (1909.0), are:

$X_{,}$	−0.1780529	$X_{,,}$	−0.1093873	$X_{,,,}$	−0.0562684
$Y_{,}$	−0.8882410	$Y_{,,}$	−0.8971793	$Y_{,,,}$	−0.9010569
$Z_{,}$	−0.3853174	$Z_{,,}$	−0.3891974	$Z_{,,,}$	−0.3908799

The velocities of the solar coördinates for the middle date are:

$\log X_0'$	0.004615
$\log Y_0'$	8.994638n
$\log Z_0'$	8.632210n

II.

$\log \alpha_0'$	8.148227	$\log \delta_0'$	9.883022	$\log (\tan \delta)_0'$	0.130182
$\log \alpha_0''$	9.568359n	$\log \delta_0''$	9.932162n	$\log (\tan \delta)_0''$	9.468466

Also:

$\log (t_0 - t_{,})$	0.598559	$\log \theta_{,,,}$	8.834140	$\log (t_{,,,} - t_{,})$	0.846082
$\log (t_{,,,} - t_0)$	0.484016	$\log \theta_{,}$	8.719597		

Determination of $p_\alpha \rho$ and $p_\delta \rho$ (general parallax factors). Footnote, page 313.

	I	II	III
α	94° 24′	94° 30′	94° 31′
$\alpha_{\odot}$	251 14	307 33	348 59
$\alpha - \alpha_{\odot}$	203 10	146 57	105 32
$\delta_{\odot}$	−22 49	−23 31	−10 24
$\log d_{,}$	5.4943		
$\cos \delta_{\odot}$	9.9646	9.9623	9.9928
$\cos \alpha_{\odot}$	9.5075n	9.7849	9.9919
$\log \Delta_1 X$	4.9964	5.2415n	5.4790n
$\sin \alpha_{\odot}$	9.9763n	9.8992n	9.2812n
$\log \Delta_1 Y$	5.4352	5.3558	4.7083
$\sin \delta_{\odot}$	9.5886n	9.6010n	9.2565n
$\log \Delta_1 Z$	5.0829	5.0953	4.7508

	I	II	III
$\cos \delta$	9.8959	9.8765	9.8610
$\sin (\alpha - \alpha_{\odot})$	9.5948n	9.7367	9.9838
$\log p_\alpha''' \rho$	0.4722n	0.6312	0.9243
$\log \mathrm{I}$	9.4845n	9.4775n	9.1175n
$\cos (\alpha - \alpha_{\odot})$	9.9635n	9.9233n	9.4278n
$\sin \delta$	9.7903	9.8187	9.8373
$\log \mathrm{II}$	9.7184n	9.7043n	9.2579n
sub	9.8535	9.8362	9.5817
$\log \{ \; \}$	9.3380n	9.3137n	8.6992n
$\log p_\delta''' \rho$	0.1407n	0.1224n	9.5079n

The geocentric parallax factors ($p^g\rho$) were taken from *Publications of the Lick Observatory*, Volume I, all of the observations having been made at the Lick Observatory.

log $p_\alpha^{m}\rho$	5.1578n	5.3168	5.6099	log $p_\delta^{m}\rho$	4.8323	4.8080n	4.1935n
log $p_\alpha^{n}\rho$	5.5187n	5.5810n	5.4505n	log $p_\delta^{n}\rho$	4.8223	4.8617	
add	0.1570	9.9230	9.6470	add	8.3670	9.1193	
log $p_\alpha\rho$	5.6757n	5.2398n	5.0975	log $p_\delta\rho$	3.1893n	3.9273	4.1935n

Formulæ page 313, footnote.

	I	II	III
cos δ	9.8959	9.8765	9.8610
sin α	9.9987	9.9987	9.9986
log $p_\alpha\rho$	5.6757n	5.2398n	5.0975
log I	5.5703n	5.1150n	4.9571
sin δ	9.7903	9.8187	9.8373
cos α	8.8849n	8.8946n	8.8962n
log $p_\delta\rho$	3.1893n	3.9273	4.1935n
log II	1.8645n	2.6406n	2.9270
add	.0000	0.0015	0.0040
log $\Delta_2 X$	5.5703n	5.1165n	4.9611
log I	4.4565n	4.0109n	3.8547
log II	2.9783n	3.7447	4.0294n
add	0.0142	9.9273	9.6950
log $\Delta_2 Y$	4.4707n	3.6720n	3.5497n
log $\Delta_2 Z$	3.0852	3.8038n	4.0545
$\Delta_1 X$	+ 0.00000926	− 0.00001744	− 0.00003013
$\Delta_2 X$	− 3178	- 1308	+ 914
ΔX	− 279	− 305	− 210
$\Delta_1 Y$	+ 0.00002724	+ 0.00002269	− 0.00000587
$\Delta_2 Y$	− 296	− 47	− .35
ΔY	+ 243	+ 222	+ 55
$\Delta_1 Z$	+ 0.00001210	+ 0.00001245	+ 0.00000563
$\Delta_2 Z$	+ 12	− 64	+ 113
ΔZ	+ 122	+ 118	+ 68

For Middle Date

(R) cos D cos A	9.039088n
(R) cos D sin A	9.952869n
tan A	0.913781
A	263° 2′ 47″.5
$A - \alpha_n$	168 32 48 .3
sin A	9.996794n
cos A	9.083012n
(R) cos D	9.956075
(R) sin D	9.590157n
tan D	9.634082n
sin D	9.597145n
cos D	9.963063
log (R)	9.993012
sin δ_n	9.818730
log I	9.415875n
cos δ_n	9.876420
cos $(A - \alpha_n)$	9.991264n
log II	9.830747n
add	0.141358
log c = cos ψ	9.972105n
ψ	159° 41′ 0″.0
log s = sin ψ	9.540590

	I	II	III
(X)	−0.1780808	−0.1094178	−0.0562894
(Y)	−0.8882167	−0.8971571	−0.9010514
(Z)	−0.3853052	−0.3891856	−0.3908731

Formulæ page 313, footnote.

P	X	Y	Z
log $\Delta_2 P_{\prime\prime\prime}$	4.9611	3.5497n	4.0545
log I	5.5597	4.1483n	4.6531
log $\Delta_2 P_\prime$	5.5703n	4.4707n	3.0852
log II	6.0543n	4.9547n	3.5692
add	9.8324	0.0630	0.0344
log numer.	5.8867n	5.0177n	4.6875
log $\Delta_2 P_{\prime\prime}$	5.1165n	3.6720n	3.8038n
log denom.	5.9626n	4.5181n	4.6499n
log d_x ; d_y ; d_z	9.2051n	0.3343	0.3202n

Auxiliary Quantities, j, a, etc.

Formulæ pages 313 and 315, footnote.

colog $(R)^3$	0.0210
$\log 2 \frac{d_x}{\theta_, \theta_{,,,}}$	1.9524n
add	9.9949
log [ ]	1.9473n
$\log \Delta_2 X_{,,}$	5.1165n
$j \cos a$	7.0638
$\log 2 \frac{d_y}{\theta_, \theta_{,,,}}$	3.0816
add	0.0004
log [ ]	3.0820
$\log \Delta_2 Y_{,,}$	3.6720n
$j \sin a$	6.7540n
$\log 2 \frac{d_z}{\theta_, \theta_{,,,}}$	3.0675n
add	9.9996
log [ ]	3.0671n
$\log \Delta_2 Z_{,,}$	3.8038n
$j \operatorname{tg} d$	6.8709
$\tan a$	9.6902n
a	333° 54′
$\sin a$	9.6435n
$\cos a$	9.9533
$\log j$	7.1106
$\tan d$	9.7604

$\log X_{,,} \Delta_2 X_{,,}$	4.1556
$\log Y_{,,} \Delta_2 Y_{,,}$	3.6249
add	0.1121
sum	4.2677
$\log Z_{,,} \Delta_2 Z_{,,}$	3.3940
add	0.0545
$\log R_{,,} \Delta_2 R_{,,}$	4.3222
$\log \Delta_2 R_{,,}$	4.3292
$a - \alpha$	239° 24′
$\log c_2$	9.9349n
$\cos (a - \alpha)$	9.7068
$\log c_3$	6.8174n
$\tan \delta_{,,}$	9.9423
$\log \tan \delta_{,,} \cos (a - \alpha)$	9.6491n
$\tan d$	9.7604
sub	0.2489
$\log c_1$	0.0093n
$\log \gamma$	9.9256

From formulæ A III we have

$\log \Gamma$	9.666908n
$\log \Phi$	9.787231n
then	
$\log \frac{\Gamma}{\gamma}$	9.7413n
$\log\left(1 - \frac{\Gamma}{\gamma}\right)$,	0.1907
$\log j$	7.1106
$\log c_2$	9.9349n
colog (2Φ)	9.9118n
$\log \beta$	7.1480n

Formulæ page 315, foot-note.

Computing $(\Delta X)'$ from $\Delta X_,$, $\Delta X_{,,}$, and $\Delta X_{,,,}$ from the same formulæ used in getting α_0' from $\alpha_,$, $\alpha_{,,}$ and $\alpha_{,,,}$, and similarly for $(\Delta Y)'$ and $(\Delta Z)'$ their values are $\log (\Delta X)' = 5.9341$, $\log (\Delta Y)' = 6.2867n$ and $\log (\Delta Z)' = 5.7519n$.

Then

P	X	Y	Z
$\log P_0'$	0.004615	8.994638n	8.632210n
$\log (\Delta P)'$	5.9341	6.2867n	5.7519n
add	0.000037	0.000850	0.000572
sum	0.004652	8.995488n	8.632782n
log III	6.0426	7.1467n	7.0903n
add	0.000048	0.006109	0.012278
$\log [P]'$	0.004700	9.001597n	8.645060n

From III we have $\log N = 9.702125$, and $\log m = 9.823715$; hence

$\log c_1 \alpha_0'$	8.1575n
$\log c_2 (\tan \delta)_0'$	0.0651n
add	0.0053
log [ ]	0.0704n
$\log j$	7.1106
colog N	0.2979
$\log \Delta \kappa$	7.4789

IVc

$\log 3^{\tau} \Delta_2 R_{,,}$	4.8063
$\log R_{,,}$	9.993[illegible]
$\log \frac{3 \Delta_2 R_{,,}}{R_{,,}}$	4.8133
$\log \left(1 + \frac{3 \Delta_2 R_{,,}}{R_{,,}}\right)$	0.0000
$\log m$	9.823715
log I	9.823715
$\log II = \frac{\Delta \kappa}{R_{,,} \cos \delta_{,,}}$	7.6[illegible]95
add	0.002644
$\log (m)$	9.826359

This completes the computation of the auxiliary quantities necessary for the complete elimination of the parallax. The values of both (m) and m are used in the solution of the general equation when solution is made without hypothesis

regarding the eccentricity as indicated in the *Synopsis of Formulæ*, page 317. For a parabolic solution this last quantity is not needed, and the computation of the auxiliaries stops with the determination of $[X]'$, $[Y]'$ and $[Z]'$. The remaining portions of the computation are exactly like those already given in previous examples and will, therefore, be omitted here.

SUPPLEMENTARY DISCUSSION.

Following the lines of the complete discussion of Example No. 1, we have:

$t_{II} - t_{I}$	3.9679	$\alpha_{II} - \alpha_{I}$	375″.9	$\delta_{II} - \delta_{I}$	11165″.4
$t_{III} - t_{II}$	3.0480	$\alpha_{III} - \alpha_{II}$	47 .2	$\delta_{III} - \delta_{II}$	8018 .7

On the basis of a 3.5 day interval these give:

f^{I}_{α}	f^{II}_{α}	f^{I}_{δ}	f^{II}_{δ}
331″.6		9208″.6	
	276″.8		638″.8
54 .2		9848 .8	
Mean 193		Mean 9529	

The means 193″ and 9529″ in f^{I}_{α} and f^{I}_{δ} respectively are the principal quantities upon which the velocities depend. The ratios of f^{II} to f^{I} are then 1.4 and 0.07 respectively. Assuming a like ratio to connect f^{III} and f^{II} we would have $f^{\text{III}}_{\alpha} = 393''$ and $f^{\text{III}}_{\delta} = 43''$. The velocity and acceleration are more determinate in δ than in α. The error in α', due to the neglected third differences is $\frac{f^{\text{III}}}{6} = 66''$. Comparing this with f^{I} it is seen that the error in α' is about $\frac{1}{3}\,\alpha'$. In α'' the error is

$$\frac{\theta_{III} - \theta_{I}}{3}\,\frac{f^{\text{III}}}{\theta} = \frac{0.92}{3}\,\frac{f^{\text{III}}}{3.5} = 36''.$$

Comparing this with f^{II}, the principal part of α'', it is seen that the error in α'' is about $\frac{1}{8}\,\alpha''$. Similarly the error in δ' is about $\frac{1}{1360}\,\delta'$, and in δ'' it is about $\frac{1}{160}\,\delta''$.

On account of the great uncertainty in α' we can conclude here that it is not worth while to take into account the *complete* elimination of the parallax. It is to be noted here, moreover, that, for the mere representation of the observations used in determining the orbit, it will be very seldom that the *complete* elimination should be used. It has been given here, however, in complete detail merely for the purpose of showing by an example how it is used. In first orbits these considerations are of more theoretical than practical value.

A glance at the parallax factors shows that even for a small ρ the p_{δ} would not exceed the errors of observation, and could, therefore, be omitted from the computation without loss of accuracy.

Further, the differences between the successive values of $p_{\alpha}\rho$ are constant, viz., 6″.2, hence their second differences equal 0. But advantage has not been taken of these ideas here, for, as noted before, we give the complete elimination of the parallax for the sake of an example.

Let us now investigate the influence of parallax upon the accuracy of z. The computation for N A IIIa shows

$$N = 0.0000 + 0.4995 + 0.0041.$$

The accuracy of N and therefore of $\frac{1}{m}$ depends upon that of the second term which involves α''. Since the second differences of $p_a\rho$ are 0, the *accuracy* of z will ***not*** be affected by the neglect of parallax. (The actual numerical value may be affected).

In order to test the workings of the complete parallax elimination numerically three orbits of this comet were computed from these same observations (those given at the beginning of this example). One was computed neglecting parallax entirely; in the second the parallax was completely eliminated as illustrated here; and in the third, the three observations were corrected for parallax at the start on the basis of the values of ρ found for the dates of observation from the second orbit.

All three will represent the observations upon which they are based, on account of the short intervals. While the accelerations are not affected by neglecting parallax, the velocities will be affected by neglecting the f^1 of the parallax factors, viz, 6″.2. This, however, will not become appreciable until we try to represent a place with a long interval, as x_0', y_0' and z_0' are multiplied by g in such representation.

Let us call the three orbits (a), (b), and (c) respectively in the order given previously. The values of z resulting for these are:

	z	
(a)	0.4283	(Parallax neglected)
(b)	0.4364	(Parallax completely eliminated)
(c)	0.4370	(Observations corrected for parallax).

The second and third agree within the uncertainty of calculation. By neglecting parallax, however, we get a difference in z of 0.0084. (This difference would have been greater but for the fact that f^1 in the parallax factors is constant, and $p_b\rho$ is so small for all three places.)

The resulting representations of the first and the third observations for the three orbits are:

O - C	(a)	(b)	(c)
$\partial_,\alpha_,$	+11″	+13″	+12″
$\partial\,\delta_,$	-36	-38	-38
$\partial_,\alpha_{,,,}$	- 8	- 7	- 6
$\partial\,\delta_{,,,}$	+17	+15	+14

On account of the short intervals (small g) the representations are practically the same for all three orbits. But compared with an observation taken 1910, March 3, two and one half months after the date of the middle observation (the epoch), the residuals are:

(O - C)	(a)	(b)	(c)
$\partial_,\alpha$	-9′.8	-5′.4	-5′.0
$\partial\,\delta$	+6.5	-2.0	-2.7

The last two are practically the same and are much better than the first. Therefore we would get a much better ephemeris by making the complete elimination of the parallax. This, however, is only of theoretical interest as we would have a second or later orbit from long intervals for an extended ephemeris.

EXAMPLE NO. 7.

DIRECT SOLUTION IN THE CASE OF MULTIPLE MATHEMATICAL SOLUTIONS, AND APPLICATION OF THE CRITERIA FOR THE DETERMINATION OF THE PHYSICAL SOLUTION.

This is an example for the direct solution of an orbit in the case of three mathematical solutions, showing the application of the criteria, page 291, for the determination of the physical solution. The object chosen is Comet *a* 1910. The orbit has been derived from short intervals by Miss SOPHIA H. LEVY.

The observations referred to 1910.0, inclusive of the aberration terms, are:

1910 Gr. M. T.	α (1910.0)	δ (1910.0)	Place of Observation
Jan. 18.1287	303° 32′ 52″	−20° 53′ 27″	Rome
Jan. 19.0166	307 1 0	−17 43 31	Algier
Jan. 20.0266	310 11 24	−14 25 38	Algier

Exactly as in Example No. 1 the following values were found for the auxiliary quantities:

Ia

X_u	+0.470804	$\log X_0'$	9.951520	$\log R$	9.992992
Y_u	−0.792680	$\log Y_0'$	9.645695	A	300° 42′ 28″
Z_u	−0.343865	$\log Z_0'$	9.283051	D	−20 27 15
ψ	6° 33′ 20″	$\log c$	9.997151	$\log s$	9.057539

IIa

$\log \alpha_0'$	0.557023	$\log \delta_0'$	0.541683	$\log (\tan\delta)_0'$	0.583929
$\log \alpha_0''$	1.670860n	$\log \delta_0''$	1.252009n	$\log (\tan\delta)_0''$	1.450666n

IIIa

$\log n$	1.51032n	$\log \Phi$	0.60412n	$\log N$	1.79951
$\log C_1$	8.74264	$\log \frac{\lambda}{\kappa}$	0.33778	$\log \kappa$	7.51196
$\log C_2$	9.04095n				
$\log a_x$	0.62217	$\log a_y$	9.63653	$\log a_z$	0.49703

IVa

$\log a^2$	1.39873	q'^2	+0.01176	c'	+0.31794
$\log b$	0.63606	$\log p'$	9.24434	$\frac{h}{s}$	+0.73413
$\log G_0^2$	0.01382	$\log h$	8.92332	$\frac{1}{m}$	−274.72

Since $p' > 0$ and $c > 0$, according to (49) page 288, three parabolic solutions are possible if $\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2$ is negative. The computation gives

$$\left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2 = -0.00006.$$

We have then a case where three mathematical parabolic solutions are

possible. The graphical solutions for z (denoting the three values of z by z_1^p, z_2^p and z_3^p) give

$$z_1^p = 1.0167 \; ; \qquad z_2^p = 0.8643 \; ; \qquad z_3^p = 0.6348 \, .$$

Owing to the difficulty of interpolation in the parts of the Table (Part 7) where $\frac{1}{m} = -274$ and ψ is less than 10°, it is more convenient to determine the values of z corresponding to the *general* solution through the solution of equation (27), page 240. From the Table, taking account of third differences the two values of z are 1.09 and 0.88. The solution of the equation gives (denoting the two solutions by z_1^g and z_2^g)

$$z_1^g = 1.0899 \; ; \qquad z_2^g = 0.8810 \, .$$

We have now to determine whether one (and which one, if any) of the three parabolic solutions agrees with one of the general solutions, within the uncertainty of the solution. At first glance it appears that the best agreement is between z_2^p and z_2^g. The parabola resulting from z_2^p is therefore the preliminary, assuming that the uncertainty of z_2^g, which is to be investigated at once, is not materially less than the difference between z_2^p and z_2^g, otherwise the parabola must be completely discarded. Or, in case the next difference, that between z_1^p and z_1^g, should also come into consideration from its being smaller than the uncertainty in z_1^g, naturally, a definite decision would not, in general, be possible. *Such an uncertainty, however, is excluded in consequence of the discussion in Part 7.*

The geocentric motion being large, the uncertainty of the solution depends (*cf.* page 270) upon the errors of observation (in this case parallax has been neglected), and *upon the third and higher differences of the geocentric motion, which were neglected in determining* α_0', δ_0', α_0'', δ_0''.

On account of the great difficulty in observing this comet the resulting error, error of observation combined with the error arising from neglecting the geocentric and barycentric parallax, can be placed at about 15″.

Reducing the differences of the observed α and δ to correspond to 1 day interval we have

f_α^{I}	f_α^{II}	f_δ^{I}	f_δ^{II}
14064″.7	2860″.5	12834″.7	1090″.3
11355 .8		11802 .2	
12710		12318	

The numbers 12710 and 12318 are the average values of f_α^{I} and f_δ^{I}, respectively, and are the principal values in determining α_0' and δ_0', respectively.

Let us denote the errors of observation for the three observations by $e_{,}$, $e_{,,}$, and $e_{,,,}$, respectively. Under the assumption of $e = e_{,} = e_{,,} = e_{,,,} = 15''$, and also with a choice of sign such that the errors do not neutralize each other, but numerically augment, we have, according to page 272 (case of equal or nearly equal intervals), as the *maximum error* of α' and δ' the value e or 15″, and for the *maximum error* of α'' and δ'' the value $4e$ or 60″, since the constant interval i is unity.

There now remains the estimation of the error caused by the neglected third differences f'''_α and f'''_δ (*cf.* page 274). The ratios of f'' to f' in α and δ are 0.225 and 0.888, respectively. If we assume, further, the same ratios of f''' to f'', we shall have $f'''_\alpha = 644''$ and $f'''_\delta = 91''$, signs not being taken into consideration. The differences here being referred to a one day interval, we have, then, from (21), page 275, $\frac{f'''_\alpha}{6} e = 122''$ to compare with $f'_\alpha = 12710''$, and $\frac{f'''_\delta}{6} e = 30''$ with $f'_\delta = 12318''$, in order to ascertain the ratios as the error $\partial\alpha'_0$ to α'_0, and $\partial\delta'_0$ to δ'_0, respectively. These gave $\partial\alpha'_0 = \frac{1}{104}\alpha'_0$, and $\partial\delta'_0 = \frac{1}{411}\delta'_0$.

In order to estimate the error in α''_0, we have to compare, (17) page 274, $\frac{\vartheta_{III} - \vartheta_I}{3} \cdot \frac{f'''_\alpha}{4} \cdot 4e = \frac{0.12k}{3} \cdot \frac{644''}{k} = 60''$ $85''$ with $f''_\alpha = 2860''$. Hence the error $\partial\alpha''_0 = \frac{1}{33}\alpha''_0$. Similarly we find the error $\partial\delta''_0 = \frac{1}{17}\delta''_0$.

Further, the computation gives

$$N = \alpha_0'^3 \tan\delta_0 - \alpha''_0(\tan\delta)'_0 + \alpha'_0(\tan\delta)''_0 = -14.987 - 179.800 + 101.786 = -63.027.$$

It is seen that the last two terms are the decisive ones for determining the accuracy of N. And in these the most uncertain quantities are α''_0 and δ''_0. The principal part of the uncertainty of the second term is then $\frac{1}{33}$ of its value, or a little more than 5 units. Similarly the uncertainty of the last term is $\frac{1}{17}$ of its value, or about 6 units. The maximum uncertainty in N is from this assumed to be about 11 units, or 0.17 of its value.

For $\frac{1}{m}$ the computation gives the value -274.72, and for ψ the value $6^\circ.56$. Since N and $\frac{1}{m}$ are proportional to each other, the maximum uncertainty of $\frac{1}{m}$ is $\pm 0.17\,(274.72) = \pm 47$ units. Since several orders of differences would have to be taken into account in the Table, it is not convenient, in this case, to draw upon it for an approximate value of the differential coefficient $\frac{\partial\frac{1}{m}}{\partial z}$. But a formula for this can be derived through the differentiation of the equation (27) page 240. Putting, for brevity, $z^2 - 2cz + 1 = \frac{r^2}{R^2} = u^2$ we get

$$\frac{\partial\frac{1}{m}}{\partial z} = \frac{1}{z}\left[\frac{3(z-c)}{u^2} - \frac{1}{m}\right].$$

For the two *general* solutions, the values of $\log r$ corresponding to z_1^0 and z_2^0 are given by $\log r_1^0 = 9.16743$ and $\log r_2^0 = 9.19766$, respectively. We have here then for $\partial z_1 = \partial z_2 = 0.01$

$$1.\quad \partial\frac{1}{m} = 38.15\,; \qquad 2.\quad \partial\frac{1}{m} = -33.16\,.$$

The sign of $\partial\frac{1}{m}$ is naturally of no consequence. Since the approximate uncertainty of $\Delta\frac{1}{m}$, derived above, is equal to 47, we get for the uncertainty of z for the two general solutions

$$\Delta z_1^0 = 0.0123\,; \qquad \Delta z_2^0 = 0.0142\,.$$

Therefore, having regard to the uncertainty of solutions, there is agreement only between the SECOND PARABOLIC *solution, for which* $z_2^p = 0.8643$, *with the* SECOND GENERAL *solution, for which* $z_2^g = 0.8810$, *since the difference* $z_2^g - z_2^p = 0.0167$ *is but slightly greater than the uncertainty* $\Delta z_2^g = 0.0142$. *Both of the other parabolic solutions are therefore to be discarded.*

It is true that the parabolic solution z_2^p lies a little outside of the limits for agreement with the general solution z_2^g; but, since the uncertainty in the general solution can only be approximated, as above, there is no reason at hand for discarding the only parabolic solution that can come into consideration.

In accordance with the present custom in astronomy of considering the parabola the solution when the observations can be represented by both a parabola and a general orbit, the computation was continued with the value z_2^p exactly as illustrated in Example No. 1.

EXAMPLE NO. 8.

EXAMPLE SHOWING CHANGE FROM PARABOLA TO GENERAL ORBIT DURING A DIFFERENTIAL CORRECTION.

In order to illustrate the ease with which the change from a parabolic hypothesis to the hypothesis without assumption regarding the eccentricity can be made, the orbit of Comet *e* 1906 (KOPFF) is chosen. This affords also an example of what may be termed "cooking", an arbitrary process by means of which one or more differential corrections may be avoided. The computations were made by R. T. CRAWFORD and A. J. CHAMPREUX.

The observations upon which the work is based are those of 1906, August 24, September 5, and September 15 given under example No. 2, page 399. Using the velocities and accelerations found in that example by the aid of five observations a parabola was computed giving the following data:

$\log \rho_0$	0.181836		
$\log x_0$	0.374444	$\log x_0'$	9.856309
$\log y_0$	9.900377	$\log y_0'$	9.620749
$\log z_0$	9.116388	$\log z_0'$	9.519014
$\log r_0$	0.398224	$\log r_0'$	9.751731
$\log f_{,}$	9.999383	$\log g_{,}$	9.313470n
$\log f_{,,,}$	9.999600	$\log g_{,,,}$	9.238024

and the representation of the First and the Third Places

		I	III
(O - C)	$\partial\alpha$	$-\ 4'\ 23''.5$	$+\ 0'\ 20''.0$
	$\partial\delta$	$-\ 5\ \ 54\ .1$	$-\ 3\ \ 40\ .7$

The computation of the differential correction by [VII], using the series for ∂f and ∂g gave

$\log P_{,}$	9.01776n	$\log Q_{,}$	9.56035n
$\log P_{,,,}$	8.63205	$\log Q_{,,,}$	9.50890
$\log P_{z,}$	7.93127n	$\log Q_{z,}$	6.73217n
$\log P_{z,,,}$	8.43132n	$\log Q_{z,,,}$	8.49383

Using $P_{z,,,}$ and $Q_{z,,,}$ for P and Q the solution of the parabolic equation for $\partial\rho_0$ gives

$\log \partial\rho_0$	9.39320
$\log \partial x_0'$	8.15586n
$\log \partial y_0'$	8.56771n
$\log \partial z_0'$	8.54042n

and since

$\log \frac{\xi_0}{\rho_0}$	9.96627
$\log \frac{\eta_0}{\rho_0}$	9.53361n
$\log \frac{\zeta_0}{\rho_0}$	9.21661

we have

$\log \partial x_0$	9.35947
$\log \partial y_0$	8.92681n
$\log \partial z_0$	8.60981

Applying these corrections to get the new heliocentric coördinates and their velocities and computing the representation we find

		I	III
(O - C)	$\partial\alpha$	$-\ 0'\ 39''.3$	$+\ 0'\ 4''.1$
	$\partial\delta$	$-\ 11\ \ 49\ .7$	$-\ 0\ \ 30\ .7$

Comparing these residuals with those we tried to remove it is seen that on the whole there is no improvement. But the material difference is that in the last set most of the error has been forced into the first declination. Seeing, therefore, that we can expect no improvement in the residuals under the parabolic hypothesis, this has to be abandoned and solution made without hypothesis regarding the eccentricity. *At this point is seen one of the most important features of Leuschner's Method.* All that has to be done now is to return to the P and Q expressions and solve for $\partial\rho$ by $\partial\rho = \dfrac{P_{z_{III}} - P_{z_I}}{Q_{z_{III}} - Q_{z_I}}$. Doing this we have

$\log \partial\rho_0$	9.76496n		
$\log \partial x_0'$	9.49924n	$\log \partial x_0$	9.73123n
$\log \partial y_0'$	9.36311	$\log \partial y_0$	9.29857
$\log \partial z_0'$	7.94697n	$\log \partial z_0$	8.98157n

Applying these we have

$\log x_0$	0.262400	$\log x_0'$	9.604906
$\log y_0$	9.775354n	$\log y_0'$	9.811791
$\log z_0$	8.542685	$\log z_0'$	9.507221
$\log r_0$	0.284378	$\log r_0'$	9.273669

which give

		I	III
	$\log f$	9.998681	9.999103
	$\log g$	9.313245n	9.237867
(O – C)	$\partial\alpha$	+ 2′ 22″.1	+ 0′ 2″.9
	$\partial\delta$	+ 2 22 .5	+ 2 1 .3

These residuals are due to the fact that the initial residuals were too large to be removed entirely by the linear relations.

By comparing them with those to be removed at the start, and assuming that the linear relations in the equations for $\partial\alpha$ and $\partial\delta$ are hereafter sufficient, it is seen that, if we now apply to our last values of x_0, y_0, z_0, and their velocities corrections which come from a $\partial\rho$, $\partial x_0'$, $\partial y_0'$ and $\partial z_0'$ amounting to [9.54449n] of the corrections $\partial\rho$, etc., from the foregoing solution, the residuals will be considerably reduced.

Doing this we arrive at an orbit which gives residuals

		I	III
(O – C)	$\partial\alpha$	– 0′ 41″.7	+ 0′ 6″.7
	$\partial\delta$	– 1 12 .1	– 0 34 .3

These were then removed by the application of two more differential corrections. The first left residuals of

		I	III
(O – C)	$\partial\alpha$	+ 3″.0	+ 0″.1
	$\partial\delta$	+ 2 .5	+ 1 .9

The last differential correction removed these. This final differential correction was very simple as it was performed with the same values of the coefficients A, B, and C as were derived for the preceding differential correction.

The eccentricity of the resulting orbit is 0.52 and the period $6\frac{2}{3}$ years.

This orbit was computed before Professor LEUSCHNER had derived either his

criteria for the feasibility of a parabolic solution or the closed expressions for ∂f and ∂g so that on the one hand a direct parabolic solution was performed and on the other hand the solution of the differential corrections was made with the series forms for ∂f and ∂g. Had the criteria and the closed expressions been available, a general solution would have been made in the first place and the numerous approximations to remove the parabolic residuals would have been unnecessary.

EXAMPLE NO. 9.

DIFFERENTIAL CORRECTION OF A DIRECT SOLUTION OF THE ORBIT OF A MINOR PLANET USING THE SERIES FOR f, g, ∂f, AND ∂g.

For an example of a differential correction of a direct solution of the orbit of a minor planet using the series for f, g, ∂f, and ∂g the computation is here given for the orbit of 1909 HC. The direct solution for the orbit of this minor planet has been given as Example No. 5. Instead of using the same observations as for the direct solution the arc is extended so as to use observations of November 7 and December 18 for the first and third places, respectively. The middle place (November 26) of the direct solution is here retained as the middle place for the differential correction. The computations were made by R. K. YOUNG and checked by O. W. LANZENDORF.

The observations used here were made by Mr. H. C. WILSON at the Lick Observatory. Reduced to 1910.0 they are:

1910	Gr. M. T.	α (1910.0)	δ (1910.0)
I	Nov. 7.8205	3° 50′ 24″.3	+25° 11′ 10″.5
II	Nov. 26.7480	3 13 3 .0	22 29 31 .3
III	Dec. 18.6262	4 54 19 .5	+20 14 51 .9

Solar Coördinates.

$X_{\prime}$	−0.7000428	$X_{\prime\prime}$	−0.4306623	$X_{\prime\prime\prime}$	−0.0628037
$Y_{\prime}$	−0.6429179	$Y_{\prime\prime}$	−.08143311	$Y_{\prime\prime\prime}$	−0.9007035
$Z_{\prime}$	−.02788955	$Z_{\prime\prime}$	−0.3532487	$Z_{\prime\prime\prime}$	−0.3907160

The direct solution (Example No. 5) gave the following quantities:

$\log x_0$	0.446074	$\log x_0'$	9.42066n	$\log \rho_0$	0.41425
$\log y_0$	9.976418	$\log y_0'$	9.68296	$\log R' \cos \psi$	9.75941n
$\log z_0$	0.124811	$\log z_0'$	8.93672	$\log \frac{\xi_0}{\rho_0}$	9.9649
$\log r_0$	0.510072	$\log r_0'$	8.70503n	$\log \frac{\eta_0}{\rho_0}$	8.7148
				$\log \frac{\zeta_0}{\rho_0}$	9.5827

Computing the residuals for the first and third places (November 7 and December 18, respectively), we find:

	I	III
$\log \vartheta$	9.51265	9.57556
$\log f$	9.999325	9.999088
$\log g$	9.51243n	9.57526
α_c	3° 50′ 23″.8	4° 54′ 47″.9
α_o	3 50 26 .8	4 54 19 .8
$(O-C)\,\partial\alpha$	+ 3 .0	− 28 .1

	I	III
δ_c	+25° 9′ 53″.6	+20° 14′ 49″.1
δ_o	+25 11 11 .5	+20 14 52 .8
$(O-C)\,\partial\delta$	+ 77 .9	+ 3 .7
$\log \rho$	0.381618	0.448477
p_α	+ 2″.5	+ 0″.3
p_δ	+ 1 .0	+ 0 .9

The complete computation of the differential correction to remove these residuals is given in the following:

DIFFERENTIAL CORRECTION.

A VIIc

log ρ_0	0.4084
log R cos ψ	9.7594n
sub	0.0879
log $(\rho - R \cos \psi)$	0.4963
log r_0	0.5101
cos β	9.9862

log $4 \frac{r_0'}{r_0}$	8.7970
log $\frac{\cos \beta}{2 r_0^4}$	7.6449

log $3 \theta_{,,,}^2$	9.5024
log $3 \theta_{,}^2$	9.6282
log $\theta_{,,,}^3$	8.5380
log $\theta_{,}^3$	8.7267

ω	x	y	z
log ω_0'	9.4207n	9.6830	8.9367
log $4 \frac{r_0'}{r_0} \omega_0$	9.2431	8.7734	8.9218
sub	0.2212	9.9429	8.5430
log [ ]	9.6419n	9.6259	7.4648
log II	8.1799n	8.1639	6.0028
log (I = $3 \theta_{,,}^2 \omega_0$)	9.9485	9.4788	9.6272
sub	0.0073	9.9784	9.9999
log ( )	9.9558	9.4572	9.6271
log II	7.6007	7.1021	7.2720
log I	9.9643	8.7142	9.5820
add	0.0019	0.0105	0.0021
log $f_{\omega,}$	9.9662	8.7247	9.5841

ω	x	y	z
log II	8.3686n	8.3526	6.1915
log (I = $3 \theta_{,}^2 \omega_0$)	0.0743	9.6046	9.7530
add	9.9913	0.0236	0.0001
log ( )	0.0656	9.6282	9.7531
log II	7.7105	7.2731	7.3980
log I	9.9640	8.7139	9.5818
add	0.0024	0.0155	0.0028
log $f_{\omega,,,}$	9.9664	8.7294	9.5846

	I	III
log sin α	8.8259	8.9327
log cos α	9.9990	9.9984
log f_y cos α	8.7237	8.7278
log f_x sin α	8.7921	8.8991
sub	9.2319	9.6844
log [ ]	7.9556n	8.4122n
log ρ	0.3816	0.4485
log A	7.5740n	7.9637n
log g	9.5124n	9.5753
log C	9.1308n	9.1268

	I	III
log f_y sin α	7.5506	7.6621
log f_x cos α	9.9652	9.9648
add	0.0017	0.0022
log ( )	9.9669	9.9670
sin δ	9.6286	9.5392
log ( ) sin δ	9.5955	9.5062
log f_z cos δ	9.5408	9.5569
sub	9.1278	9.0928
log [ ]	8.6686	8.5990n
log ρ	0.3816	0.4485
log B	8.2870n	8.1505

Determination of $P_{z_,}$, $P_{z_{,,,}}$, $Q_{z_,}$, and $Q_{z_{,,,}}$.

$\alpha_{,,,} - \alpha_,$	+1° 4′ 24″.1
sin $(\alpha_{,,,} - \alpha_,)$	8.2691
$C_, C_{,,,}$ sin $(\alpha_{,,,} - \alpha_,)$	6.5627n
$C_{,,,}$ cos $\alpha_{,,,}\ \partial_, \alpha_,$	4.2456
$C_,$ cos $\alpha_,\ \partial_, \alpha_{,,,}$	5.2364
sub	9.9532
log numer.	5.1896n
log P_x	8.6629
$A_,\ C_{,,,}$ cos $\alpha_{,,,}$	6.6992n
$A_{,,,} C_,$ cos $\alpha_,$	7.0935
sub	0.1472
log numer.	7.2407n
log Q_x	0.7140

log $\partial_, \alpha_,$	5.1194
log $\partial_, \alpha_{,,,}$	6.1066n
$C_{,,,}$ sin $\alpha_{,,,}\ \partial_, \alpha_,$	3.1789
$C_,$ sin $\alpha_,\ \partial_, \alpha_{,,,}$	4.0633
sub	9.0393
log numer.	4.0026n
log P_y	7.4759
$A_,\ C_{,,,}$ sin $\alpha_{,,,}$	5.6335n
$A_{,,,} C_,$ sin $\alpha_,$	5.9204
sub	0.1809
log numer.	6.1013n
log Q_y	9.5746

	I	III
log $Q_z \cos\alpha$	0.7130	0.7124
log $Q_y \sin\alpha$	8.4005	8.5073
add	0.0021	0.0027
log ( )	0.7151	0.7151
log $C \sin\delta$	8.7594n	8.6660
log II	9 4745n	9.3811
log (I = B)	8.2870n	8.1505
add	0.0273	0.0248
log numer.	9.5018n	9.4059
log $C \cos\delta$	9.0875n	6.0991
log Q_z	0.4143	0.3068

	I	III
log $P_z \cos\alpha$	8.6619	8.6613
log $P_y \sin\alpha$	6.3018	6.4086
add	0.0019	0.0024
log ( )	8.6638	8.6637
log $C \sin\delta$	8.7594n	8.6660
log II	7.4232n	7.3297
log (I = $\partial\delta$)	6.5761	5.2537
add	9 9335	0.0036
log numer.	7.3567n	7.3333
log $C \cos\delta$	9.0875n	9.0991
log P_z	8.2692	8.2342

Determination of the Correction $\partial\rho_0$, etc.

log $P_{z_{,,,}}$	8.2342
log $P_{z_{,}}$	8.2692
sub	8.9240
log ($P_{z_{,,,}} - P_{z_{,}}$)	7.1582n
log $Q_{z_{,,,}}$	0.3068
log $Q_{z_{,}}$	0.4143
sub	9.4485
log ($Q_{z_{,,,}} - Q_{z_{,}}$)	9.7553n
log $\partial\rho_0$	7.4029
log ∂x_0	7.3678
log ∂y_0	6.1178
log ∂z_0	6.9856

log P_z	8.6629	log P_y	7.4759
log $Q_z\,\partial\rho_0$	8.1169	log $Q_y\,\partial\rho_0$	6.9775
sub	9.8546	sub	9.8342
log $\partial x_0'$	8.5175	log $\partial y_0'$	7.3101
log $P_{z_{,}}$	8.2692	log $P_{z_{,,,}}$	8 2342
log $Q_{z_{,}}\,\partial\rho_0$	7.8172	log $Q_{z_{,,,}}\,\partial\rho_0$	7.7097
sub	9.8108	sub	9.8458
log $\partial z_0'$	8.0800	log $\partial z_0'$	8.0800

Application of Corrections.

ω	x	y	z
Old log ω_0	0.446074	9.976418	0.124811
log $\partial\omega_0$	7.3678	6.1178	6.9856
add	0.000362	0.000060	0.000316
log ω_0	0.446436	9.976478	0.125127

ω	x	y	z
Old log ω_0'	9.420660n	9.682960	8.936720
log $\partial\omega_0'$	8.5175	7.3101	8.0800
add	9.942018	0.001837	0.056556
log ω_0'	9.362678n	9.684797	8.993276

With these corrected values of x_0, y_0, z_0, x_0', y_0', and z_0' the representation of the first and third places is now computed. This gives

		I	III
	log f	9.999323	9.999090
	log g	9.512425n	9.575261
	α_c	3° 50′ 26″.3	4° 54′ 20″.0
	δ_c	+25 11 11 .7	+20 14 53 .1
(O − C)	$\partial\alpha$	+ 0″.5	− 0″.2
	$\partial\delta$	− 0 .2	− 0 .3

The representation being satisfactory the elements are computed from the formulæ A VIII. They are

Epoch 1910 Nov. 26.7332 Gr. M. T.

M_0	220° 55′ 20″.1	
ω	267 4 48 .6	1910.0
Ω	260 40 32 .2	1910.0
i	18 29 37 .0	1910.0
φ	2 48 15 .0	
log a	0.494229	
μ	643″.673	

EXAMPLE NO. 10

DIFFERENTIAL CORRECTION OF A DIRECT SOLUTION OF A COMET ORBIT USING THE SERIES FOR f, g, ∂f, AND ∂g.

For an example of a differential correction of a direct solution of a comet orbit using the series for $f, g, \partial f$, and ∂g the case of Comet e 1905 (GIACOBINI) is chosen. It may be remarked here that such a differential correction is rarely necessary. The computations were made by R. T. CRAWFORD.

Using the series for $f, g, \partial f$, and ∂g the computation for a differential correction of a parabolic orbit is the same as for a general orbit except for the solution of $\partial\rho_0$. The details of a solution for a differential correction to a general orbit have been given in Example No. 9. Therefore, only the results of the solution are given for this example as far as the determination of $\partial\rho_0$.

The observations upon which the direct solution was based are:

1905 Gr. M. T.			α (1905.0)			δ (1905.0)			Observer.
I	Dec.	6.6837	215°	24′	40″	+20°	59′	39″	GIACOBINI, Nice
II	Dec.	7.9267	216	52	39	20	26	45	DUGAN, Princeton
III	Dec.	8.0311	215	0	4	−20	23	55	SMITH, Lick

Solar Coordinates.

X_1	−0.264668	X_0	0.243650	X_{III}	0.241892
Y_1	−0.870396	Y_0	0.875416	Y_{III}	−0.875790
Z_1	0.377585	Z_0	−0.379770	Z_{III}	−0.379924

NOTE.—The solar coordinates for the middle date have been corrected so as to eliminate the geocentric parallax for that date.

The direct solution gave the following quantities:

$\log \tau_1$	9.93819n	$\log \tau_1'$	8.61736	$\log \rho_0$	0.17092
$\log \tau_3$	8.62269	$\log \tau_3'$	0.03792n	$\log R \cos\psi$	9.73417
$\log r_0$	9.95306	$\log r_0'$	9.80582n	$\log \frac{\xi_0}{\rho_0}$	9.87479n
$\log r_1$	0.09651	$\log r'$	9.72018n	$\log \frac{\eta_0}{\rho_0}$	9.74997n
				$\log \frac{\zeta_0}{\rho_0}$	9.54322

Computing the residuals for the first and third places we find:

	I	II		I	III
$\log \theta$	8.33006	7.25432	δ	+21° 0′ 8″	+20° 23′ 58″
$\log f$	9.99995	0.00000	δ_c	+20 59 42	+20 23 58
$\log g$	8.33004n	7.25432	(O − C) $\partial\delta$	−26	0
α	215° 24′ 11″	216° 59′ 59″	$\log \rho$	0.17280	0.17077
α_c	215 24 36	216 59 59	p_α	−4″	−5″
(O − C) $\partial\alpha$	+25	0	p_δ	+3	+3

Exactly as in Example No. 9 the following quantities are found:

A VIIa

	I	III		
$\log f_x$	9.87485n	9.87479n	$\log P_x$	9.35369
$\log f_y$	9.74984n	9.74997n	$\log P_y$	9.23080
$\log f_z$	9.54342	9.54323	$\log P_{z_{\prime}}$	8.99603n
$\log A$	8.20580	7.12955n	$\log Q_x$	9.67083n
$\log B$	7.80277n	6 73992	$\log Q_y$	0.01664
$\log C$	8.15727n	7.08355	$\log Q_{z_{\prime}}$	9.59009

In cases where the intervals are nearly equal it is better practice to use $P_{z_{\prime\prime\prime}}$ and $Q_{z_{\prime\prime\prime}}$ for P_z and Q_z, respectively, in forming P and Q so as to represent the last declination and force whatever residual, that may still remain, into the first declination. In this case, however, the second interval is very short (about 0.10 of a day), so that a better representation will be had by representing the first declination and forcing the remaining residual into the third declination. This is accomplished by calling $P_z = P_{z_{\prime}}$ and $Q_z = Q_{z_{\prime}}$. Doing this we then form P and Q as follows:

$\log \frac{\cos\beta}{r_0^2}$	9.68360	$\log x_0' P_x$	7.97105
$\log x_0' Q_x$	8.28819n	$\log y_0' P_y$	9.26872n
$\log y_0' Q_y$	0.05456n	add	9.97754
$\log z_0' Q_z$	9.39591n	sum	9.24626n
		$\log z_0' P_z$	8.80185
$\frac{\cos\beta}{r_0^2}$	+0.48261	add	9.80658
		$\log P$	9.05284
$-x_0' Q_x$	+0.01942		
$-y_0' Q_y$	+1.13387	For the first approximation	
$-z_0' Q_z$	+0.24884	$\log \partial\rho_{\prime}$	8.77759
Q	+1.88474		
$\log Q$	0.27525		

Solution for $\partial\rho_0$.

	1	2	3
$\log \partial\rho_{\prime}$	8.77759	8.55757	8.56061
$\log P_x$	9.35369	9.35369	9.35369
$\log Q_x \partial\rho$	8.44842n	8.22840n	8.23144n
sub	0.05091	0.03138	0.53159
$\log \partial x_0'$	9.40460	9.38507	9.38528
$\log P_y$	9.23080	9.23080	9.23080
$\log Q_y \partial\rho$	8.79423	8.57421	8.57725
sub	9.80212	9.89182	9.89095
$\log \partial y_0'$	9.03292	9.12262	9.12175
$\log P_z$	8.99603n	8.99603n	8.99603n
$\log Q_z \partial\rho$	8.36768	8.14766	8.15070
sub	0.09177	0.05759	0.05796
$\log \partial z_0'$	9.08780n	9.05362n	9.05399n

	1	2	3
$\log \frac{(1-3\cos^2\beta)}{2r_0^3}$	9.25431n	9.25431n	9.25431n
$\log (\partial\rho)^2$	7.55518	7.11514	7.12122
$\log V$	6.80949n	6.36945n	6.37553n
P	+0.112938	+0.112938	+0.112938
$-\frac{1}{2}(\partial x_0')^2$	− 32223	− 29451	− 29480
$-\frac{1}{2}(\partial y_0')^2$	− 5819	− 8794	− 8759
$-\frac{1}{2}(\partial z_0')^2$	− 7492	− 6400	− 6411
$-V$	+ 645	+ 234	+ 237
$Q\,\partial\rho$	+0.068049	+0.068527	+0.068525
$\log Q\,\partial\rho$	8.83282	8.83586	8.83585
$\log Q$	0.27525	0.27525	0.27525
$\log \partial\rho_{\prime}$	8.55757	8.56061	8.56060

The result of the third trial being practically the same as that of the second the solution is completed and we have

$\log \partial\rho_0$	8.56060		
$\log \partial x_0$	8.43539n	$\log \partial x_0'$	9.38528
$\log \partial y_0$	8.31057n	$\log \partial y_0'$	9.12175
$\log \partial z_0$	8.10382	$\log \partial z_0'$	9.05399n

Applying these corrections to the heliocentric coördinates and their velocities given by the direct solution we have the corrected orbit.

The representation of the first and third places for the corrected orbit gives:

$$(O - C)\begin{cases} \partial\alpha & 0'' & & 0'' \\ \partial\delta & 0 & & -2 \end{cases}$$

The elements were not computed from the data derived from this differential correction as an observation of December 9 became available. The orbit published in *Lick Observatory Bulletin No. 87* was based upon the first two observations given above and that of December 9.

EXAMPLES B. ORBITS BASED UPON PREVIOUS APPROXIMATIONS USING CLOSED EXPRESSIONS FOR f, g, ∂f, AND ∂g.

EXAMPLE NO. 11.

CASE OF AN INITIAL PARABOLIC ORBIT.

For an example of the computation of an orbit based upon a previous approximation, where this previous approximation is a parabola, the case of Comet a 1910 has been selected. This example shows also the use of the closed expressions for $f, g, \partial f$ and ∂g for a parabolic orbit. The computations were made by W. F. MEYER and Miss S. H. LEVY.

B I

The observations upon which this work is based are:

	1910 Gr. M. T.	α (1910.0)	δ (1910.0)	Observer.
I	Jan. 18.1287	303° 32′ 51″.9	−20° 53′ 27″.0	ZAPPA (Rome)
II	Feb. 5.6211	326 41 0 .4	+ 5 35 18 .6	AITKEN (Lick)
III	March 13.0440	336 11 15 .4	+15 38 53 .9	AITKEN (Lick)

The solar coördinates for these dates are

$X_{\prime}$	+0.4570857	$(X)_{\prime\prime}$	+0.7149449	$X_{\prime\prime\prime}$	+0.9847873
$Y_{\prime}$	−0.7993384	$(Y)_{\prime\prime}$	−0.6232607	$Y_{\prime\prime\prime}$	−0.1255285
$Z_{\prime}$	−0.3467534	$(Z)_{\prime\prime}$	−0.2703833	$Z_{\prime\prime\prime}$	−0.0544576

The coördinates for the middle date, $(X)_{\prime\prime}$, etc., are those corrected for the partial elimination of the geocentric parallax.

The Elements and Constants for the Equator 1910.0 of the initial parabolic orbit which are used for the previous approximation are:

$$T = 1910 \text{ Jan. } 16.7838 \text{ Gr. M. T.}$$
$$\log q = 9.067800$$
$$x = r\,[9.880210] \sin (321^\circ\ 55'\ 16''.0 + v)$$
$$y = r\,[9.985880] \sin (\ 64\ \ 48\ \ 26\ .6 + v)$$
$$z = r\,[9.843920] \sin (350\ \ 16\ \ 53\ .8 + v)$$

From the computation of the preliminary orbit we take

$\log \rho_{\prime} = 9.91944$ $\log \rho_{\prime\prime} = 0.17717$ $\log \rho_{\prime\prime\prime} = 0.37814$

so that the three dates corrected for aberration become

$t_{\prime}$ = Jan. 18.1239 t_0 = Feb. 5.6124 $t_{\prime\prime\prime}$ = March 13.0302

Computation of the Constants of the Artificial Initial Orbit.

$t_0 - T$	19.8186	$\log \sqrt{\frac{2}{q}} \cos \frac{1}{2} v_0$	0.225693	$\log x_0'^2$	0.104200
$\log (t_0 - T)$	1.297073			$\log y_0'^2$	0.053904
$\log q^{3}/_{2}$	8.601700	$\sin a$	9.880210	add	0.276610
$\log M_0$	2.695373	$\cos (A' + \frac{1}{2} v_0)$	9.946197	sum	0.380810
v_0	132° 1′ 39″.4	$\log x_0'$	0.052100	$\log z_0'^2$	9.627672
$\frac{1}{2} v_0$	66 0 49 .7			add	0.070609
$A' + \frac{1}{2} v_0$	27 56 5 .7	$\sin b$	9.985880	$\log G_0^2$	0.451419
$B + \frac{1}{2} v_0$	130 49 16 .3	$\cos (B' + \frac{1}{2} v_0)$	9.815379n		
$C + \frac{1}{2} v_0$	56 17 43 .5	$\log y_0'$	9.026952n		
		$\sin c$	9.843920		
		$\cos (C' + \frac{1}{2} v_0)$	9.744223		
		$\log z_0'$	9.813836		

Computation of the Constants of the Artificial Initial Orbit—*Continued.*

log ρ_0	0.177170
cos δ_u	9.997931
log σ_0	0.175101
cos α_u	9.922024
log ξ_0	0.097125
log $(X)_u$	9.854273
sub	9.874628
log x_0	9.728901
sin α_u	9.739782n
log η_0	9.914883n
log $(Y)_u$	9.794669n
sub	9.503663
log y_0	9.298332n
tan δ_u	8.990553
log ζ_0	9.165654
log $(Z)_u$	9.431980n
sub	0.187970
log z_0	9.619950

log $x_0'^2$	6.457802
log $y_0'^2$	8.596664
add	0.056019
sum	9.513821
log $z_0'^2$	9.239900
add	0.185317
log $r_0'^2$	9.699138
log r_0	9.849569

Test.

log $\frac{2}{r_0}$	0.451461
log G_0^2	0.451419
sub	5.99
log $\frac{1}{a}$	6.44

log $x_0 x_0'$	9.781001
log $y_0 y_0'$	9.325284
add	0.130390
sum	9.911391
log $z_0 z_0'$	9.433786
add	0.124818
log $r_0 r_0'$	0.036209
log r_0'	0.186640

The ρ_0, x_0', y_0', and z_0' for our middle date are now the same elements as in the initial orbit. But since we are here using the observed α_0 and δ_0 to get x_0, y_0, and z_0 these will be different from the corresponding heliocentric coördinates for this date from the preliminary orbit. The elements to be corrected, then, are the heliocentric coördinates and their velocities just found here for the middle date.

Computation of the Residuals for the First and Third Dates by Means of the Closed Expressions for f, g, ∂f, and ∂g.

A VIIIa

Elements T, q and v_0.

log $2\,r_0$		0.150599
log $(r_0 r_0')^2$		0.072418
sub		9.294994
log p		9.367412
log q		9.066382
log $\frac{p}{r_0}$		9.517843
e cos v_0		9.826405n
log $\sqrt{p}$		9.683706
e sin v_0		9 870346
tan v_0		0.043941n
v_0		132° 6′ 23″.1
sin v_0		9.870346
cos v_0		9.826405n
log e		0.000000
log M_0		2.697188
log $q^{3/2}$		8.599573
log $(t_0 - T)$		1.296761
$t_0 - T$		19.80436
t_0	Jan.	36.61240
T	Jan.	16.80804
log r_0		9.849569
sub		9.921819
log $(r_0 - q)$		9.771388

B II

Determination of f and g for First and Third Places.

	I	III
$t - T$	1.31586	55 22216
log $(t - T)$	0.119210	1.742113
log M	1.519637	3.142540
v	41° 58′ 33″.4	147° 29′ 52″.7
$\frac{1}{2} v$	20 59 16 .7	73 44 56 .3
sec $\frac{1}{2} v$	0.029813	0.553080
sec² $\frac{1}{2} v$	0.059626	1.106160
log r	9.126008	0.172542
sub	9.167808	9.964583
log $(r - q)$	8.234190	0.137125
log $\sqrt{r - q}$	9.117095	0 068562
log $\sqrt{r_0 - q}$	9.885694	9.885694
sub	9.918883	9.718991
log γ_{i}; log γ_{iii}	9.804577n	9.604685
log γ^2	9.609154	9.209370
log $\frac{\gamma^2}{r_0}$	9.759585	9.359801
sub	9.868916	
log f_{i}; log f_{iii}	9.628501	9.887065
log $2\,r r_0$	9.276607	0.323141
log $p\,\gamma^2$	8.976566	8.576782
sub	9.998020	9.992141
log [ ]	8.974586	0.315282
log g^2	8.583740	9.524652
log g_{i}; log g_{iii}	9.291870n	9.762326

B II—*Continued.*

Representation of First and Third Places.

	I	III
$\log f x_0$	9.357402	9.615966
$\log f y_0$	8.926833n	9.185397n
$\log f z_0$	9.248451	9.507015
$\log g x_0'$	9.343970n	9.814426
$\log g y_0'$	9.318822	9.789278n
$\log g z_0'$	9.105706n	9.576162
X	+0.4570857	+0.9847873
$f x_0$	+0.2277205	+0.4130150
$g x_0'$	−0.2207853	+0.6522683
ξ	+0.4640209	+2.0500706
Y	−0.7993384	−0.1255285
$f y_0$	−0.0844954	−0.1532489
$g y_0'$	+0.2083638	−0.6155712
η	−0.6754700	−0.8943486
Z	−0.3467534	−0.0544576
$f z_0$	+0.1771948	+0.3213771
$g z_0'$	−0.1275574	+0.3768442
ζ	−0.2971160	+0.6437637

	I	III
$\rho \cos \delta \cos \alpha$	9.666538	0.311769
$\rho \cos \delta \sin \alpha$	9.829606n	9.951507n
$\tan \alpha$	0.163068n	9.639738n
α_c	304° 29′ 15″.1	336° 25′ 50″.3
α_o	303 32 58 .3	336 11 12 .3
$(O-C)\,\partial\alpha$	− 56 16 .8	− 14 38 .0
$\sin \alpha$	9.916059n	9.601907n
$\cos \alpha$	9.752990	9.962169
$\rho \cos \delta$	9.913548	0.349600
$\rho \sin \delta$	9.472926n	9.808727
$\tan \delta$	9.559378n	9.459127
δ_c	−19° 55′ 42″.7	+16° 3′ 25″.5
δ_o	−20 53 19 .1	+15 38 56 .1
$(O-C)\,\partial\delta$	− 57 36.4	− 24 29.4
$\sin \delta$	9.532561n	9.441844
$\cos \delta$	9.973183	9.982718
$\log \rho$	9.940365	0.366882
$\log p_\alpha''\rho$	0.745	0.862n
$\log p_\delta''\rho$	0.838	0.713
$\log p''_\alpha$	0.805	0.495n
$\log p''_\delta$	0.898	0.346
p_α	+6″.4	−3″.1
p_δ	+7 .9	+2 .2

B III 2

Differential Correction.

$R_0 \cos D_0 \cos A_0$	9.854273	$\log \rho_0$	0.17717
$R_0 \cos D_0 \sin A_0$	9.794669n	$\log R_0 \cos \psi$	9.95856
$\tan A_0$	9.940396n	sub	9.81577
A_0	−41° 4′ 50″.0	$\log r_0 \cos \beta$	9.77433
$\sin A_0$	9.817644n	$\cos \beta$	9.92476
$\cos A_0$	9.877248		
$R_0 \cos D_0$	9.977025	$\log x_0' \xi_0$	0.14922
$R_0 \sin D_0$	9.431980n	$\log y_0' \eta_0$	9.94184
$\tan D_0$	9.454955n	add	0.20961
		sum	0.35883
$\sin D_0$	9.437988n	$\log z_0' \zeta_0$	8.97949
$\cos D_0$	9.983033	add	0.01777
$\log R_0$	9.993992	$\log \rho_0 \varphi_0$	0.37660
		$\log \varphi_0$	0.19943
$\sin \delta_{\prime\prime}$	8.988484		
$\log \mathrm{I}$	8.426472n	$\log \dfrac{\xi_0}{\rho_0}$	9.91996
$(A_0 - \alpha_{\prime\prime})$	−7° 45′ 50″.4		
$\cos (A_0 - \alpha_{\prime\prime})$	9.996001	$\log \dfrac{\eta_0}{\rho_0}$	9.73771n
$\cos \delta_{\prime\prime}$	9.997931		
$\log \mathrm{II}$	9.976965	$\log \dfrac{\zeta_0}{\rho_0}$	8.98848
add	9.987598		
$\cos \psi$	9.964563		

	I	III
log φ_0	0.19943	0.19943
log [2 $\frac{\cos\beta}{\gamma}$	0.27070n	0.47059
add	9.25127	0.18628
log Φ	9.45-70n	0.65687
log $x_0' r_0 \gamma$	9.70625n	9.50635
log [2 x_0	9.87942	9.87942
add	9.69015	0.15338
log [ ]	9.39640	0.03280
log $\frac{\rho}{rr_0}$	0.43815n	8.79194
log g_x	9.83455n	8.82474
log $y_0' r_0 \gamma$	9.68110	9.48120n
log [2 y_0	9.44885n	9.44885n
add	9.84946	0.28515
log [ ]	9.29831	9.70635n
log g_y	9.73646n	8.55829n
log $z_0' r_0 \gamma$	9.46799n	9.26809
log [2 z_0	9.77046	9.77046
add	0.00288	0.11876
log [ ]	9.47087	9.88922
log g_z	9.90902n	8.68116
log I $f \frac{\xi_0}{\rho_0}$	9.54846	9.80702
log γ x_0	9.33805	8.93827
log II	8.56367	9.16389
log III $g_x \Phi$	9.28525	9.48161
I	+0.35356	+0.64124
II	+0.36616	+0.14584
III	+0.19286	+0.30311
f_x	+0.91258	+1.09019
log $f_{x_{\prime}}$, log $f_{x_{\prime\prime\prime}}$	9.96027	0.03750
log I $f \frac{\eta_0}{\rho_0}$	9.36621n	9.62477n
log γ y_0	8.90748n	8.50770n
log II	9.13310n	8.73332n
log III $g_y \Phi$	9.18716	9.21516n
I	−0.23238	−0.42147
II	−0.13586	−0.05411
III	+0.15387	−0.16412
f_y	−0.21437	−0.63970
log $f_{y_{\prime}}$, log $f_{y_{\prime\prime\prime}}$	9.33116n	9.80598n
log I $f \frac{\zeta_0}{\rho_0}$	8.61698	8.87554
log γ' z_0	9.22910	8.82932
log II	9.45472	9.05494
log III $g_z \Phi$	9.35972	9.33803
I	+0.04140	+0.07508
II	+0.28492	+0.11348
III	+0.22894	+0.21779
f_z	+0.55526	+0.41635
log $f_{z_{\prime}}$, log $f_{z_{\prime\prime\prime}}$	9.74450	9.60890

	I	III
log f_x cos α	9.08415n	9.76815n
log f_y sin α	9.87033n	9.63941n
sub	9.92357	9.53788
log [ ]	9.79990	9.17729n
log n	9.94036	0.36688
log I	9.85954	8.81041n
log (p_{α})	5.431	5.548n
cos δ	9.973	9.983
colog n'	0.119	9.266
log II	5.523	4.797n
add	0.00002	0.00004
log $A_{\prime}$; log $A_{\prime\prime\prime}$	9.85956	8.81045n
log f_x sin α	9.74722	9.40789
log f_y cos α	9.71326	9.99967
add	0.12774	0.09898
log (	9.84100	0.09865
sin δ	9.53256n	9.44184
log sin δ ( )	9.37356n	9.54049
log f_z cos δ	9.71768	9.59162
sub	0.16220	9.09673
log	9.87988n	8.63722n
log I	9.93952	8.27034
log (p_{δ})	5.524	5.399
log II	5.643	4.665
add	0.00002	0.00011
log $B_{\prime}$, log $B_{\prime\prime\prime}$	9.93954	8.27045
log g_x cos α	9.48945n	8.52046n
log g_y sin α	9.75061	8.42665n
sub	0.18979	9.38221
log [ ]	9.94040n	7.80886n
log $A_{g_{\prime}}$, log $A_{g_{\prime\prime\prime}}$	0.00004n	7.44198n
log g_x sin α	9.65252	8.16020
log g_y cos α	9.58754n	8.78691
add	9.20786	0.09210
log (	8.79540	8.87901
log sin δ (	8.32796n	8.32085
log g_z cos δ	9.85220n	8.06388
sub	9.98771	0.08029
log [ ]	9.86991	8.40114n
log $B_{g_{\prime}}$, log $B_{g_{\prime\prime\prime}}$	9.92955n	8.03426
log g	9.29187n	9.76233
log $C_{\prime}$, log $C_{\prime\prime\prime}$	9.35151n	9.39545
log A $A_{\prime}$	7.30154n	
log $A_{\prime}$ $A_{g'}$	8.81049	
sub	0.01325	
log A	8.82374n	

	For a_i	For b_i
log $C_{,} A_{g_{,,,}}$	6.79349	
log I	6.70955n	6.54648
log $C_{,,,} A_{g_{,}}$	9.39549n	
log II	8.99740	9.35766n
sub	0.00223	0.00067
log a_1 ; log b_1	8.99963n	9.35833n
log $C_{,} A_{f_{,,,}}$	8.16196	
log I	8.07802n	7.91495
log $C_{,,,} A_{f_{,}}$	9.25501	
log II	8.85692n	9.21718
log III	8.55264n	8.12207
I	−0.011968	+0.008221
−II	+0.071932	−0.164885
± III	−0.035698	−0.013245
a_2 ; $-b_2$	+0.024266	−0.169909
log a_2 ; log b_2	8.38500	9.23021
log $C_{,} \sin \delta_{,}$	8.88407	
log I	8.63706	8.80013n
log II	9.65845n	9.22788
sub	0.03949	0.13782
log a_3 ; log b_3	9.69794	9.36570n
log $C_{,,,} \sin \delta_{,,,}$	8 83729	
log I	8.79946	8.43920n
log II	7.76316	7.33259n
sub	9.95809	9.96462
log a_4 ; log b_4	8.75755	8.40382n

	For c_i
log c_2	8.44369n
log $C_{,} \cos \delta_{,}$	9.32469n
log $z_0 B_{g_{,}}$	9.54950n
add	0.20301
log c_3	9.75251
log $C_{,,,} \cos \delta_{,,,}$	9.37817
log $z_0 B_{g_{,,,}}$	7.65421
add	0.00812
log c_4	9.38629n

	For d_i
log d_1	8.82374n
log d_3	9.93954
log d_4	8.27045

	For e_i
log $\partial\alpha_{,}$	3.52850n
sin 1″	4.68558
cos $\delta_{,}$	9.97318
log $\partial_{,} \alpha_{,}$	8.18726n
log $\partial\alpha_{,,,}$	2.94350n
cos $\delta_{,,,}$	9.98272
log $\partial_{,} \alpha_{,,,}$	7.61180n
log $A_{g_{,}}$ $\partial_{,} \alpha_{,,,}$	7.61184
log $A_{g_{,,,}}$ $\partial_{,} \alpha_{,}$	5.62924
sub	9.99546
log $e_{,}$	7.60730
log $A_{f_{,}}$ $\partial_{,} \alpha_{,,,}$	7.47136n
log $A_{f_{,,,}}$ $\partial_{,} \alpha_{,}$	6.99771
sub	0.12581
log e_2	7.59717n
log $\partial\delta_{,}$	3.53862n
log e_3	8.22420
log $\partial\delta_{,,,}$	3.16714n
log e_4	7.85272

For a, b, c.

log $\frac{c_3}{c_2}$	1.30882n		log $\frac{d_3}{d_{,}}$	1.11580n	
log II	9.69382n	log II	0.53903n	log II	8.90599
log III	0.11543	log III	0.47413	log III	8.72310n
I = a_3	+0.49881	I = b_3	−0.23211	I = e_3	+0.016757
−II	+0.49411	−II	+3.45962	−II	−0.080536
−III	−1.30445	−III	−2.97940	−III	+0.052857
a	−0.31153	b	+0.24811	e	−0.010922
log a	9.49350n	log b	9.39465	log e	8.03830n

As the residuals of the first place are much greater than those of the third we shall take, in formulæ A VIIa, $P_z = P_{z_\prime}$ and $Q_z = Q_{z_\prime}$, so that $P_{z_{\prime\prime\prime}}$ and $Q_{z_{\prime\prime\prime}}$ are not needed.

log $a_1 b$	8.39428n
log $a b_1$	8.85183
sub	0.12992
log denom.	8.98175n
log $b e_1$	7.00195
log $b_1 e$	7.39663
sub	0.17094
log numer.	7.17259n
log P_x	8.19084
log $a_1 e$	7.03793
log $a e_1$	7.10080n
sub	0.27073
log numer.	7.37153
log P_y	8.38978n
log $b d_\prime$	8.21839n
log Q_x	9.23664n
log $a d_\prime$	8.31724
log Q_y	9.33549n

log $a_3 P_x$	7.88878
log $b_3 P_y$	7.75548
e_3	+0.0167573
$-a_3 P_x$	−0.0077407
$-b_3 P_y$	−0.0056949
$c_3 P_{z_\prime}$	+0.0033217
log $c_3 P_{z_\prime}$	7.52136
log $P_{z_\prime}$	7.76885
log $a_3 Q_x$	8.93458n
log $b_3 Q_y$	8.70119
d_3	+0.870040
$a_3 Q_x$	−0.086016
$b_3 Q_y$	+0.050256
$-c_3 Q_{z_\prime}$	+0.83428
log $c_3 Q_{z_\prime}$	9.92131n
log $Q_{z_\prime}$	0.16880

A VII a

log $\frac{1}{2} a_0$	6.14
log $x_0' P_x$	8.24294
log $y_0' P_y$	8.41673
log $z_0' P_z$	7.58269
$\frac{1}{2a_0}$	+0.000138
$-x_0' P_x$	−0.017496
$-y_0' P_y$	−0.026105
$-z_0' P_z$	−0.003825
P	−0.047288
log P	8.67475n
log $\frac{\cos\beta}{r_0^2}$	0.22562
log $x_0' Q_x$	9.28874n
log $y_0' Q_y$	9.36244
log $z_0' Q_z$	9.98264n
$\frac{\cos\beta}{r_0^2}$	+1.68119
$-x_0' Q_x$	+0.19442
$-y_0' Q_y$	−0.23038
$-z_0' Q_z$	+0.96082
Q	+2.60605
log Q	0.41598

First approximation for $\partial\rho$, viz., $-\frac{P}{Q}$ gives

log $\partial\rho_1$ 8.25877n

Solution for $\partial\rho_0$, $\partial x_0'$, $\partial y_0'$, $\partial z_0'$

Trials.

	1	2
log $\partial\rho_1$	8.25877n	8.26036n
log P_x	8.19084	8.19084
log $Q_x \partial\rho$	7.49541	7.49700
sub	9.90221	9.90180
log $\partial x_0'$	8.09305	8.09264
log P_y	8.38978n	8.38978n
log $Q_y \partial\rho$	7.59426	7.59585
sub	0.06451	0.06473
log $\partial y_0'$	8.45429n	8.45451n
log P_z	7.76885	7.76555
log $Q_z \partial\rho$	8.42757	8.42916
sub	9.89242	9.89287
log $\partial z_0'$	8.31999n	8.32203n
log $(\partial x_0')^2$	6.18610	6.18528
log $(\partial y_0')^2$	6.90858	6.90902
log $(\partial z_0')^2$	6.63998	6.64406

	1	2
log coef. of $(\partial\rho)^2$	0.20005n	0.20005n
log $(\partial\rho)^2$	6.51754	6.52072
log V	6.71759n	6.72077n
log coef. of $(\partial\rho)^3$	9.9545n	9.9545n
log $(\partial\rho)^3$	4.7763n	4.7811n
log VI	4.7308	4.7356
P	−0.047288	−0.047288
$-\frac{1}{2}(\partial x_0')^2$	− 77	− 77
$-\frac{1}{2}(\partial y_0')^2$	− 405	− 405
$-\frac{1}{2}(\partial z_0')^2$	− 218	− 220
− V	+ 522	+ 526
+ VI	+ 5	+ 5
$Q\,\partial\rho$	−0.047461	−0.047459
log $Q\,\partial\rho$	8.67634n	8.67632n
log Q	0.41598	0.41598
log $\partial\rho_\prime$	8.26036n	8.26034n

log $\frac{\xi_0}{\rho_0}$	9.91996
log ∂x_0	8.18030n
log $\frac{\eta_0}{\rho_0}$	9.73771n
log ∂y_0	7.99805
log $\frac{\zeta_0}{\rho_0}$	8.98848
log ∂z_0	7.24882n

In spite of the large starting residuals the first approximation for $\partial\rho$ is much less than $\frac{r_0}{\sqrt{2}}$, hence the solution is convergent.

Application of Corrections.

w	ρ	x	y	z
Old log w_0	0.177170	9.728901	9.298332n	9.619950
log ∂w_0	8.26034n	8.18030n	7.99805	7.24882n
add	9.994708	9.987543	9.977684	9.998149
log w_0	0.171878	9.716444	9.276016n	9.618099
w'		x'	y'	z'
Old log w_0'		0.052100	0.026952n	9.813836
log $\partial w_0'$		8.09264	8.45451n	8.32203n
add		0.004742	0.011471	9.985774
log w_0'		0.056842	0.038423n	9.799610

log x_0^2	9 432888	log $x_0'^2$	0.113684	log $x_0\,x_0'$	9.773286
log y_0^2	8.552032	log $y_0'^2$	0.076846	log $y_0\,y_0'$	9.314439
add	0.053680	add	0.283002	add	0.129580
sum	9.486568	sum	0.396686	sum	9.902866
log z_0^2	9.236198	log $z_0'^2$	9.599220	log $z_0\,z_0'$	9 417709
add	0.193643	add	0.064240	add	0.122944
log r_0^2	9.680211	log G_0^2	0 460926	log $r_0\,r_0'$	0.025810
log r_0	9.840105			log r_0'	0.185705
		log $\frac{2}{r_0}$	0.460925 check.		

A VIII a, Elements T, q, v_0

log 2 r_0	0.141135	sin v_0	9.891334
log $(r_0\,r_0')^2$	0.051620	cos v_0	9.797599n
sub	9.359637	log e	9.999999
log p	9.411257 -		
log q	9.110227 -	log M_0	2.625391
		log $q^{3/2}$	8.665340
log $\frac{p}{r_0}$	9.571152	log $(t_0 - T)$	1.290731
		$t_0 - T$	19.53130
e cos v_0	9.797598n	t_0	Jan. 36.61253
		T	Jan. 17.08123
log $\sqrt{p}$	9.705628		
e sin v_0	9.891333	log r_0	9.840105
tan v_0	0.093735n	log q	9.110227
v_0	128° 51′ 51″.4	sub	9.910485
		log $(r_0 - q)$	9.750590

	I	II	III
Old log ρ	9 9404		0.3669
log $\partial\rho$	8.2603n		8.2603n
add	9.9908		9.9966
log ρ	9.9312	0.1719	0.3635
log $\alpha\,\rho$	7.6925n	7.9332n	8.1248n
$\alpha\,\rho$	- 0.00493	- 0.00857	- 0.01333
t	Jan. 18.1287	36.6211	72.0440
cor. t	Jan. 18.12377	36.61253	72.03067

Determination of f and g for First and Third Places.

	I	III
$t - T$	1.04254	54.94944
$\log (t - T)$	0.018093	1.739964
$\log M$	1.352753	3.074624
v	29° 58′ 5″.7	145° 34′ 43″.8
$\frac{1}{2} v$	14 59 2 .8	72 47 21 .9
$\sec \frac{1}{2} v$	0.015024	0.528878
$\sec^2 \frac{1}{2} v$	0.030048	1.057756
$\log r$	9.140275	0.167983
sub	8.855143	9.960210
$\log (r - q)$	7.965370	0.128193
$\log (r + q)$	8.982685	0.064096
$\log (r_0 + q)$	9.875295	9.875295
sub	9.940490	9.736034
$\log f_{\prime}$; $\log f_{\prime\prime\prime}$	9.815785n	9.611329

	I	III
$\log y'$	9.631570	9.222658
$\log \frac{y'}{r_0}$	9.791465	9.382553
sub	9.789826	
$\log t_{\prime}$, $\log t_{\prime\prime\prime}$	9.581291	9.880072
$\log 2 r r_0$	9.281410	0.309118
$\log p q$	9.042827	8.633915
sub	9.864595	9.990727
$\log [\ ]$	8.907422	0.299845
$\log g$	8.538992	9.522503
$\log g_{\prime}$, $\log g_{\prime\prime\prime}$	9.269496n	9.761252

Representation of First and Third Places.

	I	III
$\log fx_0$	9.297735	9.596516
$\log fy_0$	8.857307n	9.156088n
$\log fz_0$	9.199390	9.498171
$\log gx_0'$	9.326338n	9.818094
$\log gy_0'$	9.307919	9.799675n
$\log gz_0'$	9.069106n	9.560862
X	+0.4570857	+0.9847873
fx_0	+0.1984882	+0.3949264
gx_0'	0.2120010	+0.6578000
ξ	−0.4435729	+2.0375137
Y	−0.7993384	0.1255285
fy_0	−0.0719958	0.1432477
gy_0	+0.2031977	0.6304857
η	0.6681365	−0.8902619
Z	−0.3467534	−0.0544576
fz_0	+0.1582665	+0.3148986
gz_0	−0.1172481	+0.3637992
ζ	0.3057347	+0.6242402

	I	III
$\rho \cos \delta \cos \alpha$	9.646965	0.309101
$\rho \cos \delta \sin \alpha$	9.824865n	9.953886n
$\tan \alpha$	0.177900n	9.644785n
α	303° 34′ 48″.0	336° 11′ 8″.2
α_0	303 32 58 .4	336 11 12 .3
$(O - C)$	1 49 .6	+ 0 4 .1
$\sin \alpha$	9.920705n	9.600140n
$\cos \alpha$	9.742805	9.961354
$\rho \cos \delta$	9.904160	0.347747
$\rho \sin \delta$	9.485345n	9.795352
$\tan \delta$	9.581185n	9.447605
δ_c	20° 52′ 5″.7	+ 15 39′ 27″.3
δ_0	20 53 19 .0	+ 15 38 56 .1
$(O - C)\delta$	1 13 .3	0 31 .2
$\sin \delta$	9.551718n	9.431183
$\cos \delta$	9.970533	9.983578
$\log \rho$	9.933627	0.364169
$\log p_\alpha''\rho$	0.745	0.862n
$\log p_\delta''\rho$	0.838	0.713
$\log p_\alpha''$	0.811	0.498n
$\log p_\delta''$	0.904	0.349
p_α	+6″.5	−3″.1
p_δ	+8 .0	+2 .2

The computations given illustrate in detail all of the work necessary for a differential correction. Generally, one can substitute the new residuals into the expressions c_1, c_2, c_3 and c_4, keeping all of the other quantities the same and resolving for a further $\partial\rho_0$, $\partial x_0'$, $\partial y_0'$ and $\partial z_0'$. This process will generally give the final ρ_0, x_0', y_0', z_0' which will remove the residuals and thus save the computation of all the quantities for a second differential correction. Such a procedure is hardly sufficient in this problem. Since the original residuals which were to be removed were unusually large and combined with unequal intervals with small heliocentric

distance, it is necessary to recompute the differential coefficients based on the corrected values of the geocentric distance and of the heliocentric velocities at the middle date and the last set of residuals.

This second differential correction produced the following results:

$\log \rho_0$	0.171757				
$\log x_0$	9.716156	$\log x_0'$	0.057116	$\log \frac{2}{r_0}$	0.461142
$\log y_0$	9.275494n	$\log y_0'$	0.038560n	$\log G_0^2$	0.461142
$\log z_0$	9.618056	$\log z_0'$	9.799089		
$\log r_0$	9.839888	$\log r_0'$	0.185700		

For the representation

		I	III
$\log f$		9.579983	9.879909
$\log g$		9.268867n	9.761224
(O - C)	$\partial_{\prime}\alpha$	- 0″.9	- 0″.3
	$\partial\delta$	- 1 .0	+ 19 .7

The residual in the third declination is due to the use of $P_z = P_{z_{\prime}}$ and $Q_z = Q_{z_{\prime}}$ for the determination of $\partial\rho_0$, and also contains any errors in the observations and any slight deviation from a parabola. An observation taken by Professor BARNARD June 7th was represented with the foregoing quantities. This observation gave for residuals $\partial_{\prime}\alpha$ - 0″, $\partial\delta + 1'\ 3''$. This may be considered satisfactory for an observation taken three months after our third date on which the orbit is based.

The resulting Elements and Constants for the Equator are:

Elements.

T	1910 Jan. 17.08880 Gr. M. T.	
ω	320° 57′ 51″.4	1910.0
Ω	88 49 28 .8	1910.0
i	138 46 42 .5	1910.0
$\log q$	9.111002	

Constants for the Equator 1910.0.

$x = r\,[9.876385] \sin (322°\ 31'\ 36''.1 + v)$

$y = r\,[9.981342] \sin (\ 67\ \ 44\ \ \ 3\ .9 + v)$

$z = r\,[9.856490] \sin (354\ \ 35\ \ \ 3\ .1 + v)$

EXAMPLE NO. 12.

CASE OF AN INITIAL ELLIPTIC ORBIT OF A COMET OR A MINOR PLANET.

For an example of the computation of an orbit based upon a previous approximation, where this previous approximation is an ellipse, the case of Comet *e* 1909 (DANIEL) has been selected. This example shows also the use of the closed expression for f, g, ∂f, and ∂g for an elliptic orbit. The computations were made by S. EINARSSON and R. K. YOUNG.

I.

The observations upon which this work is based are:

	Date Gr. M. T.	α (1909.0)	δ (1009.0)	Observer
I	1909 Dec. 7.66050	94° 09′ 31″.4	+34° 44′ 21″.5	BARNARD
II	1909 Dec. 18.75589	90 30 46 .4	+43 25 59 .3	AITKEN
III	1910 Mar. 3.47726	111 14 26 .1	+52 54 37 .2	RAMBAUD

The preliminary orbit upon which this computation is based is one that had been computed here. Its elements and the corresponding Constants for the Equator are:

Elements.

Epoch	1909 Dec. 18.75330 Gr. M. T.	
M_0	2° 50′ 0″.9	
ω	3 2 1 .7	1909.0
Ω	71 9 32 .0	1909.0
i	19 54 39 .0	1909.0
e	0.625247	
log a	0.569603	
μ	496.″1411	

Constants for the Equator 1909.0.

$$x = r\,[9.976182] \sin (163° \quad 5' \quad 12''.6 + v)$$
$$y = r\,[9.944456] \sin (\;83 \quad 40 \quad 45 .4 + v)$$
$$z = r\,[9.758980] \sin (\;44 \quad 2 \quad 4 .4 + v)$$

From the data of the computation in the preliminary orbit we take the following:

log ρ_0 9.651336

v_0 15° 35′ 48″.2 $\qquad$ E_0 7° 31′ 30″.6

X_I	−0.2477581	X_{II}	−0.0562888	X_{III}	+0.9458926
Y_I	−0.8745050	Y_{II}	−0.9010553	Y_{III}	−0.2734097
Z_I	−0.3793684	Z_{II}	−0.3908799	Z_{III}	−0.1186062

These solar coördinates have been corrected for the elimination of the geocentric parallax, and referred to (1909.0).

Determination of the Constants of the Artificial Initial Orbit.

$A' + v_0$	178° 41′ 0″.8
$B' + v_0$	99 16 33 .6
$C' + v_0$	59 37 52 .6
$\cos A'$	9.980797n
$\cos B'$	9.041762
$\cos C'$	9.856681
$\cos (A' + v_0)$	9.999885n
$\log e \cos A'$	9.776847n
add	0.203675
sum	0.203560n
$\log \frac{\sin a}{\sqrt{p}}$	9.299048
$\log x_0'$	0.002608n
$\cos (B' + v_0)$	9.207339n
$\log e \cos B'$	8.837812
add	0.127647
sum	8.965459n
$\log \frac{\sin b}{\sqrt{p}}$	9.767322
$\log y_0'$	8.732781n
$\cos (C' + v_0)$	9.703775
$\log e \cos C'$	9.652731
add	0.276257
sum	9.980032
$\log \frac{\sin c}{\sqrt{p}}$	9.581846
$\log z_0'$	9.561878

$\log \rho_0$	9.651336
$\cos \delta_0$	9.861042
$\log \sigma_0$	9.512378
$\cos \alpha_0$	8.895883n
$\log \xi_0$	8.408261n
$\log (X)_0$	8.750422n
sub	0.078702
$\log x_0$	8.486963
$\sin \alpha_0$	9.998652
$\log \eta_0$	9.511030
$\log (Y)_0$	9.954752n
sub	0.133532
$\log y_0$	0.088284
$\tan \delta_0$	9.976235
$\log \zeta_0$	9.488613
$\log (Z)_0$	9.592043n
sub	0.252387
$\log z_0$	9.844430
$\log x_0^2$	6.973926
$\log y_0^2$	0.176568
add	0.000272
sum	0.176840
$\log z_0^2$	9.688860
add	0.122249
$\log r_0^2$	0.299089
$\log r_0$	0.149544

$\log x_0'^2$	0.005216
$\log y_0'^2$	7.465562
add	0.001252
sum	0.006468
$\log z_0'^2$	9.123756
add	0.053465
$\log G_0^2$	0.059933
$\log \frac{2}{r_0}$	0.151486
sub	9.370468
$\log \frac{1}{a}$	9.430401
$\log a$	0.569599
$\log x_0\, x_0'$	8.489571n
$\log y_0\, y_0'$	8.821065n
add	0.166172
sum	8.987237n
$\log z_0\, z_0'$	9.406308
add	0.210760
$\log r_0\, r_0'$	9.197997
$\log r_0'$	9.048453

II.

Determination of the Residuals of the First and Third Places using Closed Expressions for f and g.

	I	III
$(t - Ep)$	11.09529	74 71784
$\log (t - Ep)$	1 045138n	1.873424
$\log \mu (t - Ep)$	3.740740n	4 569026
$\mu (t - Ep)$	− 1° 31′ 44″.8	10° 17′ 50″ 3
M	1 18 16 .1	13 7 51 .2
E_1	3 28 38 9	32 14 36 0
$\log \sin E_1$	8 782874	9 727148
$\log e'' \sin E_1$	3.893351	4 837625
$e'' \sin E_1$	2° 10 22″ 6	19° 6′ 45″.8
M_1	1 18 16 3	13 7 50 .2
$M - M_1$	0 .2	+ 1 .0
$E - E_1$	0 .5	+ 2 .1
E	3 28 38 4	32 14 38 .1
$\log \cos E$	9 999200	9 927260
$\log e \cos E$	9 795252	9.723312
sub	9 779822	9 949870
$\log (1 - e \cos E)$	9.575074	9 673182
$\log r$	0.144677	0 242785
$E - E_0$	− 4° 2′ 52″.2	+ 24° 43′ 7″ 5
g	2 1 26 .1	− 12 21 33 .8
$\sin g$	8 547980n	9.330501
log [illegible] 2ω	0 435316	0 435316
$\log \gamma_1$, $\log \gamma_{III}$	8.983296n	9 765817
$\log \gamma^2$	7 966592	9 531634
$\log \frac{\gamma^2}{r_0}$	7 817048	9.382090
sub		
$\log f_1$; $\log f_{III}$	9.997141	9 880218
$\log 2rr_0$	0 595251	0.693359
$\log p \gamma^2$	8.320859	9 885901
sub	9 997685	9.926450
log [ ]	0 592936	0.619809
$\log g$	8 559528	0 151443
$\log g_1$, $\log g_{III}$	9.279764n	0.075722

	I	III
$\log f x_0$	8.484104	8 367181
$\log f y_0$	0 085425	9 968502
$\log f z_0$	9 841571	9 724648
$\log g x_0'$	9 282372	0.078330n
$\log g y_0'$	8.012545	8 808503n
$\log g z_0'$	8 841642n	9 637600
(X)	0 2477581	+ 0 9458926
$f x_0$	+ 0 0304863	+ 0.0232907
$g x_0'$	+ 0.1915896	− 1.1976500
ξ	− 0.0256822	− 0.228467
(Y)	− 0 8745050	− 0.2734097
$f y_0$	+ 1 217377	+ 0 9300400
$g y_0'$	+ 0 0102931	− 0.0643433
η	+ 0.3531651	+ 0.5922870
(Z)	0 3793684	− 0 1186062
$f z_0$	+ 0 6943383	+ 0 5304544
$g z_0'$	0 0694452	+ 0.4341100
ζ	+ 0 2455247	+ 0 8459582
$\rho \cos \delta \cos \alpha$	8 400632n	9.358823n
$\rho \cos \delta \sin \alpha$	9.547978	9 772532
$\tan \alpha$	1.138346n	0 413709n
α_c	94° 9′ 33″ 2	111° 5′ 36″.5
α_0	94 9 31 .4	111 14 26 1
$(O - C)\, \Delta\alpha$	− 1 .8	+ 8 49 .6
$\sin \alpha$	9.998855	9.969881
$\cos \alpha$	8.860509n	9 556170n
$\rho \cos \delta$	9.549123	9.802653
$\rho \sin \delta$	9 390095	9.927349
$\tan \delta$	9 840972	0.124696
δ_c	34° 44′ 12″.2	53° 6′ 53″.0
δ_0	34 44 21 .5	52 54 37 .2
$(O - C)\, \Delta\delta$	+ 9 3	− 12 15 8
$\sin \delta$	9 755728	9.903003
$\cos \delta$	9 914755	9.778307
$\log \rho$	9 634368	0.024346

III.

$R_0 \cos D \cos A$	8.750422n
$R_0 \cos D \sin A$	9 954752n
$\tan A$	1 204330
A	266 25′ 31″.4
$A - \alpha_0$	171 54 45 0
$\sin A$	9 999154n
$\cos A$	8 794823n
$R_0 \cos D$	9 955598
$R_0 \sin D$	9 592043n
$\tan D$	9.636445n
$\sin D$	9 599137n
$\cos D$	9 962692
$\log R_0$	9.992906

$\sin \delta_0$	9.837277
$\sin D$	9.599137n
log I	9.436414n
$\cos \delta_0$	9 861042
$\cos D$	9 962692
$\cos (A - \alpha_0)$	9 995660n
log II	9 819394n
add	0 150455
$\cos \psi$	9 969849n
$\log \rho_0$	9.651336
$\log R_0 \cos \psi$	9.962755n
sub	0.172656
$\log r_1 \cos \beta$	0 135411
$\cos \beta$	9.985867

$\log x_0'\, \xi_0$	8.41087
$\log y_0'\, \eta_0$	8.24381n
add	9.67128
sum	7 91509
$\log z_0'\, \zeta_0$	9.05049
add	0.03069
$\log \rho_0\, \varphi_0$	9.08118
$\log \varphi_0$	9 42984
$\log \sqrt{2} \cos \beta$	0 13638
$\log \frac{\cos \beta}{r_1'}$	9.68678

	I	III
$\log (t - t_0)$	1.04514	1.87342
$\log \theta_{\prime\prime\prime}$; $\log \theta_{\prime}$	9.28072	0.10900
$\log \frac{\gamma^2}{2a}$	7.09596	8.66100
$\log (\gamma_c)^2$	9.99946	9.97963
$\log (\gamma_c)$	9.99973	9.98982
$\log \sqrt{2} \cos \beta \, (\gamma_c)$	0.13610	0.12620
log II	1.15281n	0.36038
$\log (\mathrm{I} = \varphi_0)$	9.42984	
add	9.99170	0.04819
$\log \Phi$	1.14451n	0.40857
$\mathrm{I} = \log 2 r_0 \gamma \, (\gamma_c)$	9.43360n	0.20621
$\log \gamma^3$	7.96659	9.53163
$\mathrm{II} = \log \frac{r_0 r_0'}{\sqrt{2}} \gamma^3$	7.01407	8.57911
add	9.99835	0.01013
log sum	9.43195n	0.21634
$\mathrm{III} = \log \frac{3}{\sqrt{2}} \theta$	9.60732	0.43560
add - sub	9.69680	9.81741
log ( )	9.12874	0.03375n
$\log \gamma_c$ ( )	9.12847	0.02357n
$\log r \gamma$	9.12798n	0.00860
add	7.05000	8.54500
log [ ]	6.17798	8.55360n
$\log \gamma$	8.98330n	9.76582
colog r	9.85532	9.75722
$\log M$	5.43666	8.49670

	I	III
$\log \frac{\gamma^2}{r}$	7.82191	9.28885
$\log \left[1 - \frac{\gamma^2}{r}\right]$	9.99711	9.90609
$\log \theta \left[1 - \frac{\gamma^2}{r}\right]$	9.27783n	0.01509
$\log g$	9.27976n	0.07572
sub	7.64800	9.17554
log ( )	6.92583	9.19063
log I	7.10192	9.36672n
$\log \sqrt{2}\, r_0 \, (\gamma_c)$	0.29979	0.28988
$\log \frac{1}{2} r_0 r_0' \gamma$	7.88027n	8.66279
add	9.99835	0.01013
log ( )	0.29814	0.30001
$\log \gamma^3$	6.94990n	9.29746
colog r	9.85532	9.75722
log II	7.10336n	9.35469
add	7.52000	8.44850
log [ ]	4.62192n	7.80319n
$\log N$	5.19152	8.37279

$\log \frac{\xi_0}{\rho_0}$	8.75792n
$\log \frac{\eta_0}{\rho_0}$	9.85969
$\log \frac{\zeta_0}{\rho_0}$	9.83727

g_{ω_i}	$g_{x_{\prime}}$	$g_{y_{\prime}}$	$g_{z_{\prime}}$	$g_{x_{\prime\prime\prime}}$	$g_{y_{\prime\prime\prime}}$	$g_{z_{\prime\prime\prime}}$
log I	9.13545	7.86562	8.69472n	9.91797n	8.64814n	9.47724
log II	8.63721	0.23853	9.99468	8.62730	0.22862	9.98477
add	0.11975	0.00184	9.97767	9.97717	9.98843	0.11753
log [ ]	9.25520	0.24037	9.97235	9.89514n	0.21705	0.10230
$\log \frac{\gamma_i^3}{r_i r_0}$	6.65568n			8.90513		
$\log g_{\omega_i}$	5.91088n	6.89605n	6.62803n	8.80027n	9.12218	9.00743

m_{ω_i}	$m_{x_{\prime}}$	$m_{y_{\prime}}$	$m_{z_{\prime}}$	$m_{x_{\prime\prime\prime}}$	$m_{y_{\prime\prime\prime}}$	$m_{z_{\prime\prime\prime}}$
$\log M_i \omega_0$	3.92362	5.52494	5.28109	6.98366	8.58498	8.34113
$\log N_i \omega_0'$	5.19413n	3.92430n	4.75340	8.37540n	7.10557n	7.93467
add	9.97606	9.98897	0.11284	9.98201	9.98536	0.14371
$\log m_{\omega_i}$	5.17019n	5.51391	5.39393	8.35741n	8.57034	8.48484

f_{ω_i}	$f_{x_{\prime}}$	$f_{y_{\prime}}$	$f_{z_{\prime}}$	$f_{x_{\prime\prime\prime}}$	$f_{y_{\prime\prime\prime}}$	$f_{z_{\prime\prime\prime}}$
log I	8.75406n	9.85683	9.83441	8.63714n	9.73991	9.71749
$\log 2 m_{\omega_i}$	5.47122n	5.81494	5.69496	8.65844n	8.87137	8.78587
$\log \omega_0 \gamma_i^2$	6.45356	8.05488	7.81103	8.01860	9.61992	9.37607
add	9.95224	0.00249	0.00331	9.88696	0.07130	0.09931
log [ ]	6.40580	8.05737	7.81434	8.54540n	9.69122	9.47538
log II	6.09258	7.74415	7.50112	8.23218n	9.37800	9.16216
log III	7.05539	8.04056	7.77254	9.20884n	9.53075	9.41600
I	- 0.0567625	+ 0.719167	+ 0.682983	- 0.0433650	+ 0.549425	+ 0.521788
II	+ 0.0001237	+ 0.005548	+ 0.003170	- 0.0170680	+ 0.238784	+ 0.145263
III	+ 0.0011360	+ 0.010979	+ 0.005923	- 0.161748	+ 0.339431	+ 0.260618
f_{ω_i}	- 0.0555028	+ 0.735694	+ 0.692076	- 0.222181	+ 1.127640	+ 0.927669
$\log f_{\omega_i}$	8.74431n	9.86670	9.84016	9.34670n	0.05217	9.96739

A_{p_i}	$A_{f_{,}}$	$A_{g_{,}}$	$A_{m_{,}}$	$A_{f_{,,,}}$	$A_{g_{,,,}}$	$A_{m_{,,,}}$
log $p_{y_i} \cos \alpha_i$	8.72721n	5.75656	4.37442n	9.61122n	8.68123n	8.12939n
log $p_{x_i} \sin \alpha_i$	8.74317n	5.90974n	5.16905n	9.31615n	8.76972n	8.32686n
sub	8.57325	0.23116	9.92404	9.98800	9.35411	9.76019
log [ ]	7.30046	6.14090	5.09309	9.30415n	8.03534	7.88958
log ρ_i	9.63437			0.02435		
log I	7.66609			9.27980n		
log $\left(\text{II} = \frac{(p_{z_i} \rho_i)}{\rho_i^2} \cos \delta_i\right)$	6.10492n			5.33067		
add	9.98791			9.99995		
log A_{p_i}	7.65400	6.50653	5.45872	9.27975n	8.01099	7.86523

B_{p_i}	$B_{f_{,}}$	$B_{g_{,}}$	$B_{m_{,}}$	$B_{f_{,,,}}$	$B_{g_{,,,}}$	$B_{m_{,,,}}$
log $p_{y_i} \sin \alpha_i$	9.86556	6.89491n	5.51277	0.02162	9.09163	8.53979
log $p_{x_i} \cos \alpha_i$	7.60482	4.77139	4.03070	8.90575	8.35932	7.91646
add	0.00238	9.99672	0.01408	0.03205	0.07380	0.09274
log ( )	9.86794	6.89163n	5.52685	0.05367	9.16543	8.63253
sin δ_i	9.75573			9.90184		
log sin δ_i ( )	9.62367	6.64736n	5.28258	9.95551	9.06728	8.53437
log $p_{z_i} \cos \delta_i$	9.75492	6.54279n	5.30869	9.74776	8.78780	8.26521
sub	9.54758	9.43495	8.79217	9.78776	9.95575	9.93374
log [ ]	9.17125n	5.97774n	4.07475n	9.53552	8.74356	8.19895
log I	9.53688			9.51117n		
log $\left(\text{II} = \frac{(p_{\delta_i} \rho_i)}{\rho_i^2}\right)$	5.80176			4.56970n		
add	0.00008			0.00001		
log B_{p_i}	9.53696	6.34337	4.44038	9.51118n	8.71921n	8.17460n

	I	III
log g	9.27976n	0.07572
log ρ	9.63437	0.02435
log $C_{,}$; log $C_{,,,}$	9.64539n	0.05137

Computation of β_i, γ_i, δ_i, ν_i.

	I	III		I	III
log (I = $C \sin \alpha$)	9.64425n	0.02082	log (I = $C \cos \alpha$)	8.50590	9.61042n
log II	4.99349	6.49795	log II	6.59481	8.09927
log III	5.76236n	8.16887n	log III	4.49253n	6.89904n
I	−0.440810	+1.049098	I	+0.032055	−0.407773
−II	−0.0000099	−0.0003147	II .	+0.000394	+0.0125680
−III	+0.0000579	+0.0147527	III	−0.000003	−0.0007926
$-\beta_1$; $-\beta_3$	−0.440762	+1.063536	γ_1 ; γ_3	+0.032446	−0.395998
log β_1 ; log β_3	9.64421	0.02675n	log γ_1 ; log γ_3	8.51116	9.59770n
log I	8.26163	9.51256n	log I	9.39998n	9.92266
log II	4.83033	7.20618n	log II	6.43165	8.80750n
log III	4.74402n	8.47824	log III	3.47419n	7.20841
I	+0.0182654	−0.325285	I	−0.2511765	+0.836880
−II	−0.0000068	+0.0016076	−II	−0.0002702	+0.0641943
−III	+0.0000055	−0.0300773	−III	+0.0000003	−0.0016159
$-\beta_2$; $-\beta_4$	+0.018264	−0.353755	$-\gamma_2$; $-\gamma_4$	−0.251446	+0.899458
log β_2 ; log β_4	8.26160n	9.54871	log γ_2 ; log γ_4	9.40045	9.95398n

	I	III
log I	6.35096	7.85543
log II	5.32163	7.72814
add	0.03880	0.24204
log δ_1 ; log δ_3	6.38976	8.09747
log I	9.560151n	9.83174
log II	6.18780	8.56365n
log III	4.30329	8.03751n
I	−0.363200	+0.678800
II	+0.000154	−0.036614
III	+0.000002	−0.010902
δ_2 ; δ_4	−0.363044	+0.631284
log δ_2 ; log δ_4	9.55997n	9.80023

	I	III
log $\partial\alpha$	0.25527n	2.72395
cos δ	9.91476	9.78037
log sin 1″	4.68557	4.68557
log ν_1 ; log ν_3	4.85560n	7.18989
log $\partial\delta$	0.96848	2.86676n
log ν_2 ; log ν_4	5.65405	7.55233n

Equations for Solution (Coefficients Logarithmic).

$(7.65400)\,\partial\rho_0$	$+\ (9.64421)\,\partial x_0'$	$+\ (8.51116)\,\partial y_0'$	$+\ (6.38976)\,\partial z_0'$	$=\ (4.85560n)$
(9.53696)	$+\ (8.26160n)$	$+\ (9.40045)$	$+\ (9.55997n)$	$=\ (5.65405)$
$(9.27975n)$	$+\ (0.02675n)$	$+\ (9.59770n)$	$+\ (8.09747)$	$=\ (7.18989)$
$(9.51118n)$	$+\ (9.54871)$	$+\ (9.95398n)$	$+\ (9.80023)$	$=\ (7.55233n)$

The solution of these equations gives

log $\partial\rho_0$	8.04315n
log $\partial x_0'$	5.44027
log $\partial y_0'$	7.00605
log $\partial z_0'$	7.99555n

From the value of log $\partial\rho_0$ we have

log ∂x_0	6.80007
log ∂y_0	7.90284n
log ∂z_0	7.88042n

Applying these values of ∂x_0, ∂y_0, ∂z_0, $\partial x_0'$, $\partial y_0'$, and $\partial z_0'$ to the old values of x_0, y_0, z_0, x_0', y_0', and z_0' respectively, we have the new or corrected values

log x_0	8.495803	log x_0'	0.002596n
log y_0	0.085441	log y_0'	8.724556n
log z_0	9.839687	log z_0'	9.549926
log r_0	0.146242	log r_0'	9.027470
log ρ_0	9.640497	log G_0^2	0.057168

For the representation

	I	III
log f	9.997076	9.876614
log g	9.279742n	0.074726
(O − C) $\partial\alpha$	−0″.9	+ 2″.4
(O − C) $\partial\delta$	−1 .2	−12 .6

Then substituting these residuals to form new absolute terms in our four equations for $\partial\rho_0$, etc., keeping the other coefficients the same and resolving for further corrections $\partial\rho_0$, etc., we remove the residuals and have finally:

log x_0	8.495919	log x_0'	0.002600
log y_0	0.085404	log y_0'	8.724205n
log z_0	9.839623	log z_0'	9.549805

whence

log r_0	0.146200	log r_0'	9.027352
		log G_0^2	0.057147

IV. (A VIIIc)

Elements.

$\log \frac{2}{r_0}$	0.154830
$\log G_0^2$	0.057147
sub	9.401790
$\log \frac{1}{a}$	9.458937
$\log a$	0.541063
$\log r_0'^2$	8.054704
sub	9.995659
$\log (G^2_0 - r_0'^2)$	0.052806
$\log r_0^2$	0.292399
$\log p$	0.345205
$\log \frac{p}{r_0}$	0.199005
sub	
$\log e \cos v_0$	9.764376

$\log \sqrt{p}$	0.172602
$\log e \sin v_0$	9.199954
$\tan v_0$	9.435578
v_0	15° 15′ 0″.2
$\sin v_0$	9.420009
$\cos v_0$	9.984432
$\log e$	9.779944
$\log e^2$	9.559888
sub	
$\log (1 - e^2)$	9.804150
$\log a$	0 541055 check
$\log a^3$	1.623189
$\log a^{\frac{3}{2}}$	0.811594
Period	6.48029 years
$\log k''$	3.550007
$\log \mu''$	2.738413
μ''	547″.5362

$\frac{1}{2} v_0$	7° 37′ 30″.1
$\tan \frac{1}{2} v_0$	9.126694
$\log \sqrt{\frac{1 - e}{1 + e}}$	9.697282
$\tan \frac{1}{2} E_0$	8.823976
$\frac{1}{2} E_0$	3° 48′ 52″.8
E_0	7 37 45 .6
$\sin E_0$	9.123080
$\log e''$	5.094369
$\log e'' \sin E_0$	4.217449
$e'' \sin E_0$	16498″.7
M_0	3° 2′ 46″.9
Epoch	1909 Dec. 18.75330 Gr. M. T.

Constants for the Equator.

d w D'	a x A'	b y B'	c z C'
$\log r_0 w_0'$	0.148800	8.870405	9.696005
$\log r_0' w_0$	9.523271	9.112756	8.866975
sub	0.001027	0.196545	9.930316
log [ ]	0.149827	9.309301n	9 626321
$\sin d \cos (D' + v_0)$	9.977225n	9.136699n	9.453719
$\sin d \sin (D' + v_0)$	8.349719n	9.939204	9.693423
$\tan (D' + v_0)$	8.372494n	0.802505n	0.239704
$D' + v_0$	178° 38′ 57″.7	98° 57′ 17″.7	60° 3′ 55″.0
v_0	15 15 0 .2		
D'	163 23 57 .5	83 42 17 .5	44 48 54 .8
$\sin (D' + v_0)$	8.372373	9.994674	9.937816
$\cos (D' + v_0)$	9.999879n	9.192169n	9.698112
$\sin d$	9.977346	9.944530	9 755607

The details of the computation for the elements ω, Ω and i are not given here as they are exactly the same as shown in Example No. 1, page 395. They are given in the collection of elements.

The representation of the First and Third Places for the corrected orbit is computed exactly as shown on page 442 of this example, so the details are omitted. The principal quantities are

$\log f_{,}$	9.997075	$\log g_{,}$	9.279743n
$\log f_{,,,}$	9.876577	$\log g_{,,,}$	0.074712

		I	III
(O − C)	$\Delta\alpha$	+1″.0	−0″.4
	$\Delta\delta$	+0 .2	+0 .5

The corrected orbit then is given by the following collected

Elements.

Epoch	1909 Dec. 18.75330 Gr. M. T.	
M_0	$3^\circ\ 2'\ 46''.9$	
ω	$3\ \ 28\ \ 42\ .6$	
Ω	$70\ \ 58\ \ 54\ .4$	(1909.0)
i	$19\ \ 26\ \ 48\ .0$	
e	0.602481	
$\log a$	0.541063	
μ	$547''.5362$	

and

Constants for the Equator 1909.0.

$$x = r\,[9.977346]\sin(163''\ 23'\ 57''.5 + v)$$
$$y = r\,[9.944530]\sin(\ 83\ \ 42\ \ 17\ .5 + v)$$
$$z = r\,[9.755607]\sin(\ 44\ \ 48\ \ 54\ .8 + v)$$

EXAMPLE NO. 13.

CASE OF AN INITIAL NEARLY PARABOLIC ORBIT.

A portion of the work on HALLEY'S comet is given in this example to illustrate, in detail, the computation of a second orbit based upon a preliminary orbit. The observations used cover a range of 164 days, giving intervals of 90 and 74 days respectively. These long intervals necessitate the use of the closed expressions in f, g, ∂f, and ∂g, so that this example will show not only how a second orbit is computed but also the use of these closed expressions. Furthermore, the orbit is of the nearly parabolic type. Since the eccentricity of this orbit is well defined the formulæ 3 (Part 7, page 333) are used here instead of formulæ 2 (Part 7, page 332) which would ordinarily be used for a *very* nearly parabolic orbit.

It must be borne in mind that the computation given here is merely to show the applications of the formulæ of LEUSCHNER'S Method and is not to be taken as a final orbit for the comet as it is based upon only three observations at one apparition. The computations were made in duplicate by R. T. CRAWFORD and W. F. MEYER.

I.

The first observation is by BARNARD. The second place is a normal based upon four observations by AITKEN. The third is an observation by AITKEN.

They are:

	Date Gr. M. T. 1909		α (1909.0)			δ (1909.0)		
I	January	260.85720	94°	44′	51″.6	+17°	8′	59″.2
II	January	350.77347	49	56	33 .7	+13	59	51 .8
III	January	424.63910	8	32	55 .9	+ 7	53	50 .0

The dates of these observations, as given, have been corrected for aberration and the coördinates have been corrected for parallax, and referred to 1909.0.

The preliminary orbit upon which this computation is based is one that had been computed here. Its elements and the corresponding constants for the equator are:

Elements.

T =	1910	April	19.71304 Gr. M. T.	
ω	111°	46	31″.70	1909.0
Ω	57	15	36 .95	1909.0
i	162	12	32 .88	1909.0
e	0.9667007			
log a	1.2458385			
log q	9.7682756			

Constants for the Equator 1909.0.

$$x = r\,[9.9851611] \sin (145^\circ\ 48'\ 18''.47 + v)$$
$$y = r\,[9.9884770] \sin (239\ \ 21\ \ 52 .17 + v)$$
$$z = r\,[9.5354488] \sin (189\ \ 5\ \ 11 .47 + v)$$

From the data of the computation in the preliminary orbit we take for the middle date,

$\log \rho_0$	0.1324984	$\log X_0$	8.9582938n
v_0	239° 50′ 22.″75	$\log Y_0$	9.9536780
		$\log Z_0$	9.5909691n

Determination of Heliocentric Coördinates and Their Velocities for the Middle Date in the Artificial Orbit.

$A' + v_0$	25° 38′ 41″.2
$B' + v_0$	110 12 14 .9
$C' + v_0$	68 55 34 .2
$\cos A'$	9.9175742n
$\cos B'$	9.7072079
$\cos C'$	9.9945155n
$\cos (A' + v_0)$	9.9549631
$\log e \cos A'$	9.9028662n
add	0.9467605
sum	9.0082026
$\log \frac{\sin a}{\sqrt{p}}$	9.9541541
$\log x_0'$	8.9623567
$\cos (B' + v_0)$	9.6883510n
$\log e \cos B'$	9.6924999n
add	0.2989605
sum	9.9914604n
$\log \frac{\sin b}{\sqrt{p}}$	9.9574700
$\log y_0'$	9.9489304n
$\cos (C' + v_0)$	9.5557843
$\log e \cos C'$	9.9798075n
add	0.2052915
sum	9.7745160n
$\log \frac{\sin c}{\sqrt{p}}$	9.5044418
$\log z_0'$	9.2789578n

$\log \rho_0$	0.1324984
$\cos \delta_0$	9.9869084
$\log \sigma_0$	0.1194068
$\cos \alpha_0$	9.8085846
$\log \xi_0$	9.9279914
$\log X_0$	8.9582938n
sub	0.0442365
$\log x_0$	9.9722279
$\sin \alpha_0$	9.8838891
$\log \eta_0$	0.0032959
$\log Y_0$	9.9536780n
sub	0.2769293
$\log y_0$	0.2802252
$\tan \delta_0$	9.3966975
$\log \zeta_0$	9.5161043
$\log Z_0$	9.5909691n
sub	0.2652088
$\log z_0$	9.8561779
$\log x_0^2$	9.9444558
$\log y_0^2$	0.5604504
add	0.0941586
sum	0.6546090
$\log z_0^2$	9.7123558
add	0.0469714
$\log r_0^2$	0.7015804
$\log r_0$	0.3507902

$\log x_0'^2$	7.9247134
$\log y_0'^2$	9.8978608
add	0.0045955
sum	9.9024563
$\log z_0'^2$	8.5579156
add	0.0192132
$\log G_0^2$	9.9216695
$\log \frac{2}{r_0}$	9.9502398
sub	1.1960764
$\log \frac{1}{a}$	8.7541634
$\log a$	1.2458366
$\log x_0 x_0'$	8.9345846
$\log y_0 y_0'$	0.2291556n
add	0.0226190
sum	0.2065366n
$\log z_0 z_0'$	9.1351357n
add	0.0353655
$\log r_0 r_0'$	0.2419021n
$\log r_0'$	9.8911119n

II.

Representation of First and Third Places.

($f_{,}$, $f_{,,,}$, $g_{,}$, $g_{,,,}$ determined by closed form using the preliminary elements to determine $r_{,}$, $r_{,,,}$, $v_{,}$ and $v_{,,,}$ by OPPOLZER's nearly parabolic method).

	I	III
$t - T$	−213.85584	−50.07394
$\log (t - T)$	2.3301212n	1.6996118n
$\log M$	2.6710984n	2.0405890n
w	−130° 57′ 29″.60	−90° 3′ 46″.87
$\frac{1}{2} w$	− 65 28 44 .80	−45 1 53 .43
$\tan \frac{1}{2} w$	0.3408765n	0.0004776n
$\log x$	0.3438401n	0.0034412n
$\log x^2$	0.6876802	0.0068824
$\log \eta$	8.9161658	8.2353680
η	0.0824453	0.0171936
$\log G$	0.0147238	0.0030038
$\log H$	− 2	0
$\tan \frac{1}{2} v$	0.3585637n	0.0064450n
$\frac{1}{2} v$	− 66° 20′ 54″.29	−45° 25′ 30″.45
$v_{,}$; $v_{,,,}$	−132 41 48 .59	−90 51 0 .90
$\tan \frac{1}{2} v$	0.7171274	0.0128900
$\log \theta$	8.9458252	8.2415878
$\log (1 + \theta)$	0.0367376	0.0075095
add	0.0762073	0.2946328
$\log (1 + \tan^2 \frac{1}{2} v)$	0.7933347	0.3075228
$\text{colog} (1 + \theta)$	9.9632624	9.9924905
$\log r_{,}$; $\log r_{,,,}$	0.5248727	0.0682889
$v - v_0$	− 12° 32′ 11″.34	+29° 18′ 36″.35
$\frac{1}{2} (v - v_0)$	− 6 16 5 .67	+14 39 18 .18
$\log r r_0$	0.8756629	0.4190791
$\log \sqrt{r r_0}$	0.4378314	0.2095396
$\sin \frac{1}{2} (v - v_0)$	9.0381564n	9.4031188
$\log \gamma_{,}$; $\log \gamma_{,,,}$	9.5954958n	9.7321664
$\log \gamma^2$	9.1909916	9.4643328
$\log \frac{\gamma^2}{r_0}$	8.8402014	9.1135426
sub	0.0311507	0.0604209
$\log f_{,}$; $\log f_{,,,}$	9.9688493	9.9395791
$\sin (v - v_0)$	9.3365823n	9.6897847
$\log g_{,}$; $\log g_{,,,}$	0.1812382n	0.0778568

	I	III
$\log f x_0$	9.9410772	9.9118070
$\log f y_0$	0.2490745	0.2198043
$\log f z_0$	9.8250272	9.7957570
$\log g x_0'$	9.1435949n	9.0402135
$\log g y_0'$	0.1301686	0.0267872n
$\log g z_0'$	9.4601960	9.3568146n
X	− 1.0003516	+0.9293000
$f x_0$	+0.8731266	+0.8162196
$g x_0'$	−0.1391858	+0.1097017
ξ	−0.2664108	+1.8552213
Y	+0.0839090	−0.3157650
$f y_0$	+1.7744939	+1.6588393
$g y_0'$	+1.3494866	−1.0636218
η	+3.2078895	+0.2794525
Z	+0.0364017	−0.1369780
$f z_0$	+0.6683858	+0.6248230
$g z_0'$	+0.2885333	−0.2274126
ζ	+0.9933208	+0.2604324
$\rho \cos \delta \cos \alpha$	9.4255518n	0.2683957
$\rho \cos \delta \sin \alpha$	0.5062194	9.4463080
$\tan \alpha$	1.0806676n	9.1779123
α_c	94° 44′ 50″.79	8° 33′ 57″.89
α_0	94 44 51 .60	8 32 55 .90
(O − C) $\Delta\alpha$	+ 0 .81	− 61.99
$\sin \alpha$	9.9985075	9.1730404
$\cos \alpha$	8.9178399n	9.9951281
$\rho \cos \delta$	0.5077119	0.2732676
$\rho \sin \delta$	9.9970896	9.4156950
$\tan \delta$	9.4893777	9.1424274
δ_c	+17° 8′ 58″.38	+7° 54′ 10″.25
δ_0	+17 8 59 .20	+7 53 50 .00
(O − C) $\Delta\delta$	+ 0 .82	− 20 .25
$\sin \delta$	9.4696259	9.1382830
$\cos \delta$	9.9802482	9.9958556
$\log \rho$	0.5274637	0.2774120

The solar coördinates for these places were derived from the data of the American Ephemeris, and reduced to 1909.0.

III.

Differential Correction to Remove Residuals.

$R_0 \cos D \cos A$	8.95829n	$\sin \delta_{\prime\prime}$	9.38361	$\log x_0' \xi_0$	8.89035
$R_0 \cos D \sin A$	9.95368n	$\sin D$	9.59799n	$\log y_0' \eta_0$	9.95223n
$\tan A$	0.99539	log I	8.98160n	add	9.96061
A	264°.2288	$\cos \delta_{\prime\prime}$	9.98691	sum	9.91284n
$\alpha_{\prime\prime}$	49 .9427	$\cos D$	9.96291	$\log z_0' \zeta_0$	8.79506n
$A - \alpha_{\prime\prime}$	214 .2861	$\cos (A - \alpha_{\prime\prime})$	9.91710n	add	0.03192
$\sin A$	9.99779n	log II	9.86692n	$\log \rho_0 \varphi_0$	9.94476n
$\cos A$	9.00240n	add	0.05316	$\log \varphi_0$	9.81226n
$R_0 \cos D$	9.95589	$\cos \psi$	9.92008n	$\log \sqrt{2} \cos \beta$	0.13725
$R_0 \sin D$	9.59097n	$\log \rho_0$	0.13250	$\log \dfrac{\cos \beta}{r_0^2}$	9.28515
$\tan D$	9.63508n	$\log R_0 \cos \psi$	9.91306n		
$\sin D$	9.59799n	sub	0.20502		
$\cos D$	9.96291	$\log r_0 \cos \beta$	0.33752		
$\log R_0$	9.99298	$\cos \beta$	9.98673		

	I	III
$\log (t - t_0)$	1.95384n	1.86844
$\log \theta_{\prime\prime\prime}$; $\log \theta_{\prime}$	0.18942	0.10402
$\log \dfrac{\gamma^2}{2a}$	7.64412	7.91747
$\log (\gamma_c)^2$	9.99808	9.99639
$\log (\gamma_c)$	9.99904	9.99820
$\log \sqrt{2} \dfrac{\cos \beta}{\gamma}$	0.54175n	0.40508
log II	0.54079n	0.40328
$\log (\mathrm{I} = \varphi_0)$	9.81226n	9.81226n
add	0.07440	9.87132
$\log \Phi$	0.61519n	0.27460
log I	0.12037n	9.80046
$\log 2 r_0 \gamma$	0.24732n	0.38399
log II	0.24540n	0.38039
$\log \gamma^2$	9.19099	9.46433
$\log \dfrac{r_0 r_0'}{\sqrt{2}}$	0.09139n	0.09139n
log III	9.28142n	9.55392n
log IV	0.51507	0.42883
I	−1.31939	+0.63163
II	−1.75954	+2.40100
III	−0.19117	−0.35803
±IV	+3.27392	−2.68431
[ ]	+0.00382	−0.00971
log [ ]	7.58206	7.98722n
$\log \gamma$	9.59550n	9.73217
colog r	9.47513	9.93171
$\log M$	7.54774	8.54615

	I	III
$\log \dfrac{\gamma^2}{r}$	8.66612	9.39604
$\log \left[1 - \dfrac{\gamma^2}{r} \right]$	9.97938	9.87570
$\log \theta \left[1 - \dfrac{\gamma^2}{r} \right]$	0.16880n	9.97972
$\log g$	0.18124n	0.07786
sub	8.46333	9.40405
log ( )	8.63213	9.38377n
log I	8.80822	9.55986n
$\log \sqrt{2}\, r_0 (\gamma_c)$	0.50035	0.49951
$\log \frac{1}{2} r_0 r_0' \gamma$	9.53637	8.67304n
add	0.04479	9.92987
log ( )	0.54514	0.42938
$\log \gamma^3$	8.78649n	9.19650
colog r	9.47513	9.93171
log II	8.80676n	9.55759
add	7.52667	7.72000
log [ ]	6.33343	7.27759n
$\log N$	7.57927n	8.52343
$\log \dfrac{\xi_0}{\rho_0}$	9.79549	
$\log \dfrac{\eta_0}{\rho_0}$	9.87080	
$\log \dfrac{\zeta_0}{\rho_0}$	9.38361	

g_{ω_i}	g_{x_I}	g_{y_I}	g_{z_I}	$g_{x_{III}}$	$g_{y_{III}}$	$g_{z_{III}}$
log I	8.90865n	9.89522	9.22525	9.04532	0.03189n	9.36192n
log II	0.12179	0.42979	0.00574	0.12095	0.42895	0.00490
add	9.97257	0.11127	0.06661	0.03504	0.17462	9.88788
log [ ]	0.09436	0.54106	0.07235	0.15599	0.20651	9.89278
log $\frac{\gamma_i^3}{r_i r_0}$	7.91083n			8.77742		
log g_{ω_i}	8.00519n	8.45189n	7.98318n	8.93341	8.98393	8.67020

m_{ω_i}	m_{x_I}	m_{y_I}	m_{z_I}	$m_{x_{III}}$	$m_{y_{III}}$	$m_{z_{III}}$
log $M_i\ \omega_0$	7.51997	7.82797	7.40392	8.51838	8.82638	8.40233
log $N_i\ \omega_0'$	6.54163n	7 52820	6.85823	7.48579	8.47236n	7.80239n
add	9.95177	0.17651	0.10879	0.03853	0.10021	9.87435
log m_{ω_i}	7.47174	8.00448	7.51271	8.55691	8.57257	8.27668

f_{ω_i}	f_{x_I}	f_{y_I}	f_{z_I}	$f_{x_{III}}$	$f_{y_{III}}$	$f_{z_{III}}$
log I	9.76434	9.83965	9.35246	9.73507	9.81038	9.32319
log $2m_{\omega_i}$	7.77277	8.30551	7.81374	8.85794	8.87360	8.57771
log $\omega_0\,\gamma_i^2$	9.16322	9.47122	9.04717	9.43656	9.74456	9.32051
add	0.01732	0.02869	0.02466	0.10170	0.05484	0.07218
log [ ]	9.18054	9.49991	9.07183	9.53826	9.79940	9.39269
log II	8.46569	8.78506	8.35698	8.82341	9.08455	8.67784
log III	8.62038	9.06708	8.59837	9.20801	9.25853	8.94480
I	+0.58121	+0.69127	+0.225142	+0.54334	+0.64622	+0.210470
II	+0.02922	+0.06096	+0.022750	+0.06659	+0.12149	+0.047626
III	+0.04172	+0.11670	+0.039662	+0.16144	+0.18135	+0.088064
f_{ω_i}	+0.65215	+0.86893	+0.287554	+0.77137	+0.94906	+0.346160
log f_{ω_i}	9.81434	9.93898	9.45872	9.88726	9.97729	9.53928

A_{p_i}	A_{f_I}	A_{g_I}	A_{m_I}	$A_{f_{III}}$	$A_{g_{III}}$	$A_{m_{III}}$
log p_{y_i} cos α_i	8.85682n	7.36973	6.92232n	9.97242	8.97906	8.56770
log p_{x_i} sin α_i	9.81285	8.00370n	7.47025	9.06030	8.10645	7.72995
sub	0.04558	0.09072	0.10829	9.94329	9.93747	9.93182
log [ ]	9.85843n	8.09442	7.57854n	9.91571	8.91653	8.49952
log ρ_i	0.52746			0.27741		
log A_{p_i}	9.33097n	7.56696	7.05108n	9.63830	8.63912	8.22211

B_{p_i}	B_{f_I}	B_{g_I}	B_{m_I}	$B_{f_{III}}$	$B_{g_{III}}$	$B_{m_{III}}$
log p_{y_i} sin α_i	9.93749	8.45040n	8.00299	9.15033	8.15697	7.74561
log p_{x_i} cos α_i	8.73218n	6.92303	6.38958n	9.88239	8.92854	8.55204
add	9.97205	9.98691	9.98929	0.07384	0.06789	0.06302
log ( )	9.90954	8.43731n	7.99228	6.95623	8.99643	8.61506
sin δ_i	9.46963			9.13828		
log I	9.37917	7.90694n	7.46191	9.09451	8.13471	7.75334
log II	9.43897	7.96343n	7.49296	9.53514	8.66606	8.27254
sub	9.16915	9.14275	8.86986	9.80445	9.84868	9.84352
log [ ]	8.54832n	7.04969	6.33177n	9.33959	8.51474n	8.11606n
log B_{p_i}	8.02086	6.52223n	5.80431	9.06218	8.23733	7.83865

	I	II
log g	0.18124n	0.07786
log ρ	0.52746	0.27741
log C_I ; log C_{III}	9.65378n	9.80045

For β_i, γ_i, δ_i, ν_i.

	I	III
log (I = C sin α)	9.65229n	8.97349
log II	7.53919	8.61135
log III	6.31447n	7.48550
I	−0.44904	+0.094078
−II	− 346	− 40865
−III	+ 21	− 3058
$-\beta_1$; $-\beta_3$	−0.45229	+0.050155
log β_1; log β_3	9.65542	8.70032n
log I	8.04125	8.93386
log II	6.49446n	8.20956
log III	5.06770	7.10204
I	+0.0109964	+0.085874
−II	+ 3122	− 16202
−III	− 117	− 1265
$-\beta_2$; $-\beta_4$	+0.0112969	+0.068407
log β_2; log β_4	8.05296n	8.83510n
log I	7.42314	8.49530
log II	6.63107	7.80210n
add	0.06499	9.90163
log δ_1; log δ_3	7.48813	8.39693
log I	9.63403n	9.79631
log II	6.37841n	8.09351
log III	5.38430n	7.41864n
I	−0.43056	+0.62561
II	− 24	+ 1240
III	− 2	− 262
δ_2; δ_4	−0.43082	+0.63539
log δ_2; log δ_4	9.63430n	9.80304

	I	III
log (I = C cos α)	8.57162	9.79558
log II	7.84719	8.91935
log III	7.30104	8.47207n
I	+0.0372925	+0.62457
II	+ 70338	+ 8305
III	+ 20000	− 2965
γ_1; γ_3	+0.0463263	+0.67797
log γ_1; log γ_3	8.66583	9.83121
log I	9.12192n	8.11177
log II	6.80246n	8.51756
log III	6.05427n	8.08861n
I	−0.132409	+0.0129351
−II	+ 635	− 329277
−III	+ 113	+ 122634
$-\gamma_2$; $-\gamma_4$	−0.131661	−0.0077292
log γ_2; log γ_4	9.11946	7.88813
log $\partial\alpha$	9.90849	1.79232n
cos δ	9.98025	9.99586
log sin 1″	4.68558	
log ν_1; log ν_3	4.57432	6.47376n
log $\partial\delta$	9.92428	1.30643n
log ν_2; log ν_4	4.60986	5.99201n

Equations for Solution (Coefficients Logarithmic).

$$\begin{array}{lllll}
(9.33097\text{n})\,\partial\rho_0 & + (9.65542\ )\,\partial x_0' & + (8.66583)\,\partial y_0' & + (7.48813\ )\,\partial z_0 & = (4.57432\ ) \\
(8.02086\ ) & + (8.05296\text{n}) & + (9.11946) & + (9.63430\text{n}) & = (4.60986\ ) \\
(9.63830\ ) & + (8.70032\text{n}) & + (9.83121) & + (8.39693\ ) & = (6.47376\text{n}) \\
(9.06218\ ) & + (8.83510\text{n}) & + (7.88813) & + (9.80304\ ) & = (5.99201\text{n})
\end{array}$$

The solution of these equations gives

log $\partial\rho_0'$	5.37967
log $\partial x_0'$	5.82046
log $\partial y_0'$	6.64756n
log ∂z_0	6.16537n

From the value of log $\partial\rho_0$ we have

log ∂x_0	5.17516
log ∂y_0	5.25047
log ∂z_0	4.76328

Applying these values of ∂x_0, ∂y_0, ∂z_0, $\partial x_0'$, $\partial y_0'$, and $\partial z_0'$ to the old values of x_0, y_0, z_0, x_0', y_0', and z_0', respectively, we have the new values

log x_0	9.9722348	log x_0'	8.9626699
log y_0	0.2802293	log y_0'	9.9491473n
log z_0	9.8561814	log z_0'	9.2792920n
log r_0	0.3507948	log r_0'	9.8913326n
		log G_0^2	9.9221155

IV.

From these the following quantities are derived by A VIII (the details of the computation are not given):

ω	239° 53′ 36″.95
log e	9.9855082
log a	1.2525224
log q	9.7686346
T	1910 April 19.67760 Gr. M. T.

Using these quantities the new values of $f_{,}$, $f_{,,,}$, $g_{,}$, and $g_{,,,}$ were derived as on page 450 of this example. They are:

log $f_{,}$	9.9688565
log $f_{,,,}$	9.9395596
log $g_{,}$	0.1812403n
log $g_{,,,}$	0.0778450

With these and the corrected heliocentric coördinates and their velocities the residuals for the first and the third places were computed. They are:

		I	III
(O − C)	$\Delta\alpha$	0″.0	−0″.1
	$\Delta\delta$	0 .0	0 .0

SHORT METHODS OF DETERMINING ORBITS.

THIRD PAPER.

THE DIRECT SOLUTION OF ORBITS OF DISTURBED BODIES.

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BERKELEY ASTRONOMICAL DEPARTMENT.

UNIVERSITY OF CALIFORNIA PUBLICATIONS.

PUBLICATIONS OF THE LICK OBSERVATORY, VOL. VII, PART 9.

SHORT METHODS OF DETERMINING ORBITS.

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CONTENTS.

*This paper, with the exception of the references to the orbit of the Eighth Satellite of Jupiter, was written in 1905 and read in abstract under the title of *An Analytical Method of Determining the Orbits of New Satellites* at the seventh meeting of the *Astronomical and Astrophysical Society of America*, New York, December, 1905. *Publications*, Vol. 1, Page 249.

INTRODUCTION.

The recent[1] discoveries by W. H. PICKERING and C. D. PERRINE of distant satellites of Saturn and Jupiter, and the fact that the powerful photographic methods of search now in use undoubtedly will lead to the further discovery of satellites, make it imperative that an adequate method should be established for the determination of the orbits of new satellites from a limited number of observations. Any method proposed for this purpose should be simple in its application and should permit of the ready correction of the preliminary elements on the basis of additional observations. It is also of the greatest importance that the principal perturbations of the satellite, during the period covering the observations on which the orbit is based, should be taken into account in the first approximation of the elements. The graphical methods, which are patterned in the main after the methods in use for the determination of double star orbits, seem to be wholly inadequate for the solution of an orbit to the degree of accuracy which the observations are capable of yielding even in the case of a short arc. Solutions which are partially analytic but involve arbitrary assumptions or variations of certain elements, as the period, position of the orbit plane, or eccentricity, also should be avoided. In the case of satellites which are subject to large perturbations, the essential object to be aimed at is the immediate determination of a set of osculating elements of the greatest possible accuracy. On the basis of such a set of elements together with the perturbations, comparison may be made with observations and the elements improved. A graphical construction of what may be termed an average orbit through the given observations, or a partially analytical solution on the basis of certain arbitrary assumptions, is very likely to result in a set of elements which differ to such an extent from the osculating elements that any perturbations based on the same will be greatly in error, nor can they be confounded with mean elements owing to the shortness of the arc. But, if the perturbations are greatly in error, the comparison of theory with observation will lead to residuals which are also erroneous, and the improvement of the elements on the basis of such residuals may be questioned. In fact, the whole process may turn out to be non-convergent.

My attention was first directed to this problem through W. H. PICKERING'S valuable memoir *On the Ninth Satellite of Saturn*, *Annals of Harvard College Observatory*, LIII, No. III. A direct analytical method of solution of the general problem involved in this case seemed to be eminently desirable, particularly as it

[1]This paper was read at the Seventh Meeting of the Astronomical and Astrophysical Society of America, Columbia University, New York, December, 1905; *cf. Publications of the Astronomical and Astrophysical Society of America*, Volume I, page 249.

might be expected that such a method would readily reveal all the solutions possible on the basis of the given series of observations; but no serious thought was given to the matter until after the appearance in September of this year[1] of *Lick Observatory Bulletins* 78 and 82 giving PERRINE'S account of his discovery of the sixth and seventh satellites of Jupiter, the first series of observations secured at the Lick Observatory, and the results of the investigation of the orbits by PERRINE and ROSS. On page 130 of *Lick Observatory Bulletin* No. 78, PERRINE calls attention to the fact that no method of deriving the elements of a satellite directly from three or more observations has been published, and that consequently his solutions were derived by graphical methods and on the basis of arbitrary assumptions. The method used by ROSS in the case of the seventh satellite was that of arbitrary variation of the position of the plane of the orbit. No reference is given, however, as to the method used in the case of his determination of the orbit of the sixth satellite of Jupiter. The method used by PICKERING in the case of the ninth satellite of Saturn is entirely graphical and based upon the assumption that the plane of the orbit is inclined to the ecliptic at an angle of about five degrees, which assumption was obtained by inspection.

From correspondence with Professor NEWCOMB and Dr. ROSS during the summer, it appears that considerable difficulty was encountered in determining the orbits of the sixth and seventh satellites of Jupiter. In all of these determinations, the object of the investigators was to satisfy the observations by an *elliptic* orbit, *irrespective of any perturbations* which might have prevailed during the period of observation, although ROSS fully realized that the perturbations of the Sun were quite appreciable. The *preliminary* orbits obtained by PICKERING, PERRINE and ROSS, therefore, leave residuals which considerably exceed the possible errors of observation, although ROSS'S results are the most satisfactory in this respect.

My own *original* intention, also, was to set up a method which would satisfy the observations by *elliptic orbits irrespective of perturbations*, but by *direct analytical solution without any arbitrary assumption*, so that the numerical values of none of the constants would be prejudiced beforehand. My further aim was to arrange the formulæ in such a manner that all possible solutions would reveal themselves at one and the same time within the degree of certainty that a limited arc would permit. In deriving my solution of the problem, I had the able assistance of Dr. RUSSELL TRACY CRAWFORD and Mr. A. J. CHAMPREUX, who participated in a seminar which was formed for the purpose of considering this problem. They also volunteered to perform the necessary numerical work under my immediate direction. Their results are given in Part 10 of this volume and will be freely referred to by me in this discussion. For a first application of the analytical method of solution, *irrespective of the perturbations*, the seventh satellite of Jupiter

[1] 1905.

was selected, partly because ROSS encountered greater difficulties in this case than with the sixth satellite, in spite of the experience gained with the latter and Phoebe, and partly because PERRINE had settled on the retrograde motion as being more probable than the direct motion. In order to make the analytical results immediately comparable with those obtained by ROSS, it was decided to base the determination of the orbit on the same observations that ROSS had used, namely, on the observations of January 3, the first of the two secured on February 8, and March 6. The observations were taken from *Lick Observatory Bulletin* No. 78, page 133. CRAWFORD and CHAMPREUX found by direct computation that these observations could be satisfied by two and only two distinct and well-determined orbits,[1] one placing the satellite at a nearer distance from the Earth than Jupiter's distance, with a retrograde motion; and the other at a farther stance with direct motion. A comparison of these orbits with those obtained by PERRINE and ROSS is printed on page 498 of this volume. In the mean time, Mr. ALBRECHT had secured positions of the satellite with the Crossley reflector on August 8, 9, and 10. A comparison with ROSS's ephemeris in *Lick Observatory Bulletin* No. 82 showed the following approximate residuals, O — C: in position angle, — 4°.6; in distance, 29′.6, the computed position angle and distance being respectively, 294° and 25′. CRAWFORD and CHAMPREUX found the following residuals from their *first approximation irrespective of the perturbations*, to the two possible solutions in the sense O — C:

Date, August 9	p	s
Direct motion:	— 5°.6	+ 33′.6
Retrograde motion	— 270	+ 22 .8

An improvement of the first approximation, irrespective of perturbations, was not attempted, as it was decided to compute a new orbit by taking immediate account of the solar perturbations. The agreement of the residuals for the first approximation to direct motion with those of ROSS's orbit is very marked. But the representation of either orbit is by no means satisfactory. The lack of representation for retrograde motion does not necessarily imply at this stage of the investigation that the motion is direct rather than retrograde, although further investigation shows this to be the case. The magnitude of the residuals which the August observations leave for all the orbits derived on the basis of undisturbed motion, plainly indicates that no satisfactory orbit can be obtained without taking into immediate account the principal perturbations to which the satellite was subjected during the period covered by the observations on which the orbits were based. Fortunately only a simple generalization of the original analytical method for determining an orbit irrespective of perturbations was necessary for the derivation of formulæ for the *direct computation of osculating elements* from three

[1] *Cf.* the footnote on page 501.

or more observations in cases where the satellite is strongly disturbed by the attraction of one or more bodies. The modifications arising from the oblateness of the central body, etc., as yet have not been formulated in this connection.

In the case of comets and asteroids, it has been found possible to approximate an osculating orbit closely by neglecting the perturbations in the first solution. Such elements are used for the determination of special perturbations on the basis of which the original elements are changed into real osculating elements. At most, the process of determining the special perturbations and correcting the elements is repeated until the perturbations no longer change and the residuals of the normal places are satisfactory. But let us consider the case of an asteroid suffering large periodic perturbations during the discovery opposition. The process referred to above, if applied properly, would be very laborious. It may even happen in extreme cases that the successive corrections to the special perturbations and the osculating elements do not approach zero. I am not aware that such a case has hitherto actually been encountered. But it does occur that the normal places of the discovery opposition can be satisfactorily represented by an orbit in which the perturbations are neglected, whereas, if taken account of, there result osculating elements which differ considerably from the adopted elements. If, then, the adopted elements are made the basis of an ephemeris for the next opposition, with or without the inclusion of special perturbations, the asteroid may be very far from the predicted place and even lost. Such has been the case with (132) Æthra, discovered by WATSON at Ann Arbor on June 13, 1873. The planet was observed from June 13 to July 5. During this time and shortly after it was disturbed by Mars, but WATSON computed two sets of elements irrespective of the perturbations. WILHELM LUTHER also has published, particularly in the *Astronomische Nachrichten*, 3101, several sets of elements and opposition ephemerides up to 1896. But in all these calculations, the perturbations during the period covered by the observations, as well as the subsequent ones, have been neglected. The planet was of the eleventh magnitude at the time of discovery and, in spite of LUTHER'S extensive calculations and a careful photographic search instituted by various observers, has never again been identified. It is extremely desirable that the orbit should be investigated again by including the perturbations in the computation of an orbit from the available observations by the method outlined in these pages. The computation of such an orbit already has been undertaken at this Observatory.

In cases of new satellites, asteroids and comets which are strongly disturbed at the time of the observations, which must form the basis of the computation, it is, therefore, essential that the disturbing forces be taken into account in the original determination of the orbit from the observations. This applies even to arcs covering only a few days.

It is proposed to accomplish this by an extension of the *Short Method of Deter-*

mining Orbits from Three Observations, given in Parts I and VII of this volume, to the case of more than one attracting body. *The analytical method of determining the orbit of a satellite* from three or more observations *irrespective of perturbations*, referred to above, consists in an adaptation of the *Short Method* to the solution of satellite orbits. The most general solution is that which gives an osculating orbit of an asteroid, comet, or satellite, considered as material points, from three or more observed positions by taking immediate account of the appreciable attraction of other bodies and of their figures besides the attraction of the supposed central body. The ordinary cases of an asteroid or comet moving solely under the Sun's attraction, or of a satellite moving solely under the attraction of its primary are merely special cases of the former.

In applying the general solution to the seventh satellite of Jupiter only the attraction of Jupiter and the Sun have been considered by CRAWFORD and CHAMPREUX, and this has been found entirely sufficient for a representation of the available observations. The resulting elements osculating for February 8 are printed on page 498 of this volume and those derived from the solution irrespective of the perturbations, are printed alongside for comparison. The differences between the two sets of elements are due solely to solar perturbations. The osculating elements represent the actual Jovicentric position and velocity of the satellite at the epoch on the assumption that the satellite was subject only to perturbations by the Sun from the first to the last dates covered by the geocentric arc on which the elements are based. *These elements are elements that would have resulted from a solution irrespective of the perturbations, if the observed right ascensions and declinations could have been corrected in advance for the solar perturbations, starting with February 8 as epoch of osculation.* The *first approximation of such osculating elements* gave as residuals for the direct orbit for August 9 in position angle $+2^{\circ}.7$, in distance, $+5'.2$, all perturbations being neglected in the computation of the position, as compared to $-5^{\circ}.6$ and $+33'.6$ for the solution irrespective of the perturbations. The fact that the *preliminary osculating* elements represent the August observation so much better than the elements computed *irrespective of perturbations* clearly shows that shortly after discovery the satellite was subjected to periodic perturbations of considerable magnitude and that, therefore, the latter solution is insufficient. CRAWFORD and CHAMPREUX have also computed the residuals on the basis of the *final osculating elements and of the perturbations* resulting therefrom for all of the observations from January 3 to March 6 printed in *Lick Observatory Bulletin* No. 82. These residuals are entirely comparable with the errors of observation, so that the result must be considered eminently satisfactory.

[1]The discovery of the eighth satellite of Jupiter has led Messrs. COWELL and

[1]These concluding paragraphs of the *Introduction* were added in June, 1911.

CROMMELIN to derive a method of dealing with this problem which has given eminently satisfactory results as indicated by the smallness of the residuals of the observations taken during the discovery opposition and by the accuracy of their predictions of the positions for subsequent oppositions. The process adopted, however, by Messrs. COWELL and CROMMELIN appears to be very laborious in application. It is not possible, at this writing, to enter upon a full discussion of the practical and theoretical merits of their method. A brief statement of the principal features of the process adopted by them in dealing with the orbit of the eighth satellite of Jupiter, as compared with the method developed in these pages, must therefore suffice.

In my *Direct Solution of the Orbits of Disturbed Bodies*, the first step consists in computing the velocities and accelerations α', δ'; α'', δ'' at the adopted epoch. The number of observations to be utilized for this purpose is readily determined by inspection on the basis of the run of the differences of the observed α, δ, the given intervals, and the available arc.

The next step consists in the direct solution of the distances of the disturbed body from the Earth and from the attracting bodies, together with the velocity and acceleration in the line of sight.

The final step consists in the computation of the osculating elements. But evidently, after the coördinates of the disturbed body and their velocities become known, osculating elements may be computed with reference to either attracting mass, *e. g.*, if there be two such masses. In the case of two solutions for the geocentric distance in a problem where two attracting masses are being considered, it is therefore possible to compute simultaneously four sets of osculating elements on the basis of the initial values of α', δ', α'', δ''. The selection of the physical solution in the usual sense is then accomplished by the representation of the observations. In some cases it may be necessary to base the selection on observations subsequent to the observations utilized in the solution of the problem.

The computation of an ephemeris is accomplished in the usual way by applying the special perturbations to the undisturbed positions computed from the osculating elements. The improvement of the orbit on the basis of subsequent observations is accomplished by differential correction of the osculating elements on the basis of the differences between theory and observation.

The first step in COWELL and CROMMELIN'S procedure[1] consists in interpolating three places for equidistant dates from the observed places. The next step consists in the determination of the geocentric distances by *trial and error* and the subsequent correction of these distances on account of the previously neglected terms of the fourth order with respect to the time. Unfortunately, the details of this solution by trial and error, etc., have not been published, and it

[1] *Monthly Notices of the Royal Astronomical Society*, Vol. LXVIII, No 8, page 576.

is, therefore, not possible to make an immediate comparison of the numerical processes involved with those fully printed for my method in Part 10 of this volume, but the description given of the procedure leaves little doubt that the numerical operations are exceedingly laborious.

The next step in COWELL and CROMMELIN'S procedure consists in the application of mechanical quadratures to the *disturbed* coördinates. This process is a modification of the process of special perturbations in rectangular coördinates. The subsequent disturbed coördinates depend upon the initial coördinates. Any inaccuracy in these initial coördinates, which may reveal itself later by comparison with subsequent observations, appears to require for the improvement of the orbit a complete revision of all the preceding work. Furthermore, according to the theory of interpolation, the irregularities in the differences of the undisturbed rectangular coördinates on the one hand and of the perturbations in these coördinates on the other, will combine in the differences of the disturbed coördinates, so that, in general, not only much smaller intervals of time are required in applying the process of mechanical quadratures to the *disturbed* coördinates than would be required, if it were applied to the *perturbations alone*, but the process must also be continued over periods for which the perturbations are insignificant.

It would seem, therefore, that the direct solution, as proposed in these pages, combined with the process of special perturbations after preliminary osculating elements have been obtained, presents many points of advantage as compared with the process adopted by COWELL and CROMMELIN.

At the time when my direct method was applied by CRAWFORD and MEYER[1] to the solution of the preliminary osculating orbit of the eighth satellite of Jupiter, I had not as yet derived the closed expressions for ∂f and ∂g (*cf.* page 247), so that the removal of the original residuals was attempted with the aid of the series for ∂f and ∂g. These series had proved entirely sufficient in the case of the seventh satellite of Jupiter, but were insufficient in the case of the eighth satellite. Final adjustment of the orbit was accomplished by the laborious process of variation of constants. Since then MEYER has applied my closed expressions for ∂f and ∂g to the improvement of the orbit immediately after the direct solution and thereby has been enabled to correct the orbit as indicated in Part 10 in a remarkably short time.

In the next section the formulæ will be developed for the determination of the distances of the disturbed body from the Earth and the attracting bodies, the derivation being limited to two attracting mass-points, for simplicity's sake. The formulæ may be readily extended, if the occasion demands, for more than two attracting bodies and for taking account of their figures.

The derivation of the formulæ may be accomplished in various ways accord-

[1] *L. O. Bulletin* 137.

ing to the choice of coördinates. They may be derived very conveniently by considering the problem the converse of ENCKE'S method of special perturbations. The developments are, therefore, based on the equations of motion in rectangular coördinates. In the problem of special perturbations, the object is to determine disturbed coördinates on the basis of osculating elements. In the problem to be considered here the object is to determine osculating elements or equivalent constants, on the basis of disturbed positions furnished by observation.

Only the case of a disturbed satellite has been fully developed, the solution of the osculating orbit of a disturbed comet or planet being merely indicated. But the computer need experience no difficulty in applying the proposed methods to the latter cases. Two such applications, one to the determination of the osculating orbit of a comet, the other to that of a disturbed planet, are now under way at the Students' Observatory.

THE VELOCITIES AND ACCELERATIONS α', δ', α'', δ'', AT THE EPOCH.

The computation of the initial velocities and accelerations at the epoch from five or more observations is accomplished as in Part 7, page 229, *et seq.* The procedure is fully illustrated by Example 2 in Part 8. The velocities and accelerations are most conveniently determined in mean solar days as the unit of time. In case of disturbed comets or minor planets the unit is then changed to $\frac{1}{k}$ mean solar days by dividing the velocities by k and the accelerations by k^2, where k^2 is the solar constant of attraction. In case of a disturbed satellite or also in case of the determination of the orbit of a satellite irrespective of the perturbations, it is most convenient to change the unit to $\frac{1}{(k)}$ mean solar days, where $(k)^2$ is the constant of attraction of the primary. If the mass of the planet in terms of the Sun's mass be (m), then $(k)^2 = (m)k^2$, the numerical ratio of the constants of attraction of the bodies being the same as the numerical ratio of their masses, if the same units of time, mass, and length be chosen. Thus, since according to NEWCOMB, Jupiter's mass $(m) = \frac{1}{1047.355}$, we have for the logarithm of Jupiter's constant of attraction $\log (k)^2 = 3.4510689 - 10$.

If in the case of a newly discovered body, it can not be decided beforehand whether the Sun or the major planet is the primary, then the velocities and accelerations may be expressed either in $\frac{1}{k}$ or $\frac{1}{(k)}$ mean solar days as the unit of time.

If the geocentric motion of the disturbed body be very irregular, then it may be advantageous to depart from the analytical or graphical methods of determining the velocities and accelerations, outlined in Part 7, page 229, *et seq.*, by deriving the velocities and accelerations in position angle and distance and then properly transforming them into velocities and accelerations in the geocentric coördinates.

In this connection it is to be observed that a complete elimination of parallax may be performed in the solution of the orbits of disturbed bodies in exactly the same manner as in Part 7, page 233, *et seq.* But as any minor inaccuracies committed in the derivation of the preliminary osculating orbit are removed by differential correction in the final representation of the observations, the parallax may be either entirely neglected or, in case of a disturbed satellite, the observations may be corrected for parallax on the basis of the distance of the primary.

The planetary aberration is accounted for by a procedure similar to that adopted in Part 7 in the determination of the orbits of undisturbed bodies. This procedure will be considered in detail after the derivation of the fundamental formulæ, which follows.

DERIVATION OF THE FORMULÆ FOR THE DISTANCES OF THE DISTURBED BODY FROM THE EARTH AND FROM THE ATTRACTING BODIES.

Osculating Satellite Orbits.

The velocities and accelerations in right ascension and declination of the body of which the orbit is sought having been determined in the preceding section, expressions shall next be developed for the geocentric distance, its velocity, and acceleration at the time $t = 0$. No restriction need be placed on the number of attracting masses. For the sake of simplicity, however, it will be assumed that the body is attracted by only two masses. The terms depending on additional masses may then be written by analogy. If it be possible to judge in advance which of the attracting bodies is the central body, it is convenient to place the origin of coördinates in the center of that body. Thus, in the case of a suspected satellite, subject to perturbations by the Sun, the center of the primary should be chosen as origin so as to avoid a subsequent transformation. Similarly we should choose the center of the Sun as origin in cases of asteroids or comets which are greatly disturbed by one or more planets. This choice, however, is not essential to the solution of the problem, as the same results will ultimately be obtained, no matter which mass be chosen as origin of coördinates.

If it can not be decided in advance which is the central body, the center of either mass may be chosen as origin, as would necessarily be the case with equal attractions. We shall first consider the case of a satellite, which is disturbed by the Sun. Taking the Earth's equator as fundamental plane, we may denote by

$\alpha, \delta, \rho, \sigma = \rho \cos \delta, \xi, \eta, \zeta$	the coördinates of the satellite referred to the Earth,
$a, d, r, s = r \cos d, x, y, z$	the coördinates of the satellite referred to the primary,
$[a], [d], [r], [s] = [r] \cos [d], [x], [y], [z]$	the coördinates of the satellite referred to the Sun,
$(\alpha), (\delta), (\rho), (\sigma) = (\rho) \cos (\delta), (\xi), (\eta), (\zeta)$	the coördinates of the primary referred to the Earth,
$(a), (d), (r), (s) = (r) \cos (d), (x), (y), (z)$	the coördinates of the primary referred to the Sun,
$A, D, R, S = R \cos D, X, Y, Z$	the coördinates of the Sun referred to the Earth.

Then the equations of motion of the satellite referred to the center of mass of the primary are:

$$\begin{aligned} \frac{d^2 x}{dt^2} + mk^2 \frac{x}{r^3} &= k^2 \left\{ \frac{-(x) - x}{[r]^3} - \frac{-(x)}{(r)^3} \right\} \\ \frac{d^2 y}{dt^2} + mk^2 \frac{y}{r^3} &= k^2 \left\{ \frac{-(y) - y}{[r]^3} - \frac{-(y)}{(r)^3} \right\} \qquad (1) \\ \frac{d^2 z}{dt^2} + mk^2 \frac{z}{r^3} &= k^2 \left\{ \frac{-(z) - z}{[r]^3} - \frac{-(z)}{(r)^3} \right\}. \end{aligned}$$

Introducing

$$
\begin{gathered}
mk^2 = (k)^2\ ; \qquad \frac{k^2}{(k)^2} = \frac{1}{m} = \gamma \\
\frac{1}{(k)^2}\frac{d^2 x}{dt^2} = x'', \quad \frac{1}{(k)^2}\frac{d^2 y}{dt^2} = y'', \quad \frac{1}{(k)^2}\frac{d^2 z}{dt^2} = z'',
\end{gathered}
\tag{2}
$$

so that the unit of time is $\frac{1}{(k)}$ mean solar days, we obtain after dividing equations (1) throughout by $(k)^2$:

$$
\begin{aligned}
x'' &= -x\left[\frac{1}{r^3} + \frac{\gamma}{[r]^3}\right] - \gamma\frac{(x)}{[r]^3} + \gamma\frac{(x)}{(r)^3} \\
y'' &= -y\left[\frac{1}{r^3} + \frac{\gamma}{[r]^3}\right] - \gamma\frac{(y)}{[r]^3} + \gamma\frac{(y)}{(r)^3} \\
z'' &= -z\left[\frac{1}{r^3} + \frac{\gamma}{[r]^3}\right] - \gamma\frac{(z)}{[r]^3} + \gamma\frac{(z)}{(r)^3}\,.
\end{aligned}
\tag{3}
$$

After substituting

$$
\begin{aligned}
x &= \xi - (\xi), & y &= \eta - (\eta), & z &= \zeta - (\zeta) \\
x'' &= \xi'' - (\xi)'', & y &= \eta'' - (\eta)'', & z &= \zeta'' - (\zeta)'',
\end{aligned}
\tag{4}
$$

and remembering that

$$
(\xi) - (x) = X, \quad (\eta) - (y) = Y, \quad (\zeta) - (z) = Z, \tag{5}
$$

equations (3) become

$$
\begin{aligned}
\xi'' + \xi\left[\frac{1}{r^3} + \frac{\gamma}{[r]^3}\right] &= \frac{(\xi)}{r^3} + \gamma\frac{X}{[r]^3} + \gamma\frac{(x)}{(r)^3} + (\xi)'' \\
\eta'' + \eta\left[\frac{1}{r^3} + \frac{\gamma}{[r]^3}\right] &= \frac{(\eta)}{r^3} + \gamma\frac{Y}{[r]^3} + \gamma\frac{(y)}{(r)^3} + (\eta)'' \\
\zeta'' + \zeta\left[\frac{1}{r^3} + \frac{\gamma}{[r]^3}\right] &= \frac{(\zeta)}{r^3} + \gamma\frac{Z}{[r]^3} + \gamma\frac{(z)}{(r)^3} + (\zeta)''.
\end{aligned}
\tag{6}
$$

$(\xi)''$, $(\eta)''$, $(\zeta)''$ are the accelerations of the primary in the geocentric coördinates. If we neglect the perturbations of the primary, then $(\xi)''$, $(\eta)''$, $(\zeta)''$ may be expressed as follows:

$$
(\xi)'' = (x)'' + X'', \quad (\eta)'' = (y)'' + Y'', \quad (\zeta)'' = (z)'' + Z'' \tag{7}
$$

$$
\begin{aligned}
(x)'' &= -\gamma\frac{(x)\,(1+m)}{(r)^3}, & (y)'' &= -\gamma\frac{(y)\,(1+m)}{(r)^3}, & (z)'' &= -\gamma\frac{(z)\,(1+m)}{(r)^3} \\
(X)'' &= -\gamma\frac{X}{R^3}, & Y'' &= -\gamma\frac{Y}{R^3}, & Z'' &= -\gamma\frac{Z}{R^3},
\end{aligned}
\tag{8}
$$

where the mass of the Earth is neglected because it would affect only the seventh place of decimals. When the approximations (8) are not sufficient $(\xi)''$, $(\eta)''$, $(\zeta)''$ may be accurately computed as indicated later (page 473).

From (7) and (8) we obtain for the sum of the last three terms on the right hand side of the first of equations (6)

$$
\gamma\frac{X}{[r]^3} + \gamma\frac{(x)}{(r)^3} + \xi'' = \gamma X\left(\frac{1}{[r]^3} - \frac{1}{R^3}\right) - \frac{(x)}{(r)^3} \tag{9}
$$

and similar expressions for the last two equations (6).

In order that equations (6), after introducing expressions (9), may be made available for the solution of ρ, ρ', ρ'', or σ, σ', σ'', in terms of the observed right ascension and declination at the time $t=0$, their velocities, and accelerations, it will be necessary to introduce polar coördinates throughout. We have

$$\xi = \sigma\cos\alpha\,,\quad \eta = \sigma\sin\alpha\,,\quad \zeta = \sigma\tan\delta\,, \tag{10}$$

and after differentiating twice with respect to t and dividing by $(k)^2$

$$\begin{aligned} \xi'' &= \sigma''\cos\alpha - 2\,\sigma'\sin\alpha\alpha' - \sigma\sin\alpha\alpha'' - \sigma\cos\alpha\,\alpha'^2 \\ \eta'' &= \sigma''\sin\alpha + 2\,\sigma'\cos\alpha\alpha' + \sigma\cos\alpha\alpha'' - \sigma\sin\alpha\,\alpha'^2 \\ \zeta'' &= \sigma''\tan\delta + 2\,\sigma'(\tan\delta)' + \sigma(\tan\delta)''\,. \end{aligned} \tag{11}$$

We also have

$$\begin{aligned} (\xi) &= (\sigma)\cos(\alpha)\,, & (\eta) &= (\sigma)\sin(\alpha)\,, & (\zeta) &= (\sigma)\tan(\delta) \\ (x) &= (\varkappa)\cos(a)\,, & (y) &= (\varkappa)\sin(a)\,, & (z) &= (\varkappa)\tan(d) \\ X &= S\cos A\,, & Y &= S\sin A\,, & Z &= S\tan D\,. \end{aligned} \tag{12}$$

On substituting, as indicated above, expressions (9) in (6) and then introducing polar coördinates in accordance with (10) to (12), we obtain the following equations for the determination of σ, σ', σ'':

$$\left.\begin{aligned} &\sigma''\cos\alpha - 2\,\sigma'\sin\alpha\alpha' - \sigma\sin\alpha\alpha'' - \sigma\cos\alpha\,\alpha'^2 + \sigma\cos\alpha\left[\frac{1}{r^3} + \frac{\gamma}{[r]^3}\right] = \\ &\qquad \frac{(\sigma)\cos(\alpha)}{r^3} + \gamma S\cos A\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right] - \frac{(\varkappa)\cos(a)}{(r)^3} && \text{(a)} \\ &\sigma''\sin\alpha + 2\,\sigma'\cos\alpha\alpha' + \sigma\cos\alpha\alpha'' - \sigma\sin\alpha\,\alpha'^2 + \sigma\sin\alpha\left[\frac{1}{r^3} + \frac{\gamma}{[r]^3}\right] = \\ &\qquad \frac{(\sigma)\sin(\alpha)}{r^3} + \gamma S\sin A\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right] - \frac{(\varkappa)\sin(a)}{(r)^3} && \text{(b)} \\ &\sigma''\tan\delta + 2\,\sigma'(\tan\delta)' + \sigma(\tan\delta)'' + \sigma\tan\delta\left[\frac{1}{r^3} + \frac{\gamma}{[r]^3}\right] = \\ &\qquad \frac{(\sigma)\tan(\delta)}{r^3} + \gamma S\tan D\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right] - \frac{(\varkappa)\tan(d)}{(r)^3}\,. && \text{(c)} \end{aligned}\right\} \tag{13}$$

Multiply (a) by $\cos\alpha$ and (b) by $\sin\alpha$ and add. Then multiply (a) by $-\sin\alpha$ and (b) by $\cos\alpha$ and add, and copy (c). We then obtain the following equations in σ, σ', and σ'':

$$\left.\begin{aligned} &\sigma'' + \sigma\left[\frac{1}{r^3} - \alpha'^2 + \frac{\gamma}{[r]^3}\right] = \frac{(\sigma)}{r^3}\cos[(\alpha)-\alpha] + \gamma S\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right]\cos[A-\alpha] - \frac{(\varkappa)}{(r)^3}\cos[(a)-\alpha] && \text{(a)} \\ &2\,\sigma'\alpha' + \sigma\alpha'' = \frac{(\sigma)}{r^3}\sin[(\alpha)-\alpha] + \gamma S\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right]\sin[A-\alpha] - \frac{(\varkappa)}{(r)^3}\sin[(a)-\alpha] && \text{(b)} \\ &\sigma''\tan\delta + 2\,\sigma'(\tan\delta)' + \sigma\left[\frac{\tan\delta}{r^3} + (\tan\delta)'' + \gamma\frac{\tan\delta}{[r]^3}\right] = \\ &\qquad \frac{(\sigma)}{r^3}\tan(\delta) + \gamma S\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right]\tan D - \frac{(\varkappa)}{(r)^3}\tan(d)\,. && \text{(c)} \end{aligned}\right\} \tag{14}$$

In these equations γ, α, δ, α', δ', α'', δ'', (σ), (α), (δ), (r), (a), (d), (s), R, A, D, S, are known and $(\tan\delta)'$ and $(\tan\delta)''$ are given by

$$(\tan\delta)' = \sec^2\delta_{\prime\prime}\,\delta'\,; \qquad (\tan\delta)'' = \sec^2\delta_{\prime\prime}\left[2\tan\delta_{\prime\prime}\,\delta'^2 + \delta''\right]. \tag{15}$$

The unknowns involved are σ, σ', σ'', r and $[r]$.

The equations may be conveniently solved for σ, σ', σ'' by determinants. The coefficients of σ in the first and third equations contain the unknowns r and $[r]$. But the determinant of the coefficients of σ, σ', σ'' is independent of r and $[r]$. This determinant shall be denoted by $2\varDelta$. On the right-hand side $\frac{1}{r^3}$ is a factor of the first term, $\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right]$ of the second term, and $\frac{1}{(r)^3}$ of the third term in each of the three equations. If, then, in solving for σ, we replace in $2\varDelta$ the coefficients of σ by the corresponding absolute terms, the new determinant may be written as the algebraical sum of three determinants, of which the first has the factor $\frac{1}{r^3}$, the second the factor $\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right]$ and the third the factor $\frac{1}{(r)^3}$. Hence, if we write

$$\begin{aligned}
\varDelta \quad &= \quad \alpha'^3\tan\delta - \alpha''(\tan\delta)' + \alpha''(\tan\delta)'' \\
\varDelta\,\varkappa &= -S\,\{(\tan\delta\cos[A-\alpha] - \tan D)\,\alpha' + \sin[A-\alpha](\tan\delta)'\} \\
\varDelta\,(\varkappa) &= -(\sigma)\,\{(\tan\delta\cos[(\alpha)-\alpha] - \tan(\delta))\,\alpha' + \sin[(\alpha)-\alpha](\tan\delta)'\} \\
\varDelta\,[\varkappa] &= -(s)\,\{(\tan\delta\cos[(a)-\alpha] - \tan(d))\,\alpha' + \sin[(a)-\alpha](\tan\delta)'\}\,,
\end{aligned} \tag{16}$$

then

$$\sigma = \frac{(\varkappa)}{r^3} + \gamma\,\varkappa\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right] - \frac{[\varkappa]}{(r)^3}\,. \tag{17}$$

Let

$$E = \frac{(\varkappa)}{\cos\delta}\,; \qquad F = \frac{\gamma\,\varkappa}{\cos\delta}\,; \qquad G = -\frac{\gamma}{\cos\delta}\left[\frac{\varkappa}{R^3} + \frac{m\,[\varkappa]}{(r)^3}\right] = -F\left[\frac{1}{R^3} + \frac{m\,[\varkappa]}{\varkappa\,(r)^3}\right], \tag{18}$$

then

$$\rho = \frac{E}{r^3} + \frac{F}{[r]^3} + G\,, \tag{19}$$

where E, F, and G are completely known.

As a check we evidently have $(\varkappa) = \varkappa + [\varkappa]$.

To obtain two other equations for the complete determination of the unknowns ρ, r, and $[r]$, let ψ be the angle subtended at the Earth by the satellite and the center of its primary and $[\psi]$ the angle subtended at the Earth by the satellite and the Sun. Then

$$\begin{aligned}
r^2 \quad &= (\rho)^2 + \rho^2 - 2\,(\rho)\,\rho\cos\psi \\
[r]^2 &= R^2 + \rho^2 - 2\,R\,\rho\cos[\psi]\,,
\end{aligned} \tag{20}$$

where ψ and $[\psi]$ depend on known quantities. Equations (19) and (20) may be solved conveniently by approximations in several ways, according to the conditions of the problem. In the case of a satellite, for example, let

$$\frac{F}{[r]^3} + G = H\,, \tag{21}$$

where in the first approximation we can put $[r] = (r)$. Then

$$\rho = \frac{E}{r^3} + H. \tag{22}$$

Eliminating ρ from (22) and the first of (20), we have

$$r^8 - [(\rho)^2 - 2H(\rho)\cos\psi + H^2]\, r^6 + 2E[(\rho)\cos\psi - H]\, r^3 - E^2 = 0, \tag{23}$$

or

$$r^8 - \{[(\rho)\cos\psi - H]^2 + (\rho)^2\sin^2\psi\}\, r^6 + 2E[(\rho)\cos\psi - H]\, r^3 - E^2 = 0,$$

or

$$[(\rho)\cos\psi - H]^2 r^6 - 2E[(\rho)\cos\psi - H]\, r^3 + E^2 = r^8 - (\rho)^2\sin^2\psi\, r^6,$$

or

$$\{[(\rho)\cos\psi - H]\, r^3 - E\}^2 = r^8 - (\rho)^2\sin^2\psi\, r^6,$$

or

$$[(\rho)\cos\psi - H]\, r^3 - E = \pm r^3 \sqrt{r^2 - (\rho)^2\sin^2\psi}, \tag{24}$$

or

$$r^3 = \frac{E \pm r^3\sqrt{r^2 - (\rho)^2\sin^2\psi}}{(\rho)\cos\psi - H}. \tag{25}$$

As a first approximation we have

$$r^3 = \frac{E}{(\rho)\cos\psi - H}. \tag{26}$$

With the value of r found from (26) we obtain ρ from (22) and then $\pm [r]$ from the second of (20). H may now be more accurately computed from (21). A closer value of r will then result from (25), if at the same time the first approximation of r obtained from (26) is introduced on the right hand side of (25). This method of approximation has proved thoroughly satisfactory in the case of the seventh satellite of Jupiter, as shown by Dr. CRAWFORD'S and Mr. CHAMPREUX'S investigation in Part 10, page 492.

Instead of substituting successively for a given value of H a previous approximation of r in the second term of the numerator of the right hand member of (25), it is advantageous to apply the method of differential correction. Let

$$f(r) = r^3 - \frac{E \pm r^3\sqrt{r^2 - (\rho)^2\sin^2\psi}}{K}, \tag{27}$$

where

$$K = (\rho)\cos\psi - H. \tag{28}$$

It is required to find the values of r such that $f(r) = 0$. Let r_i be any approximation, *e. g.*, the first, if obtained from (26), and let $f(r_i) = M_i$. Then the correction to be applied to r_i, so that $f(r) = f(r_i + \partial r_i) = 0$, is

$$\partial r = \frac{-M_i}{\left(\frac{\partial f(r)}{\partial r}\right)_{r = r_i}}. \tag{29}$$

By differentiating (27) with respect to r and substituting the differential coefficient for $r = r_i$ in (29), we readily derive:

$$\partial r = 3\, r_i^2 \left[1 \mp \frac{4\, r_i^2 - 3\,(\rho)^2 \sin^2 \psi}{3\, K \sqrt{r_i^2 - (\rho)^2 \sin^2 \psi}} \right]. \tag{30}$$

Two solutions, r_1 and r_2, result after the first approximation and these are then continued separately until $f(r_1) = 0$ and $f(r_2) = 0$. The values of H and K may then be improved for both solutions by the second equation of (20), by (21), and (28). The approximations for r are then repeated with the new value of K, etc. This method of approximation was successfully applied by Messrs. CRAWFORD and MEYER in the case of the eighth satellite of Jupiter, as shown in Part 10, page 500.

It should be observed, however, that by proper transformation of the second of equations (20) and of (23), tables similar to those given at the end of Part 7, might be constructed so as to serve for interpolating the values of the geocentric distances corresponding to the two solutions, for successive values of H or K.

The foregoing solution is based on approximate values of $(\xi)''$, $(\eta)''$, $(\zeta)''$, the perturbations of the primary being neglected. More accurate values of these accelerations may be obtained by numerical differentiation from the data given in the *Astronomisches Jahrbuch*, the *American Ephemeris and Nautical Almanac*, or other similar publications. Let us assume for the present that accurate numerical values of $(\xi)''$, $(\eta)''$, $(\zeta)''$, are available. Then, after introducing polar coördinates in accordance with (10) to (12), equations (6) become:

$$\begin{aligned}
&\sigma'' \cos\alpha - 2\,\sigma' \sin\alpha\,\alpha' - \sigma \sin\alpha\,\alpha'' - \sigma\cos\alpha\,\alpha'^2 + \sigma\cos\alpha \left[\frac{1}{r^3} - \frac{\gamma}{[r]^3}\right] = \\
&\qquad \frac{(\sigma)\cos(\alpha)}{r^3} + \gamma S \cos A \frac{1}{[r]^3} - \gamma \frac{(\kappa)\cos(\alpha)}{(r)^3} + (\xi)'' \qquad \text{(a)}\\
&\sigma'' \sin\alpha + 2\,\sigma' \cos\alpha\,\alpha' + \sigma \cos\alpha\,\alpha'' - \sigma\sin\alpha\,\alpha'^2 + \sigma\sin\alpha \left[\frac{1}{r^3} - \frac{\gamma}{[r]^3}\right] = \\
&\qquad \frac{(\sigma)\sin(\alpha)}{r^3} + \gamma S \sin A \frac{1}{[r]^3} - \gamma \frac{(\kappa)\sin(\alpha)}{(r)^3} + (\eta)'' \qquad \text{(b)} \qquad (31)\\
&\sigma'' \tan\delta + 2\,\sigma' (\tan\delta)' + \sigma(\tan\delta)'' + \sigma\tan\delta \left[\frac{1}{r^3} - \frac{\gamma}{[r]^3}\right] = \\
&\qquad \frac{(\sigma)\tan(\delta)}{r^3} + \gamma S \tan D \frac{1}{[r]^3} - \gamma \frac{(\kappa)\tan(d)}{(r)^3} + (\zeta)''. \qquad \text{(c)}
\end{aligned}$$

It will be noted that the terms divided by $\frac{1}{(r)^3}$ and $\frac{1}{[r]^3}$ appear in consequence of the solar perturbations.

As before, multiply (a) by $\cos\alpha$ and (b) by $\sin\alpha$ and add. Then multiply (a) by $-\sin\alpha$ and (b) by $\cos\alpha$ and add, and copy (c). We then obtain the following new equations in σ, σ', σ'':

$$\begin{aligned}
&\sigma'' + \sigma\left[\frac{1}{r^3} - \alpha'^2 + \gamma\frac{1}{[r]^3}\right] = \\
&\quad \frac{(\sigma)}{r^3}\cos[(\alpha)-\alpha] + \gamma S\frac{1}{[r]^3}\cos[A-\alpha] + \gamma\frac{(\kappa)}{(r)^3}\cos[(a)-\alpha] + (\xi)''\cos\alpha + (\eta)''\sin\alpha
\end{aligned} \tag{a}$$

$$\begin{aligned}
&2\sigma'\alpha' + \sigma\alpha'' = \\
&\quad \frac{(\sigma)}{r^3}\sin[(\alpha)-\alpha] + \gamma S\frac{1}{[r]^3}\sin[A-\alpha] + \gamma\frac{(\kappa)}{(r)^3}\sin[(a)-\alpha] - (\xi)''\sin\alpha + (\eta)''\cos\alpha
\end{aligned} \tag{32 (b)}$$

$$\begin{aligned}
&\sigma''\tan\delta + 2\sigma'(\tan\delta)' + \sigma\left[\frac{\tan\delta}{r^3} + (\tan\delta)'' + \gamma\frac{\tan\delta}{[r]^3}\right] = \\
&\quad \frac{(\sigma)}{r^3}\tan(\delta) \qquad + \gamma S\frac{1}{[r]^3}\tan D \qquad + \gamma\frac{(\kappa)}{(r)^3}\tan(d) \qquad + (\zeta)''.
\end{aligned} \tag{c}$$

The left-hand sides of these equations are the same as those of (14).

On the right-hand side, however, the first three terms have the common factors $\frac{1}{r^3}$, $\frac{1}{[r]^3}$ and $\frac{1}{(r)^3}$, respectively, while the remaining terms are multiplied by the geocentric accelerations of the primary. In solving for σ by determinants, we have, therefore, an additional determinant depending upon these accelerations.

Let this be denoted by $2(\Delta)$, so that

$$2(\Delta) = \begin{vmatrix} 1 & 0 & +(\xi)''\cos\alpha + (\eta)''\sin\alpha \\ 0 & 2\alpha' & -(\xi)''\sin\alpha + (\eta)''\cos\alpha \\ \tan\delta & 2(\tan\delta)' & +(\zeta)'' \end{vmatrix}, \tag{33}$$

or

$$\begin{aligned}
2(\Delta) = 2\alpha'(\zeta)'' &- 2\alpha'\tan\delta\,(\xi)''\cos\alpha - 2\alpha'\tan\delta\,(\eta)''\sin\alpha \\
&+ 2(\tan\delta)'(\xi)''\sin\alpha - 2(\tan\delta)'(\eta)''\cos\alpha,
\end{aligned}$$

or

$$\begin{aligned}
(\Delta) = (\zeta)''\alpha' &+ (\xi)''[\sin\alpha(\tan\delta)' - \cos\alpha\tan\delta\,\alpha'] \\
&- (\eta)''[\cos\alpha(\tan\delta)' + \sin\alpha\tan\delta\,\alpha'].
\end{aligned} \tag{34}$$

Then performing the solution by determinants exactly as in the case of equations (14) and letting

$$\Delta\kappa_0 = (\Delta), \tag{35}$$

we obtain

$$\sigma = \frac{(\kappa)}{r^3} + \gamma\frac{\kappa}{[r]^3} + \gamma\frac{[\kappa]}{(r)^3} + (\kappa)_0. \tag{36}$$

Then letting E and F represent the same expressions as in (18) and putting

$$G = -\frac{\gamma[\kappa]}{\cos\delta\,(r)^3} + (\kappa)_0, \tag{37}$$

we have as before in (19)

$$\rho = \frac{E}{r^3} + \frac{F}{[r]^3} + G. \tag{38}$$

This equation together with equations (20) gives the values of ρ, r, and $[r]$, the method of solution being exactly the same as in the former case.

After ρ and therefore σ have been determined by equation (17), σ' may be computed from equation (14b) and σ'' from equation (14a) or (14c).

Orbits of Satellites Irrespective of Perturbations.

If it be intended to compute the orbit of the satellite without regard to any perturbations whatsoever, formulæ (32) to (36) in connection with the first of equations (20), are at once available, if all terms involving γ be omitted, these terms appearing in consequence of the solar perturbations. The equations for σ, σ', σ' may also be written out directly by analogy, as follows:

$$\begin{aligned} &\sigma'' + \sigma\left[\frac{1}{r^3} - \alpha'^2\right] = \frac{(\sigma)}{r^3}\cos\left[(\alpha) - \alpha\right] + (\xi)''\cos\alpha + (\eta)''\sin\alpha && \text{(a)} \\ &2\,\sigma'\alpha' + \sigma\alpha'' = \frac{(\sigma)}{r^3}\sin\left[(\alpha) - \alpha\right] - (\xi)''\sin\alpha + (\eta)''\cos\alpha && \text{(b)} \\ &\sigma''\tan\delta + 2\,\sigma'(\tan\delta)' + \sigma\left[\frac{\tan\delta}{r^3} + (\tan\delta)''\right] = \frac{(\sigma)}{r^3}\tan(\delta) + (\zeta)''. && \text{(c)} \end{aligned} \tag{39}$$

From these equations we obtain, by the first and third of (16), by the first of (18), by (34) and (35)

$$\rho = \frac{E}{r^3} + \kappa_0. \tag{40}$$

This equation, together with the first of (20), gives the various solutions for r or ρ in the usual manner.

The geocentric acceleration $(\xi)''$, $(\eta)''$, $(\zeta)''$ in (39) are supposed to have been accurately computed from the data of the astronomical ephemerides. Approximate values, however, for $(\xi)''$, $(\eta)''$, $(\zeta)''$ may be obtained as in the general case from (7) and (8). We shall then have in (39):

$$\begin{aligned} (\xi)''\cos\alpha + (\eta)''\sin\alpha &= -\gamma\frac{(1+m)}{(r)^3}(S)\cos\left[(a) - \alpha\right] - \gamma\frac{S}{R^3}\cos(A - \alpha) \\ -(\xi)''\sin\alpha + (\eta)''\cos\alpha &= -\gamma\frac{(1+m)}{(r)^3}(S)\sin\left[(a) - \alpha\right] - \gamma\frac{S}{R^3}\sin(A - \alpha) \\ (\zeta)'' &= -\gamma\frac{(1+m)}{(r)^3}(S)\tan d - \gamma\frac{S}{R^3}\tan D. \end{aligned} \tag{41}$$

Solving (39) by determinants as before, we have by (16),

$$\sigma = \frac{(\kappa)}{r^3} - \gamma(1+m)\frac{[\kappa]}{(r)^3} - \gamma\frac{\kappa}{R^3}. \tag{42}$$

Now, by the first of (18) and letting

$$G = -\frac{\gamma\kappa}{\cos\delta}\left[\frac{1}{R^3} - \frac{(1+m)}{\kappa(r)^3}[\kappa]\right], \tag{43}$$

we obtain, as before

$$\rho = \frac{E}{r^3} + G. \tag{44}$$

As before, σ' is given by equation (39b) and σ'' either by (39a) or (39c), after ρ or σ have been found.

Osculating Orbits of Comets or Minor Planets.

The formulæ for the determination of the distances of a disturbed comet or asteroid from the Earth, Sun, and disturbing planet may be derived either directly from the equations of motion in rectangular coördinates by proceeding in a similar manner as above in the case of a disturbed satellite, or they may be written by analogy from the formulæ for a disturbed satellite, or they may be deduced from the latter by change of notation and transformation of the origin from the center of the planet to that of the Sun.

If the last mentioned method be adopted, we would first substitute in equation (3) $x = [x] - (x)$; $x'' = [x]'' - (x)''$, etc.; then we would multiply both sides of the equations by $\frac{(k)^2}{k^2} = m$, to change the unit of time from $\frac{1}{(k)}$ to $\frac{1}{k}$ mean solar days. Then we would substitute $(x)'' = -\frac{(1+m)(x)}{(r)^3}$, where $(x)''$ now represents the acceleration of the planet in $\frac{1}{k}$ mean solar days. Finally, we would write x, y, z, r in place of $[x], [y], [z], [r]$, and *vice versa*, so that, as in the case of disturbed satellites, the coördinates of the disturbed body with reference to the central and disturbing body, may be designated without parentheses and in square brackets, respectively.

After the introduction of geocentric coördinates, the only change in equations (6) and following would then be that m would appear in place of y, that in place of x, y, z, r we would have $[x], [y], [z], [r]$, and *vice versa*, that in place of $(\xi), (\eta), (\zeta)$ we would have X, Y, Z, and *vice versa*, and that in place of $(x), (y), (z)$ we would have $-(x), -(y), -(z)$. The right hand side of the first equation (6), for example, would then contain terms multiplied by (ξ), (x) and X. Since any one of these coördinates may be expressed in terms of the other two, the equation may be reduced so as to contain only two of these coordinates. After the coefficients of these coördinates have been written down, only two expressions (16) remain, and the final formulæ (17) and following may be written out at once.

Orbits of Comets and Planets Irrespective of Perturbations.

Formulæ for the determination of the heliocentric and geocentric distances may be written at once by merely omitting in the preceding case all terms multiplied by m, the mass of the disturbing body. The resulting formulæ will then be those of the *Short Method* as derived in Parts 1 and 7.

In case no assumption can be made regarding the nature of the newly discovered body, the decision whether the body is a disturbed satellite or disturbed minor planet, or whether it is moving under nearly equal attractions, can be made readily after the solution of the orbit has been accomplished either by the formulæ for a disturbed satellite or a disturbed planet, by representing the observations on the basis of the various resulting orbits. This is due to the fact that either set of formulæ takes rigid account of the attractions of the bodies involved, the difference being merely in the choice of the unit of time and of the origin to which the accelerations are referred.

As stated on page 467 the planetary aberration is accounted for in a similar manner as in Part 7. The aim of the preliminary computation must be to lead to the solution of the true rectangular coördinates x, y, z, and velocities x', y', z' of the disturbed body at the (reduced) epoch with respect to the primary, which may be either the Sun or a major planet. The observed (apparent) coördinates α, δ of the disturbed body, therefore, are reduced to the beginning of the year inclusive of the aberration of the fixed stars and the geocentric rectangular coördinates X, Y, Z of the Sun are interpolated from an ephemeris for the observed (uncorrected) time.

After the solution of the geocentric distance ρ of the disturbed body and the computation of $\xi = \rho \cos \alpha \cos \delta$, etc., we have $\xi - X$, etc., for the *true* heliocentric coördinates of the disturbed body at its (reduced) epoch t. It is evident that the geocentric distance ρ just referred to is the distance of the true position of the disturbed body (at the reduced epoch t) from the position of the Earth at the observed epoch. The true heliocentric coördinates of the disturbed body are designated by x, y, z, or by $[x]$, $[y]$, $[z]$, according to whether the Sun is regarded as the primary or as the disturbing body.

Similarly, if the apparent geocentric coördinates, (α), (δ), of the major planet be interpolated from an ephemeris with the observed time and reduced to the beginning of the year, inclusive of the aberration terms, then, with its *proper* geocentric distance (ρ), we have $(x) = (\rho)\cos(\alpha)\cos(\delta) - X = (\xi) - X$, etc., for the true heliocentric place of the major planet at *its* reduced time, t'.

It is evident that the reduced time t of the disturbed body, and the reduced time t' of the major planet are not identical, so that the combination of the true heliocentric coördinates of the disturbed body and of the major planet produces the coördinates of the disturbed body at the time t with respect to the major planet at the time t'. The difference between the times t and t' can not be ascertained until the solution has progressed to the determination of the geocentric distance of the disturbed body. In the direct solution of the osculating orbit of a disturbed satellite this difference in general is entirely negligible, but it should be considered later if necessary, when the residuals O − C are being determined. In

the case of a disturbed minor planet or of a comet in which the position of the major planet need not be known with absolute rigidity, the difference between t and t' in general may also be disregarded in the course of the direct solution.

The *proper* geocentric distance (ρ) of the major body referred to above is the distance of its true position at its *reduced* time t' from the position of the Earth at the observed time, but for practical purposes (ρ) may be directly interpolated with the observed time.

As a check on (x), (y), (z) or on the corresponding polar coördinates (a), (d), (r), the true heliocentric coördinates (l), (b), (r) of the major planet, referred to the mean equinox and ecliptic of date, may be interpolated with *its* reduced time t' directly from the *American Ephemeris and Nautical Almanac* and then reduced to the beginning of the year and transformed to the equator. Furthermore, the sum of these two coördinates (x), (y), (z) of the major body and of the solar coordinates X, Y, Z at the observed time will furnish a check on the apparent coördinates (α), (δ) of the major body and at the same time will produce the *proper* geocentric distance (ρ) referred to above.

Reference to these points is made for the purpose of preparing for a rigid procedure in computing the *true* rectangular coördinates x, y, z and velocities x', y', z' of the disturbed body at the (reduced) epoch with reference to its primary after the geocentric distance and its velocity have become known in the course of the solution. In the case of a disturbed satellite where a major planet is the primary, the simplest procedure appears to be to re-determine the *true* heliocentric rectangular coördinates (x), (y), (z) of the major planet, but this time with the reduced time t of the disturbed satellite. The subtraction of (x), (y), (z) from the true heliocentric rectangular coördinates $[x]$, $[y]$, $[z]$ of the disturbed satellite at the time t will then produce the true rectangular coördinates x, y, z of the disturbed satellite at the reduced time t with reference to the major planet. The true heliocentric coördinates, $[x] = \xi - X$, etc., of the disturbed satellite at its reduced time t evidently are computed exactly as are the heliocentric coördinates x, y, z of an undisturbed comet or planet in Part 7. In the case of a disturbed planet or comet the heliocentric coördinates are designated here by x, y, z, just as in Part 7 and are the only heliocentric coördinates to be computed.

A similar procedure must be adopted in the comparison between theory and observation for the first and third places.

The geocentric distances necessary for correcting the intervals for planetary aberration may be derived from the geocentric distance at the epoch and from its velocity, its acceleration, in general, being neglected. But if the geocentric distances resulting from the representation of the places does not agree within the required accuracy with the geocentric distances adopted for the determination of the reduced times $t_,$ and $t_{,,,}$ for the first and third places, then the representation should be repeated with improved aberration times.

DERIVATION OF THE ELEMENTS AND COMPUTATION OF THE EPHEMERIS.

The elements may now be determined for each solution of σ, σ', and σ''. With the aid of formulæ VIII of the *Synopsis of Formulæ* in Part 7, we first compute the geocentric coördinates ξ, η, ζ and the velocities ξ', η', ζ' of the disturbed body. From these we may then compute the coördinates with reference to the primary, x, y, z, and their velocities, x', y', z', or with reference to the disturbing body, $[x]$, $[y]$, $[z]$, and their velocities $[x]'$, $[y]'$, $[z]'$. In the case of the direct solution of the disturbed orbit of a satellite the planet is the primary and the Sun the disturbing body. In the case of the disturbed orbit of a minor planet, or of a comet, the Sun is the primary and the planet is the disturbing body. But if no decision can be made beforehand regarding the primary and disturbing body, then the coördinates or velocities should be determined with reference to both the Sun and the planet. In order to compute the velocities referred to, it is necessary to derive by numerical differentiation the geocentric velocities X', Y', Z' of the Sun, or $(\xi)'$, $(\eta)'$, $(\zeta)'$ of the planet, or both, in the doubtful case, from the data of an ephemeris.

For each set of rectangular coördinates and velocities, an osculating orbit may now be computed by the formulæ VIIIc of the *Synopsis of Formulæ*, Part 7. The computation of the constants to the equator corresponding to the osculating elements and of the corresponding ephemeris is then readily accomplished in the usual way.

In the case of the orbit of a disturbed comet or minor planet, the special perturbations are most conveniently computed by ENCKE'S method of special perturbations for the purpose of comparing the computed positions with future observations.

In the case of a disturbed satellite, ENCKE'S method of special perturbations requires slight modifications on account of the change in the constant of attraction. The necessary changes are indicated in the *Synopsis of Formulæ* which follows.

If the computations have yielded more than one solution, the physical solution is readily determined by the representation of the observations on the basis of the various orbits. For this purpose it is not necessary to compute the special perturbations for all mathematical data. The perturbations need to be computed only for the physical orbit after it has been determined.

If the representation of the observations on the basis of the physical elements and the special perturbations is not entirely satisfactory, the perturbations may be considered constant, and the osculating elements may be corrected in exactly the same manner as in Part 7, *Synopsis of Formulæ*, B.

In case of very large corrections to the osculating elements, the special perturbations may require recomputation. The necessary formulæ for the determination of the orbit of the disturbed body and for its differential correction are given in proper order, and in detail, in the *Synopsis of Formulæ* which follows.

SYNOPSIS OF FORMULÆ.

The directions for computing the orbit of a disturbed satellite are given in detail. For the computation of the orbit of a satellite irrespective of perturbations, see page 475. For the computation of the orbit of a disturbed comet or minor planet, see page 476. For the computation of the orbit of a comet or minor planet irrespective of perturbations, apply the "Short Method," Part 7.

In case no assumption has been possible regarding the nature of the newly discovered body, the orbit may be computed either by the formulæ for a disturbed satellite or by those for a disturbed minor planet.

I.

Express the observed places in terms of α and δ,[1] reduced to the beginning of the year, inclusively of the aberration of the fixed stars. Compute α', δ', α'', δ'' by the formulæ A II, footnote 2, *Synopsis of Formulæ*, Part 7.

The parallax may, in general, be neglected. Its complete elimination as set forth in the footnotes of the *Synopsis of Formulæ*, Part 7, is unwarranted on account of the additional computation involved as the solution, in general, is based on more than three observations, and also because any inaccuracies committed in the direct solution may later be removed by differential correction. In case of a suspected satellite, however, the parallax corrections may be applied on the basis of the primary's distance.

Compute with (m) the mass of the attracting planet,

$$\gamma = \frac{1}{(m)}, \qquad (k)^2 = (m)k^2, \qquad (k), \qquad \frac{\sin 1''}{(k)}, \qquad \frac{2}{(k)},$$

and if account is to be taken of the Earth's mass,

$$\frac{k^2}{(k)^2}(1 + m_{\oplus})\,; \quad \log(1 + m_{\oplus}) = 0.000001.$$

Interpolate, for the beginning of the year, from an ephemeris, the solar coördinates X, Y, Z for the observed date of the epoch (for which it is most convenient to choose the date of an observation near the middle of the available arc). Then interpolate also the apparent geocentric coördinates of the attracting body at the observed epoch (ρ), (α), (δ)[2] and reduce these to the beginning of the year, including the aberration terms. Then $(\sigma) = (\rho)\cos(\delta)$.

[1] In exceptional cases it is more advantageous to express the observed places in terms of position angle p and distance s, to derive the velocities p', s', and accelerations p'', s'', in the same manner as α', δ', α'', δ'' are derived, and then to transform p', s' into α', δ', and p'', s'' into α'', δ''.

[2] *Cf.* page 477.

Similarly, the velocities[1] $(\xi)'$, $(\eta)'$, $(\zeta)'$ may be derived, either here or under IV, in terms of $\frac{1}{(k)}$ mean solar days by numerical differentiation from the data of an ephemeris, exactly as in Part 7, *Synopsis of Formulæ* A I. For this purpose the (ξ), (η), (ζ) may be computed from the corresponding (ρ), (α), (δ).

Correct the observed date of the middle place for planetary aberration on the basis of the primary's distance. With the corrected date, take from an ephemeris the heliocentric coördinates of the planet (primary) (r), (l), (b), from which (a), (d) may be computed. Then $(s) = (r)\cos(d)$.

$$R\cos D\cos A = X, \qquad R\cos D\sin A = Y, \qquad R\sin D = Z, \qquad S = R\cos D$$

$$\cos\left\{\begin{matrix}\psi\\ [\psi]\end{matrix}\right\} = \sin\omega\sin\delta + \cos\omega\cos\delta\cos(\Omega - \alpha),$$

where ψ is computed from $\omega = (\delta)$, $\Omega = (\alpha)$ and $[\psi]$ from $\omega = D$ and $\Omega = A$.

For small values of ψ (or $[\psi]$) use

$$\cos\psi = \cos\delta\cos(\alpha - \Omega), \qquad \sin\psi\cos P = \cos\delta\sin(\alpha - \Omega), \qquad \sin\psi\sin P = \sin\delta\,.$$

II.

$$\Delta = \alpha'^2\tan\delta - \alpha''(\tan\delta)' + \alpha'(\tan\delta)'',$$

$$\varkappa = -\frac{S}{\Delta}\left\{(\tan\delta\cos[A - \alpha] - \tan D)\;\alpha' + \sin[A - \alpha](\tan\delta)'\right\},$$

$$(\varkappa) = -\frac{(\sigma)}{\Delta}\left\{(\tan\delta\cos[(\alpha) - \alpha] - \tan(\delta))\,\alpha' + \sin[(\alpha) - \alpha](\tan\delta)'\right\},$$

$$[\varkappa] = -\frac{s}{\Delta}\left\{(\tan\delta\cos[(a) - \alpha] - \tan(d))\,\alpha' + \sin[(a) - \alpha](\tan\delta)'\right\}.$$

$$E = \frac{(\varkappa)}{\cos\delta}, \qquad F = \frac{\gamma\varkappa}{\cos\delta}, \qquad G = -F\left[\frac{1}{R^3} + \frac{m[\varkappa]}{\varkappa(r)^3}\right].$$

III.

Solution of the Distances of the Disturbed Body from the Earth and Attracting Masses.

$$H = \frac{F}{[r]^3} + G, \tag{a}$$

where as a first approximation $[r] = (r)$.

Then,

$$r^3 = \frac{E \pm r^3\sqrt{r^2 - (\rho)^2\sin^2\psi}}{(\rho)\cos\psi - H}, \tag{b}$$

where in the first approximation $\pm r^3\sqrt{} = 0$. The equation (b) is then solved

[1] *Cf.* footnote 2, page 482.

successively by introducing the preceding approximation of r in $\pm r^3 t$, the two resulting solutions being kept separate. After the equation (b) has been satisfied, compute for both solutions

$$\rho = \frac{E}{r^3} + \frac{F}{[r]^3} + G \tag{c}$$

$$r^2 = (\rho)^2 + \rho^2 - 2(\rho)\rho\cos\psi, \qquad [r]^2 = R^2 + \rho^2 - 2R\rho\cos[\psi] \tag{d}$$

Then repeat the whole process with more accurate values of H from formulæ (a) until H and r remain constant.

Or, for any approximation of H, correct the first approximation of r successively by

$$\delta r = 3 r_1^4 \left[1 \mp \frac{4 r_1' - 3(\rho)^2 \sin^2\psi}{3K + r_1^2 - (\rho)^2 \sin^2\psi}\right].$$

IV.

Compute $\sigma = \rho\cos\delta$ and σ' and σ'' from

$$2\sigma'\alpha' = \frac{(\sigma)}{r^3}\sin[(\alpha) - \alpha] + \gamma S\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right]\sin[A - \alpha] + \frac{(\kappa)}{(r)^3}\sin[(\alpha) - \alpha] - \sigma\alpha'',$$

$$\sigma'' = \frac{(\sigma)}{r^3}\cos[(\alpha) - \alpha] + \gamma S\left[\frac{1}{[r]^3} - \frac{1}{R^3}\right]\cos[A - \alpha] - \frac{(\kappa)}{(r)^3}\cos[(\alpha) - \alpha]$$
$$- \sigma\left[\frac{1}{r^3} - \alpha'^2 + \frac{\gamma}{[r]^3}\right].$$

V.

Compute the residuals, the elements, and the constants to the equator as in Part 7, *Synopsis of Formulæ* A VI and VIII, using the closed expressions for f and g, if necessary as in Part 7, *Synopsis of Formulæ* B.

In case of doubt as to the nature of the disturbed body, four[1] orbits may require consideration. The orbits with reference to the Sun as primary, rest on $x = \xi - X$, $x' = \xi' - X'$, etc. The orbits with reference to the second attracting mass (planet) as primary rest on $x = \xi - (\xi)$; $x' = \xi' - (\xi)'$, etc.[2]

If an assumption regarding the nature of the disturbed body has been possible, then the orbits with reference to only one of the attracting masses (either Sun or planet) require consideration.

The selection of the physical solution in any case is made on the basis of the representation of the positions upon which the solutions were based.

VI.

To improve the physical solution, the special perturbations of the disturbing body are best computed by ENCKE's method of rectangular coordinates and applied to the positions computed from the osculating elements for the purpose of deriving the residuals to be removed.

[1] *Cf.* footnote 1, page 501, where three solutions for the distances, corresponding to six mathematical orbits were found to exist.

[2] Provided it is intended to neglect the difference of the aberration times for the planet and satellite. For greater rigidity *cf.* page 478, third paragraph which applies to the velocities as well as to the coordinates.

In the case of a disturbed satellite, the Sun being the disturbing body, ENCKE'S formulæ as given in VON OPPOLZER'S *Lehrbuch zar Bahnbestimmung*, Vol. II, pages 72 to 81, take the following form, starting with equation (3), page 73:

$$\frac{d^2 x}{(k)^2 dt^2} = -\frac{x}{r^3} - \gamma \frac{(x) + x}{[r]^3} + \gamma \frac{(x)}{(r)^3}, \qquad \gamma = \frac{1}{(m)} = \frac{k^2}{(k)^2}$$

and since

$$x = x_0 + \overline{\xi}$$

and

$$\frac{d^2 x_0}{(k)^2 dt^2} = -\frac{x_0}{r_0^3},$$

$$\frac{d^2 \overline{\xi}}{(k)^2 dt^2} = -\gamma \left\{ \frac{(x) + x}{[r]^3} - \frac{(x)}{(r)^3} \right\} + \left\{ \frac{x_0}{r_0^3} - \frac{x}{r^3} \right\},$$

or

$$\frac{d^2 \overline{\xi}}{dt^2} = -k^2 \left\{ \frac{(x) + x}{[r]^3} - \frac{(x)}{(r)^3} \right\} + (k)^2 \left\{ \frac{x_0}{r_0^3} - \frac{x}{r^3} \right\}.$$

Let

$$q = (x_0 + \tfrac{1}{2} \overline{\xi})\, \overline{\xi} + (y_0 + \tfrac{1}{2} \overline{\eta})\, \overline{\eta} + (z_0 + \tfrac{1}{2} \overline{\zeta})\, \overline{\zeta}.$$

With q as argument take f from v. OPPOLZER'S table or compute f directly. Then

$$\frac{d^2 \overline{\xi}}{dt^2} = -k^2 \left\{ \frac{(x) + x}{[r]^3} - \frac{(x)}{(r)^3} \right\} + \frac{(k)^2}{r_0^3} \left\{ f q x - \overline{\xi} \right\} - w k^2 \left\{ \frac{(x) + x}{[r]^3} - \frac{(x)}{(r)^3} \right\} = \Sigma (X);$$

or, if

$$h = \frac{w (k)^2}{r_0^3}; \quad a = \frac{x_0 + \frac{1}{2} \overline{\xi}}{r_0^2 (1 + \frac{1}{12} h)}; \quad b = \frac{y_0 + \frac{1}{2} \overline{\eta}}{r_0^2 (1 + \frac{1}{12} h)}; \quad c = \frac{z_0 + \frac{1}{2} \overline{\eta}}{r_0^2 (1 + \frac{1}{12} h)},$$

$$S_{(x)} = {}^{\text{II}}f_{(x)} (a + iw) + \tfrac{1}{12} \Sigma (X) - \tfrac{1}{240} f^{\text{II}} (a + iw) \ldots, \quad S_{(y)}, \quad S_{(z)}, \text{ etc.}$$

Then

$$q = \frac{a S_{(x)} + b S_{(y)} + c S_{(z)}}{1 - \frac{h}{12} f (ax + by + cz)}.$$

If further

$$h' = \frac{h}{1 + \frac{1}{12} h},$$

then

$$\frac{d^2 \overline{\xi}}{dt^2} = \Sigma (X) + h' \left\{ f q x - S_{(x)} \right\}, \quad \frac{d^2 \overline{\eta}}{dt^2}, \quad \frac{d^2 \overline{\zeta}^2}{dt^2}.$$

$$R^2 = r_0^2 (1 + \tfrac{1}{12} h); \qquad S (X) = [wk]^2 \left\{ \frac{(x)}{(r)^3} - \frac{[x]}{[r]^3} \right\}.$$

The residuals are removed by differential correction by applying the proper formulæ of Part 7, *Synopsis of Formulæ*, A VII or B.

APPLICATION OF LEUSCHNER'S METHOD OF DIRECT SOLUTION OF ORBITS OF DISTURBED BODIES.

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UNIVERSITY OF CALIFORNIA PUBLICATIONS.
PUBLICATIONS OF THE LICK OBSERVATORY, VOL. VII, PART 10.

APPLICATIONS OF LEUSCHNER'S METHOD OF DIRECT SOLUTION OF ORBITS OF DISTURBED BODIES.

BY RUSSELL TRACY CRAWFORD.

CONTENTS.

INTRODUCTION.

Two examples are given to show the computations necessary for the determination of an orbit according to LEUSCHNER'S method of *Direct Solution of Orbits of Disturbed Bodies*, as set forth in Part 9 of this volume. The numbers of formulæ given are those of the *Synopsis of Formulæ* in Part 9.

The computation for the Seventh Satellite of Jupiter is given in full. The only difference in the direct solutions between the orbits of the Seventh and the Eighth Satellites is in the manner of solving the equation for r, so that this will be given in detail for the case of the Eighth Satellite.

The computations for the Seventh Satellite were made by R. T. CRAWFORD and A. J. CHAMPREUX; those for the Eighth Satellite were made by R. T. CRAWFORD and W. F. MEYER.

THE ORBIT OF THE SEVENTH SATELLITE OF JUPITER.

I

The observations upon which this orbit is based were made by PERRINE at the Lick Observatory. They are published in *Lick Observatory Bulletin* No. 78. They are:

	1905 Gr. M. T.		α (1905.0)			δ (1905.0)		
I	Jan.	3.633333	19°	5′	55″.5	+ 7°	13′	58″.1
II	Jan.	28.623611	21	56	5 .4	+ 8	18	11 .6
III	Feb.	8.631944	23	40	9 .8	+ 8	55	18 .8
IV	Feb.	21.642361	26	0	45 .8	+ 9	43	52 .4
V	Mar.	6.652778	28	36	3 .4	+ 10	35	50 .7

The coördinates α and δ given here are the observed coördinates corrected for parallax corresponding to Jupiter's distance.

Use for the mass of Jupiter NEWCOMB'S value, viz., $\frac{1}{m} = 1047.355$. Then

$\log \gamma = \log \frac{1}{m}$	3.0200939	$\log \frac{\sin 1''}{(k)}$	7.9600404
$\log \sqrt{\frac{1}{m}}$	1.5100470	$\log \frac{2}{(k)}$	3.5754956
$\log \sqrt{m}$	8.4899530	$\log m_\oplus$	4.4889
$\log k$	8.2355814	$\log (1 + m_\oplus)$	0.000001
$\log (k)$	6.7255344	$\log \frac{k}{(k)} \sqrt{1 + m_\oplus}$	1.510048
$\log (k)^2$	3.4510689		

Using the first, third, and fifth observations and formulæ A II of Part 7, first approximations to the velocities and accelerations for the middle date are found to be

$$\log \alpha_0' \quad 0.729270 \qquad \log \delta_0' \quad 0.272687$$
$$\log \alpha_0'' \quad 2.095593 \qquad \log \delta_0'' \quad 1.541625$$

The more accurate determination of these quantities depending upon the five observations, according to the formulæ A II, footnote 2, of Part 7, is made exactly as shown in detail in example No. 2 of Part 8. The resulting values are:

$$\log \alpha_0' \quad 0.743547 \qquad \log \delta_0' \quad 0.289298 \qquad \log (\tan \delta)_0' \quad 0.299872$$
$$\log \alpha_0'' \quad 2.069686 \qquad \log \delta_0'' \quad 1.496778 \qquad \log (\tan \delta)_0'' \quad 1.523510$$

From the *American Ephemeris and Nautical Almanac*, we take, for the middle date, approximate $\log (\rho) = 0.72426$. Using this to correct the middle date for aberration, we have for the corrected date, Feb. 8.601357. With this date then, we take out

$$^{1}\log (\rho) = 0.724258 \qquad (\alpha) = 1^h\ 34^m\ 34^s.14 \qquad (\delta) = +\ 8°\ 41'\ 1''.8$$

Then

$$(\alpha) - \alpha_0 = -\ 6^s.51$$

$$\tan \psi = 7.621292 \qquad \cos \psi = 9.999996 \qquad \sin \psi = 7.621288$$

[1] It would have been more accurate to use (ρ), (α), (δ) for the uncorrected date, and to have reduced them to the beginning of the year, including the aberration terms. *Cf.* also, last paragraph, page 477, and the discussion on page 478.

Again, from the *American Ephemeris and Nautical Almanac*, we take for the corrected middle date the heliocentric coördinates of Jupiter reduced to 1905.0. Whence

$$(l) = 35^\circ\ 28'\ 34''.6, \qquad (b) = -1^\circ\ 10'\ 36''.0 \qquad \log (r) = 0.696347$$

Transforming these to the equator in the usual way, gives

(a)	$33^\circ\ 34'\ 39''.6$	$\log \sin (d)$	9.326503
$\log \tan (d)$	9.336497	$\log \cos (d)$	9.990006

Since the observations were reduced to the beginning of the year by including the aberration terms, the Sun's coördinates should be taken out for the middle date uncorrected for aberration. They are

$X_\odot$	+0.7529310	$Y_\odot$	− 0.5851685	$Z_\odot$	− 0.2538530
$\log X_\odot$	9.8767552	$\log Y_\odot$	9.7672810n	$\log Z_\odot$	9.4045823n

$R_\odot \cos D \cos A$	9.876755
$R_\odot \cos D \sin A$	9.767281n
$\tan A$	9.890526n
A	$322^\circ\ 8'\ 45''.9$
$A - \alpha_{\prime\prime}$	$298\ \ 28\ \ 36\ .1$
$\sin A$	9.787921n
$\cos A$	9.897395

$\log S_\odot = R_\odot \cos D$	9.979360
$R_\odot \sin D$	9.404582n
$\tan D$	9.425222n
$\sin D$	9.410354n
$\cos D$	9.985132
$\log R_\odot$	9.994228

$\sin \delta_{\prime\prime}$	9.190578
$\sin D$	9.410354n
$\log \mathrm{I}$	8.600932n
$\cos \delta_{\prime\prime}$	9.994713
$\cos D$	9.985132
$\cos (A - \alpha_{\prime\prime})$	9.678338
$\log \mathrm{II}$	9.658183
$\cos [\psi]$	9.618345

II

$\log (\alpha_\odot')^3$	2.230641
$\tan \delta_{\prime\prime}$	9.195864
$\log \mathrm{I}$	1.426505
$\log \alpha_\odot''$	2.069686
$\log (\tan \delta)_\odot'$	0.299872
$\log \mathrm{II}$	2.369558
$\log \alpha_\odot'$	0.743547
$\log (\tan \delta)_\odot''$	1.523510
$\log \mathrm{III}$	2.267057
I	+ 26.6996
− II	− 234.1844
III	+ 184.9512
$\varDelta$	− 22.5336
$\log \varDelta$	1.352830n

$p;\ q;\ s$	$A;\ D;\ S$	$(\alpha);\ (\delta);\ (\sigma)$	$(a);\ (d);\ (s)$
$\tan \delta_{\prime\prime}$	9.195864	9.195864	9.195864
$\cos (p - \alpha_{\prime\prime})$	9.678338	0.000000	9.993473
$\tan \delta_{\prime\prime} \cos (p - \alpha_{\prime\prime})$	8.874202	9.195864	9.189337
$\tan q$	9.425222n	9.183932	9.336497
sub	0.107610	8.444926	9.605662
log ( )	9 532832	7.628858	8.794999n
$\log \alpha_\odot'$	0.743547		0.743547
$\log \mathrm{I}$	0.276379	8.372405	9.538546n
$\sin (p - \alpha_{\prime\prime})$	9.943994n	6.675247n	9.235709
$\log (\tan \delta)_\odot'$	0.299872	0.299872	0.299872
$\log \mathrm{II}$	0.243866n	6.975119n	9.535581
add	8.890629	9.982244	7.835714
$\log \}\ \ \{$	9.134495	8.354649	7.371295n
$\log (\ \ - s)$	9.979360n	0.719250n	0.686353n
$\log - s\ \}\ \ \{$	9.113855n	9.073899n	8.057648
$\log \varkappa;\ \log (\varkappa);\ \log [\varkappa]$	7.761025	7.721069	6.704818n

$\log R_\odot^3$	9.982684
$\log \left(\mathrm{I} = \dfrac{\varkappa}{R_\odot^3} \right)$	7.778341
$\log m$	6.979906
$\log [\varkappa]$	6.704818n
$\mathrm{colog}\ (r)^3$	7.910959
$\log \mathrm{II}$	1.595683n − 10
add	0
log [ ]	7.778341
$\log \gamma$	3.020094
$\sec \delta_{\prime\prime}$	0.005287
$\log G$	0.803722n

$\log \gamma \sec \delta_{\prime\prime}$	3.025381
$\log F$	0.786406
$\log E$	7.726356

III

Let $H = \frac{F}{[r]^3} + G$.

For first approximation to H let $(r) = [r]$.

Solution for r.

Trial I

log F	0.786406	(ρ) cos (ψ)	0.724254
colog $(r)^3$	7.910959	sub	0.264665
log I	8.697365	log denom.	1.064974
log G	0.803722n	log E	7.726356
add	9.996587	log r^3	6.661382
log H	0.800309n	log r	8.887127

Substituting this approximate value of r in the complete right member of the equation (b) for r^3, we will hereafter get two solutions for r because of the double sign before the radical. The solution coming from the positive sign for the radical will be designated r_1 and called "first solution"; that from the negative sign r_2 and called "second solution." The trials then proceed:

	Trial II	III for r_1	III for r_2
log r^2	7.774254	7.776090	7.772408
log $(\rho)^2$ sin^2 (ψ)	6.691092	6.691092	6.691092
sub	9.962572	9.962737	9.962405
diff.	7.736826	7.738827	7.734813
log $\sqrt{\text{diff.}}$	8.868413	8.869414	8.867406
log r^3	6.661382	6.664135	6.658612
log $r^3 \sqrt{\text{diff.}}$	5.529795	5.533549	5.526018
log E	7.726356	7.726356	7.726356
add	0.002753	0.002777	
sum	7.729109	7.729133	
log r^3_1	6.664135	6.664159	
log r_1	8.888045	8.888053	
sub	9.997230		9.997253
diff.	7.723586		7.723609
log r_2^3	6.658612		6.658635
log r_2	8.886204		8.886212

Second Approximation for H by (c, d, a).

	First Solution.	Second Solution.
log r^3	6.664159	6.658635
log E	7.726356	7.726356
log $\frac{E}{r^3}$	1.062197	1.067721
log H	0.800309n	0.800309n
add	9.917836	9.929941
log ρ	0.718145	0.730250
log ρ^2	1.436290	1.460500
log 2	0.301030	
log R	9.994228	
cos $[\psi]$	9.618345	
log III	0.631748	0.643853
log R^2	9.988456	

	First Solution.	Second Solution.
I	+ 0.97377	+ 0.97377
II	+ 27.30800	+ 28.87353
− III	− 4.28300	− 4.40406
$[r]^2$	+ 23.99877	+ 25.44324
log $[r]^2$	1.380189	1.405572
log $[r]$	0.690094	0.702786
log $[r]^3$	2.070283	2.108358
log F	0.786406	0.786406
log $\frac{F}{[r]^3}$	8.716123	8.678048
log G	0.803722n	0.803722n
add	9.996436	9.996736
log H	0.800158n	0.800458n
(ρ) cos (ψ)	0.724254	0.724254
sub	0.264734	0.264597
log $[(\rho)$ cos (ψ) − $H]$	1.064892	1.065055

Using Second Approximation for H by (b).

Trial	IV for r_1	IV for r_2
$\log r^2$	7.776106	7.772424
$(\rho)^2 \sin^2 (\psi)$	6.691092	6.691092
sub	9.962738	9.962407
diff.	7.738844	7.734831
$\log \sqrt{\text{diff.}}$	8.869422	8.867415
$\log r^3$	6.664159	6.658635
$\log r^3 \sqrt{\text{diff.}}$	5.533581	5.526050
$\log E$	7.726356	7.726356
add	0.002777	
sum	7.729133	
log denom.	1.064892	
$\log r_1^3$	6.664241	
$\log r_1$	8.888080	

Trial.	IV for r_2
sub	9.997253
diff.	7.723609
log denom.	1.065055
$\log r_2^3$	6.658554
$\log r_2$	8.886185

Substituting these values of r in the equation for $[\rho]$ we find (details omitted)

$\log \rho$	0.718147	0.730251
$\log [r]$	0.690096	0.702787

which agree so closely with those found in the second approximation for H that another approximation is not necessary and the solution for r is ended.

IV

Determination of σ, σ', and σ''.

	First Solution.	Second Solution.
$\log \rho_0$	0.718147	0.730251
$\cos \delta_{''}$	9.994713	9.994713
$\log \sigma_0$	0.712860	0.724964
$\log \alpha_0''$	2.069686	2.069686
log I	2.782546	2.794650
$\operatorname{colog} r_0^3$	3.335759	3.341446
$\log (\sigma)$	0.719250	0.719250
$\sin [(\alpha) - \alpha_{''}]$	6.675247n	6.675247n
log II	0.730256n	0.735943n
$\operatorname{colog} [r]^3$	7.929712	7.891639
$\operatorname{colog} R_0^3$	0.017316	0.017316
sub	9.996436	9.996736
log [ ]	0.013752n	0.014052n
$\sin (A - \alpha_{''})$	9.943994n	9.943994n
$\log S$	9.979360	9.979360
$\log \gamma$	3.020094	3.020094
log III	2.957200	2.957500
$\log (s)$	0.686353	
$\operatorname{colog} r^3$	7.910959	
$\sin [(a) - \alpha_{''}]$	9.235709	
log IV	7.833021	
−I	−606.1029	−623.2329
+II	− 5.3735	− 5.4443
+III	+906.1500	+906.7760
−IV	− 0.0068	− 0.0068
sum	+294.6668	+278.0920
log sum	2.469331	2.444188
$\log 2\, \alpha_0'$	1.044577	1.044577
$\log \sigma_0'$	1.424754	1.399611

	First Solution.	Second Solution.
$\operatorname{colog} r_0^3$	3.335759	3.341446
$\log (\alpha_0')^2$	1.487094	1.487094
sub	9.993802	9.993884
diff.	3.329561	3.335330
$\log \dfrac{\gamma}{[r]^3}$	0.949806	0.911733
add	0.001808	0.001635
log [ ]	3.331369	3.336965
$\log \sigma_0$	0.712860	0.724964
log I	4.044229	4.061929
$\log \dfrac{(\sigma_0)}{r_0^3}$	4.055009	4.060696
$\cos [(\alpha) - \alpha_{''}]$	0.000000	
log II	4.055009	4.060696
$\log \gamma\, S \left[\dfrac{1}{[r]^3} - \dfrac{1}{R_0^3} \right]$	3.013206n	3.013506n
$\cos (A - \alpha_{''})$	9.678338	9.678338
log III	2.691544n	2.691844n
$\log \dfrac{(s)}{(r)^3}$	8.597312	
$\cos [(a) - \alpha_{''}]$	0.000000	
log IV	8.597312	
−I	−11072.08	−11532.65
+II	+11350.34	+11499.95
+III	− 491.52	− 491.86
−IV	− 0.04	− 0.04
σ_0''	− 213.30	− 524.60
$\log \sigma_0''$	2.328991n	2.719828n

Determination of the Velocities of Jupiter's Geocentric Rectangular Coördinates at the Middle Date.

1905 Gr. M. T.	Feb. 1.0	5.0	9.0	13.0	Feb. 17.0
log (ρ)	0.7147895	0.7198293	0.7246981	0.7293847	0.7338806
(α) 1905.0	22° 33′ 23″.4	23° 6′ 36″.0	23° 41′ 51″.8	24° 19′ 3″.3	24° 58′ 3″.8
(δ) 1905.0	+ 8 13 42 .1	+ 8 27 42 .8	+ 8 42 24 .5	+ 8 57 43 .6	+ 9 13 36 5
cos (δ)	9.9955060	9.9952463	9.9949661	9 9946653	9.9943442
cos (α)	9.9654377	9.9636712	9.9617430	9.9596504	9 9573897
log (ξ)	0 6757332	0.6787468	0.6814072	0.6837004	0.6856145
sin (α)	9.5838721	9.5938371	9.6041303	9.6146801	9.6254232
log (η)	0.2941676	0.3089127	0.3237945	0.3387301	0.3536480
sin (δ)	9.1556969	9.1677647	9.1800634	9.1925153	9.2050500
log (ζ)	9.8704864	9.8875940	9 9047615	9.9219000	9.9389306

	(ξ)	f^{I}	f^{II}	f^{III}	f^{IV}
Feb. 1.0	+4.739508				
		+0.033002			
5.0	4.772510		−0.003678		
		29324		−0.000223	
9.0	4.801834		3901		+0.000023
		25423		−0.000200	
13.0	4.827257		−0.004101		
		+0.021322			
Feb. 17.0	+4.848579				

	(η)	f^{I}	f^{II}	f^{III}	f^{IV}
Feb. 1.0	+1.968646				
		+0.067987			
5.0	2.036633		+0.003011		
		70998		−0.000266	
9.0	2.107631		2745		+0.000009
		73743		−0.000257	
13.0	2 181374		+0.002488		
		+0.076231			
Feb. 17.0	+2.257605				

	(ζ)	f^{I}	f^{II}	f^{III}	f^{IV}
Feb. 1.0	+0.7421410				
		+0.0298176			
5.0	0.7719586		+0.0013088		
		311264		−0.0001096	
9.0	0.8030850		11992		−0.0000042
		323256		−0.0001138	
13.0	0.8354106		+0.0010854		
		+0.0334110			
Feb. 17.0	+0.8688216				

Date: = Feb. 8.631944

$n = -0.092014$

$N_{,}^{3}(n) = -0.16243$

$N_{,}^{4}(n) = -0.0819$

(φ)	(ξ)	(η)	(ζ)
$f^{\mathrm{I}}\ (a+iw)$	+0.0273735	+0.0723705	+0.0317260
$n\,f^{\mathrm{II}}\ (a+iw)$	+ 3589	− 2526	− 1103
$N_{,}^{3}(n)\,f^{\mathrm{III}}\ (a+iw)$	+ 342	+ 424	+ 181
$N_{,}^{4}(n)\,f^{\mathrm{IV}}\ (a+iw)$	− 19	− 07	+ 3
sum	+0.0277647	+0.0721594	+0.0316341
log sum	8.443493	8.858293	8.500155
log $(\varphi)_{0}'$	1.115899	1.530699	1.172561

V

	First Solution.	Second Solution.
log σ_0	0.712860	0.724964
cos α_u	9.961837	9.961837
sin α_u	9.603641	9 603641
tan δ_u	9 195864	9.195864
log σ_0 cos α_u	0.674697	0.686801
log $(\xi)_0$	0.681177	0.681177
sub	8.177067	8.115077
log x_0	8.851764n	8.796254
log σ_0 sin α_u	0.316501	0.328605
log $(\eta)_0$	0.322421	0.322421
sub	8.137500	8.156571
log y_0	8.454001n	8 478992
log σ_0 tan δ_u	9.908724	9.920828
log $(\zeta)_0$	9.903183	9.903183
sub	8.108583	8.617692
log z_0	8.011766	8.520875
log x_0^2	7.703528	7.592508
log y_0^2	6.908002	6.957984
add	0.064507	0.090608
sum	7.768035	7.683116
log z_0^2	6.023532	7.041750
add	0.007752	0 089328
log r_0^2	7.775787	7.772444
log r_0*	8.887893	8.886222

*These values would have checked more closely with the final values resulting from the trials, if the inaccuracy indicated in the foot note 2, page 482, had not been committed.

	First Solution.	Second Solution.
log σ_0'	1.424754	1.399611
log I	1.386591	1.361448
log α_0'	0.743547	0.743547
log II	1.060048	1.072152
I	+ 24.35517	+ 22.98517
− II	− 11.48281	− 11.80733
− III = − $(\xi)_0'$	− 13.05867	− 13.05867
x_0'	− 0.18631	− 1.88083
log x_0'	9.270236n	0.274350n
log I	1.028395	1 003252
log II	1.418244	1.430348
I	+ 10.67566	+ 10.07516
II	+ 26.19653	+ 26.93694
− III = − $(\eta)_0'$	− 33.93900	− 33.93900
y_0'	+ 2.93319	+ 3.07310
log y_0'	0.467340	0.487577
log I	0.620618	0.595475
log $(\tan\delta)_0'$	0.299872	0 299872
log II	1.012732	1.024836
I	+ 4.17463	+ 3.93981
II	+ 10.29750	+ 10.58854
− III = − $(\zeta)_0'$	− 14 87855	− 14.87855
z_0'	− 0.40642	− 0 35020
log z_0'	9.608975n	9.544316n
log $x_0\,x_0'$	8.122000	9.070604n
log $y_0\,y_0'$	8.921341n	8 966569
add	9.924936	9.432450
sum	8.846277n	8.399019n
log $z_0\,z_0'$	7.620741n	8.065191n
add	0.025098	0.165431
log $r_0\,r_0'$	8.871375n	8.564450n
log r_0'	9.983295n	9.678265n

Preliminary Osculating Elements (Equatorial).

	First Solution.	Second Solution.
log $y_0 z_0'$	8.062976	8.023308n
log $z_0 y_0'$	8.479106	9.008452
sub	9.789868	0.042765
$\sqrt{p}$ sin i sin Ω	8.268974n	9.051217n
log $x_0 z_0'$	8.460739	8.340570n
log $z_0 x_0'$	7.282002n	8.795225n
sub	0.027864	9.812224
$\sqrt{p}$ sin i cos Ω	8.488603	8.607449
tan Ω	9.780371n	0.443768n
Ω	328° 54′ 24″.8	289° 47′ 45″.3
sin Ω	9.713012n	9.973546n
cos Ω	9.932641	9.529778
$\sqrt{p}$ sin i	8.555962	9.077671
log $x_0 y_0'$	9.319104n	9.283831
log $y_0 x_0'$	7.724237	8.753342n
sub	0.010900	0.112199
$\sqrt{p}$ cos i	9.330004n	9.396030
tan i	9.225958n	9.681641
i	170° 26′ 57″.8	25° 39′ 41″.7
sin i	9.219896	9.636543
cos i	9.993938n	9.954902
log $\sqrt{p}$	9.336066	9.441128
log p	8.672132	8.882256
log $\frac{p}{r_0}$	9.784052	9.996071
sub	9.809003	7.958444
e cos v_0	9.593055n	7.954515n
e sin v_0	9.319361n	9.119393n
tan v_0	9.726306	1.164878
v_0	208° 2′ 4″.4	266° 5′ 11″.3
sin v_0	9.672101n	9.998986n
cos v_0	9.945795n	8.834108n
log e	9.647260	9.120407
log e''	4.961685	4.434832
φ	26° 21′ 9″.5	7° 34′ 56″.1
½ v_0	104° 1′ 2″.2	133° 2′ 35″.6
log $(1 - e)$	9.745173	9.938545
log $(1 + e)$	0.159529	0.053827
log $\frac{1-e}{1+e}$	9.585644	9.884718

	First Solution.	Second Solution.
log x_0 cos Ω	8.784405n	8.326032
log y_0 sin Ω	8.167013	8.452538n
add	9.880054	9.529115
log r_0 cos u_0	8.664459n	7.855147n
log r_0 sin u_0	8.791870	8.884332
tan u_0	0.127411n	1.029185n
u_0	126° 42′ 48″.4	95° 20′ 30″.0
v_0	208 2 4.4	266 5 11.3
ω	278 40 44.0	189 15 18.7
sin u_0	9.903977	9.998110
cos u_0	9.776566n	8.968925n
log r_0	8.887893	8.886222
log e^2	9.294520	8.240814
log $(1 - e^2)$	9.904702	9.992372
log a	8.767430	8.889884
log a^3	6.302290	6.669652
log $a^{3/2}$	8.151145	8.334826
log $\frac{k}{(k)}\sqrt{1 + m}$	1.510048	1.510048
log 1 year (days)	2.562581	2.562581
log P (days)	2.223774	2.407455
P	167^d.4073	255^d.5376
log 360°	2.556302	2.556302
log $\mu°$	0.332528	0.148847
$\mu°$	2°.150445	1°.408793
log $\sqrt{\frac{1-e}{1+e}}$	9.792822	9.942359
tan ½ v_0	0.602671n	0.029687n
tan ½ E_0	0.395493n	9.972046n
½ E_0	111° 54′ 46″.7	136° 50′ 33″.7
E_0	223 49 33.4	273 41 7.4
sin E_0	9.840400n	9.999101n
log e'' sin E_0	4.802085n	4.433933n
e'' sin E_0	− 63399″.4	− 27160″.2
	− 17° 36′ 39″.4	− 7° 32′ 40″.2
M_0	241 26 12.8	281 13 47.6
log $ab.\ T$	8.47943n	8.49153n
$ab.\ T$	− 0.030160	− 0.031012
t_0 = Feb.	8.631944	8.631944
Epoch = Feb.	8.601784	8.600932

Constants for the Equator 1905.0.

(Preliminary Osculating Elements).

	First Solution.	Second Solution.		First Solution.	Second Solution.
$\sin a \sin A_a$	9.932641	9.529778	$\sin b \sin B_a$	9.713012n	9.973546n
$\sin a \cos A_a$	9.706950n	9.928448	$\sin b \cos B_a$	9.926579n	9.484680
$\tan A_a$	0.225691n	9.601330	$\tan B_a$	9.786433	0.488866n
A_a	120° 44′ 26″.0	21° 46′ 5″.6	B_a	211° 26′ 52″.5	287° 58′ 30″.4
ω	278 40 44 .0	189 15 18 .7	B'	130 7 36 .5	117 13 49 .1
A'	39 25 10 .0	211 1 24 .3			
$\sin A_a$	9.934241	9.569202	$\sin B_a$	9.717441n	9.978268n
$\cos A_a$	9.708550n	9.967872	$\cos B_a$	9.931008n	9.489402
$\sin a$	9.998400	9.960576	$\sin b$	9.995571	9.995278

First Solution.

$$x = r\,[9.998400] \sin(39°\ 25'\ 10''.0 + v)$$
$$y = r\,[9.995571] \sin(130\ \ 7\ \ 36.5 + v)$$
$$z = r\,[9.219896] \sin(278\ \ 40\ \ 44.0 + v)$$

Second Solution.

$$x = r\,[9.960576] \sin(211°\ 1'\ 24''.3 + v)$$
$$y = r\,[9.995278] \sin(117\ \ 13\ \ 49.1 + v)$$
$$z = r\,[9.636543] \sin(189\ \ 15\ \ 18.7 + v)$$

In order to determine which of the two solutions is the physical solution, a place other than those upon which the orbits are based was computed from each orbit. An observation of August 9 was available at the time this work was done. The position was determined without taking into account the perturbations due to the action of the Sun.

The representation of the August 9 observation in position angle and distance (with respect to Jupiter) is

		First Solution.	Second Solution.
(O – C)	Δp	+216°.9	+2°.7
	Δs	+ 15′.6	+5′.2

These residuals lead us to reject the first solution (which is retrograde) and accept the second solution (which is direct).

VI

Using the second solution, the special perturbations due to the action of the Sun were then computed by ENCKE's method for the period January 4 to March 9 (using eight day intervals).

Then with the osculating elements and these perturbations, the positions for the dates of the fundamental places, January 3, January 28, February 21 and March 6 were computed. The representation of these places is:

		Jan. 3	Jan. 28	Feb. 21	March 6
(O – C)	$\Delta\alpha$	– 9″.6	– 2″.3	+ 2″.6	+ 1″.0
	$\Delta\delta$	+23 .9	– 0 .4	– 0 .6	– 11 .3

Using the residuals of January 3 and March 6, with the use of the series for ∂f, ∂g, *Synopsis of Formulæ*, [VII], Part 7, a differential correction was computed, giving an orbit which represented these places by

		Jan. 3	March 6
(O – C)	$\Delta\alpha$	– 3″.5	– 1″.4
	$\Delta\delta$	+ 3 .8	+ 1 .2

This orbit is given by the following:

Elements.

Epoch 1905 Feb. 8.6009 Gr. M. T.

M_0	283° 4′ 4″	
ω	187 29 41	Mean Equinox
Ω	288 19 59	and Equator
i	25 39 24	1905.0.
e	0.121519	
log a	8.893716	
$\mu°$	1°.39027	
P	258.9424 days.	

Constants for the Equator 1905.0.

$$x = r\,[9.959820] \sin (207° \; 40' \; 43'' + v)$$
$$y = r\,[9.995934] \sin (114 \; 7 \; 31 + v)$$
$$z = r\,[9.636463] \sin (187 \; 29 \; 41 + v)$$

Messrs. CRAWFORD and CHAMPREUX have also computed an orbit *irrespective of the perturbations* for this satellite. For the sake of comparison all of their results are tabulated below, together with the preliminary orbit derived by PERRINE[1] and the final orbit derived by ROSS.[2]

The three orbits by CRAWFORD and CHAMPREUX are designated below respectively by (Cr. & Ch.)$_1$, (Cr. & Ch.)$_2$, and (Cr. & Ch.)$_3$. The orbit (Cr. & Ch.)$_1$ represents the *solution irrespective of the perturbations;* the orbit (Cr. & Ch.)$_2$ represents the *direct solution of the disturbed orbit;* the orbit (Cr. & Ch.)$_3$ represents the same after the removal of the residuals of the first and last places by *differential correction.* All of these orbits are freely referred to by Professor LEUSCHNER in the *Introduction* to Part 9.

[3]ELEMENTS OF THE SEVENTH SATELLITE OF JUPITER (DIRECT MOTION) REFERRED TO THE EARTH'S EQUATOR.

Computer	Epoch 1905	M	Ω	i	ω	e	Period	a
(Cr. & Ch.)$_1$	February 8.6009	83° 17′ 57″	279° 45′ 8″	26° 27′ 14″	6° 38′ 42″	0.12576	251.1415	49′ 48″
(Cr. & Ch.)$_2$	February 8.6009	281 13 48	289 47 45	25 39 42	189 15 19	0.13195	255.5376	50 20
(Cr. & Ch.)$_3$	February 8.6009	283 19 59	288 19 59	25 39 23	187 29 41	0.12152	258.9424	50 47
PERRINE (preliminary).			275 47	26 15	182 6	0.24	200	43 48
ROSS (final)			281 7.8	26 12	331 16.8	0.0246	265.0	52 54

a (Cr. & Ch.) for log (ρ) = 0.72124; a (ROSS) for log (ρ) = 0.71624.

[1]*L. O. Bulletin* No. 78.
[2]*L. O. Bulletin* No. 82.
[3]*Cf.* Addendum, page 503.

THE ORBIT OF THE EIGHTH SATELLITE OF JUPITER.

I

The observations upon which this orbit is based are the following:

1908 Gr. M. T.			α (1908.0)			δ (1908.0)			Reference
I	Jan.	27.5288	131°	27′	50″.2	+ 18°	5′	6″.3	*M. N.* Vol. LXVIII, No. 8.
II	Feb.	22.4560	128	23	42 .1	+ 19	5	54 .3	*M. N.* Vol. LXVIII, No. 8.
III	March	8.8486	127	9	49 .1	+ 19	31	36 .7	*L. O. Bulletin* No. 156.
IV	Apr.	1.7021	126	41	24 .4	+ 19	49	27 .6	*L. O. Bulletin* No. 156.
V	Apr.	29.7023	128	26	3 .9	+ 19	35	49 .6	*L. O. Bulletin* No. 156.

The coördinates α and δ are the observed coördinates corrected for parallax corresponding to Jupiter's distance.

The mass factors that are used are the same as those used for the Seventh Satellite.

The first approximation for the velocities and accelerations, using the first, third, and fifth observations, gives:

$\log \alpha_0'$	2.228880n	$\log \delta_0'$	1.857764
$\log \alpha_0''$	0.997216	$\log \delta_0''$	0.413583n

The more accurate values of these quantities depending upon the five observations are:

$\log \alpha_0'$	2.322919n	$\log \delta_0'$	1.902567	$\log (\tan \delta)_0'$	9.914059
$\log \alpha_0''$	1.039044	$\log \delta_0''$	0.447788n	$\log (\tan \delta)_0''$	1.730332n

From the *American Ephemeris and Nautical Almanac* we take, for the middle date, approximate $\log (\rho) = 0.658699$. With this (ρ), the corrected middle date becomes March 8.82230. For this date, we have the following:

[1](α)	126° 48′ 37″.6	$(\alpha) - \alpha_{\prime\prime}$	− 21′ 11″.5	$\cos \psi$	9.999978
(δ)	+ 19 59 30 .5	$\tan \psi$	7.998946n	$\sin \psi$	7.998924n

The heliocentric coördinates of Jupiter reduced to 1908.0 for the corrected middle date are

$$(l) = 131° \; 44' \; 47''.0 \qquad (b) = +0° \; 41' \; 55''.2 \qquad \log (r) = 0.725533$$

Hence

(a)	134° 24′ 47″.1	$\sin (d)$	9.488674
$\tan (d)$	9.510330	$\cos (d)$	9.978344

The Sun's coördinates for the uncorrected middle date are

$$X_\odot = +0.9728883 \qquad Y_\odot = -0.1833903 \qquad Z_\odot = -0.0795554$$

Then

$$\log R_0 = 9.997043 \qquad \cos [\psi] = 9.859229n$$

Jupiter's geocentric rectangular velocities are given by

$$\log (\xi)' = 0.569466n \qquad \log (\eta)' = 1.322412 \qquad \log (\zeta)' = 0.973888$$

II

$\log \varkappa$	7.570066n	$\log (\varkappa)$	7.001973	$\log [\varkappa]$	7.67395
$\log G$	0.624754	$\log F$	0.615886n	$\log E$	7.027699

[1] See footnote, page 490.

III

With a first approximation to H, letting $(r) = [r]$, we get for the first trial $\log r = 9.153204$.

Using this r in the complete equation for r^3,

$$r^3 = E \pm \frac{r^3 \sqrt{r^2 - (\rho)^2 \sin^2(\psi)}}{(\rho)\cos(\psi) - H},$$

we get for a second trial[1] for each solution

$\log r_1 = 9.198200$ $\qquad$ $\log r_2 = 9.087567$

and from a third trial

$\log r_1 = 9.217353$ $\qquad$ $\log r_2 = 9.121781$

The solutions for r from this point on differ from the method used in the computation for the Seventh Satellite, so the computation will be given in detail.

	Trial IV		Trial V	
Solutions.	1	2	1	2
$\log r^2$	8.43171	8.24356	8.48248	8.22483
$\log 4r^2$	9.03677	8.84562	9.08454	8.82689
$\log 3(\rho)^2 \sin^2(\psi)$	7.79237	7.79237	7.79237	7.79237
sub	9.97453	9.95978	9.97725	9.95792
log [numer.]	9.01130	8.80540	9.06179	8.78481
$\log 3K\sqrt{r^2 - (\rho)^2 \sin^2(\psi)}$	9.24541	9.13974	9.27115	9.12908
log II	9.76589	9.66566	9.79064	9.65573
sub ; add	9.85394		9.79197	
log [ ]	9.61983	0.16527	9.58261	0.16215
$\log 3r^2$	8.91183	8.72068	8.95960	8.70195
log denom.	8.53166	8.88595	8.54221	8.86410
$\log(-M)$	6.50140	6.33689n	4.6272n	4.7227n
$\log \partial r$	7.96974	7.45094n	6.0850n	5.8586n
old $\log r_1$; $\log r_2$	9.217353	9.121781	9.241241	9.112413
add	0.023888	9.990632	9.999696	9.999758
$\log r_1$; $\log r_2$	9.241241	9.112413	9.240937	9.112171
$\log r^2$	8.482482	8.224826	8.481874	8.224342
$(\rho)^2 \sin^2(\psi)$	7.315246	7.315246	7.315246	7.315246
sub	9.969398	9.942927 +	9.969353 +	9.942859 +
diff.	8.451880	8.167753 +	8.451227 +	8.167201 +
$\log \sqrt{\text{diff.}}$	9.225940	9.083877	9.225614	9.083601
$\log r^3$	7.723723	7.337239	7.722811	7.336513
$\log r^3 \sqrt{\text{diff.}}$	6.949663	6.421116	6.948425	6.420114
$\log E$	7.027699	7.027699	7.027699	7.027699
add ; sub	0.263762	9.876570	0.263199	9.876887
sum ; diff.	7.291461	6.904269	7.290898	6.904586
$\log K$	9.568086	9.568086	9.568086	9.568086
$\log r_f^3$	7.723375	7.336183	7.722812	7.336500
$\log r_1$; $\log r_2$	9.241125	9.112061	9.240937	9.112167
sub	6.9038	7.3865		5.47
$\log M$	4.6272	4.7227		2.81

log denom.	8.8641
$\log(-M)$	2.81n
$\log \partial r$	3.95n
old $\log r_2$	9.112171
add	9.999997
$\log r_2$	9.112168

[1] After the first trial the differential formulæ, the application of which follows, might have been applied at once.

With these values of $r_{,}$ and r_2, we get for a second approximation for H, as shown in detail for the Seventh Satellite, the following:

	First Solution.	Second Solution.
$\log [r]$	0.712058	0.735589
$\log H$	0.621633	0.622103
$\log K = \{(\rho)\cos(\psi) - H\}$	9.571221	9.565903

With this second approximation for H, we get new values for $r_{,}$ and r_2 by the differential method:

$$\log r_{,} = 9.238241 \qquad \log r_2 = 9.112669$$

A third approximation for H gave the following final values[1]:

$$\log r_{,} = 9.238259 \qquad \log r_2 = 9.112669$$

IV

	First Solution.	Second Solution.
$\log \sigma$	0.616736	0.644368
$\log \sigma'$	1.300139	1.542730
$\log \sigma''$	2.897837	2.850649

V

	First Solution.	Second Solution.		First Solution.	Second Solution.
$\log x_0$	8.823340	8.989527n	$\log x_0'$	0.305495n	1.026380n
$\log y_0$	9.119019n	8.929802	$\log y_0'$	9.489438n	1.075799
$\log z_0$	8.957631n	7.755080	$\log z_0'$	0.023899	0.817936
$\log r_0$	9.238270	9.112645	$\log r_0'$	0.039975n	1.207024

Preliminary Osculating Elements (Equatorial).

	First Solution.	Second Solution.		First Solution.	Second Solution.
ω	61° 40′ 10″.7	289° 46′ 19″.6	$\log a$	9.203422	7.548979n
Ω	235 55 28 .0	139 46 52 .9	P	2.06624 years	
i	144 51 14 .9	108 44 14 .9	μ_0	0°.477023	
e	0.48212	13.55153	M_0	287° 13′ 34″.4	
			Epoch =	March 8.82326	T = Feb. 24.5842

Constants for the Equator (1908.0).

	First Solution.	Second Solution.		First Solution.	Second Solution.
A'	281° 16′ 7″.4	214° 58′ 9″.6	$\sin a$	9.943989	9.898318
B'	0 37 1 .1	358 58 17 .4	$\sin b$	9.976146	9.839306
C'	61 40 10 .7	289 46 19 .6	$\sin c$	9.760166	9.976351

The observations were represented by means of the Elements and Constants for the Equator for each solution, with the following results:

Representation from the First Solution.

		Jan. 27	Feb. 22	April 1	April 29
(O − C)	$\Delta\alpha$	−46″.2	−10″.9	+34″.9	+136″.0
	$\Delta\delta$	+10 .7	+ 4 .7	−13 .3	− 46 .5

[1] By the graphical solution, *cf.* footnote 1, page 290, Dr. E. S. HAYNES has found the number of solutions to be three, the third solution being close to the second.

Representation from the Second Solution.

(O – C)	Jan. 27	Feb. 22	April 1	April 29
$\Delta\alpha$	$-4'\ 27''.4$	$+0'\ 12''.1$	$+2'\ 24''.9$	$+18'\ 17''.2$
$\Delta\delta$	$-21\ \ 4\ .4$	$2\ \ 3\ .2$	$-0\ \ 14\ .1$	$-2\ \ 49\ .2$

It is evident from a comparison of the residuals by the two solutions that the first solution is the physical one.

VI

Using the first solution the special perturbations due to the action of the Sun were computed by the modification of ENCKE'S method over the period from January 25 to April 30.

The osculating Elements with the perturbations gave the following representation:

(O – C)	Jan. 27	Feb. 22	April 1	April 29
$\Delta\alpha$	$-42''.7$	$-9''.9$	$+37''.8$	$+146''.7$
$\Delta\delta$	$+6\ .7$	$+4\ .2$	$-14\ .4$	$-51\ .2$

Using the residuals of January 27 and April 29, a differential correction by the use of the closed expressions as shown in Part 8, Example No. 10, gave the following results:

(O – C)	Jan. 27	April 29
$\Delta\alpha$	$+5''.5$	$+11''.2$
$\Delta\delta$	$-0\ .3$	$-3\ .0$

These residuals were substituted for the constant terms in the equations arising from the first differential correction. This procedure gave the final results as follows:

Representation of the Observations.

(O – C)	Jan. 27	Feb. 22	April 1	April 29
$\Delta\alpha \cos\delta$	$+4''.3$	$+4''.5$	$+0''.8$	$+2''.8$
$\Delta\delta$	$+0\ .8$	$+0\ .8$	$-0\ .4$	$+0\ .3$

Elements.

Epoch 1908 March 8.82326 Gr. M. T.

M_0	266° 3′ 51″	
ω	67 45 54	Mean equinox
Ω	240 1 9	and equator
i	144 51 21	1908.0.
e	0.351958	
log a	9.215643	
$\mu°$	0°.457308	
P	2.15532 years	

Constants for the Equator 1908.0.

$$x = r\,[9.937932] \sin(282°\ 58'\ 7'' + v)$$
$$y = r\,[9.981246] \sin(3\ \ 1\ \ 12 + v)$$
$$z = r\,[9.760148] \sin(67\ \ 45\ \ 54 + v)$$

Below is given a tabulation of the original elements derived by COWELL and CROMMELIN and of those derived above, together with the corresponding residuals, from which it would appear that the results obtained by LEUSCHNER'S method are all that could possibly be desired.

Elements.

	COWELL and CROMMELIN	CRAWFORD and MEYER
i	148°.86	144°.86
Ω	277 .46	240 .02
e	0 .333 (about)	0 .352
a	0 .1702 (about)	0 .1643
P	2 .167 years.	2 .155

Residuals (O – C).

Date 1908	$\Delta\alpha$ C. & C.	$\Delta\alpha$ C. & M.	$\Delta\delta$ C. & C.	$\Delta\delta$ C. & M.
Jan. 27	+0".6	+4".3	+0".6	+0".8
Feb. 22	+1 .0	+4 .5	+0 .3	+0 .8
March 8	−2 .4	0 .0	+0 .4	0 .0
April 1	+2 .0	+0 .8	−1 .6	−0 .4
April 29	+8 .2	+2 .8	−2 .6	+0 .3

ADDENDUM.

Before this volume was ready to go to press the following set of elements for the Seventh Satellite by ROSS appeared in *A. N.* 4175:

$$\left.\begin{aligned}\Omega &= 291°.5\\ i &= 25 .3\\ \omega &= 182 .8\end{aligned}\right\}\text{ Earth's Equator 1905.0} \qquad \begin{aligned}e &= 0.208\\ P &= 260.6\text{ days.}\end{aligned}$$

These elements depend "upon twelve observations distributed uniformly over the observed arc from January 3, 1905, to September 25, 1906. The principal perturbations have been included." This set of elements, based upon two oppositions, is to be regarded as the best set of elements, at this time, for the orbit of the seventh satellite. Comparison shows that, with the exception of the eccentricity, the elements derived above by the direct solution agree better with these last elements by ROSS than any of the others.